Asymptotic Analysis -1 (2024) 1–19
DOI 10.3233/ASY-241945
IOS Press
CORRECTED PROOF

On subelliptic equations on stratified Lie groups driven by singular nonlinearity and weak L^1 data

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Abstract. The article is about an elliptic problem defined on a *stratified Lie group*. Both sub and superlinear cases are considered whose solutions are guaranteed to exist in light of the interplay between the nonlinearities and the weak L^1 datum. The existence of infinitely many solutions is proved for suitable values of λ, p, q by using the Symmetric Mountain Pass Theorem.

Keywords: Stratified Lie Group, Subelliptic operator, Marcinkiewicz space, Singular problem

1. Introduction

In this article, we study the existence of solutions to the following problem:

$$\begin{cases} -\Delta_{p,\mathbb{G}} u = f(x)u^{-\eta} + \lambda |u|^{q-2}u & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{P}$$

where $\Delta_{p,\mathbb{G}} u := \operatorname{div}_{\mathbb{G}}(|\nabla_{\mathbb{G}} u|^{p-2}\nabla_{\mathbb{G}} u)$ is the sub elliptic p-Laplacian and Ω is any bounded domain on a stratified Lie group $\mathbb{G}$. Here $0 < \eta < 1 < p < Q$, where Q is the homogeneous dimension of $\mathbb{G}$. We consider f to be a nonnegative function in some Marcinkiewicz spaces $\mathcal{M}^r(\Omega)$ for $r \geqslant 1$. We shall establish the existence of weak solutions for two different cases: (i) $\lambda < 0$, $1 < p < q$ and (ii) $\lambda > 0$, $1 < q < p$.

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It is worth recalling that singular local semilinear elliptic problems such as

$$\begin{cases} -\Delta u = f(x)u^{-\eta} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

arise in various physical problems, such as chemical heterogeneous catalysts (see e.g., Aris [1]) and non-Newtonian fluids (see e.g. Emden–Shaker [7]). The term u^{η} describes the resistance of the material and is used in the modelling of the heat conduction in electrically conducting materials (see e.g., Fulks–Maybee [9] and Nachman–Callegari [21]).

However, our treatment of the subject will be confined within the boundaries of mathematics. No discussion of singular semilinear PDEs can be complete without mentioning the seminal work of Lazer–McKenna [18]. The reader can also refer to the work of Arora et al. [2], Biswas et al. [3], Molica Bisci–Ortega [20], Sbai–Hadfi [25], Zuo et al. [30], and the references therein. The readers can consult Giacomoni et al. [14] for their approach to handling the singular term.

More recently, researchers turned their attention to PDEs on *stratified Lie groups* (see the definition in Section 2). One may wonder as to why this is an interesting consideration? The challenge here lies in checking if the results that hold for Euclidean domains (commutative setup) continue to hold for a noncommutative setup for a domain. Ghosh et al. [13] have recently reproduced a theory of fractional Sobolev spaces that opened the gateway to considering nonlocal elliptic PDEs.

The main results proved in this paper are as follows.

Theorem 1.1. *For every $0 < \eta < 1 < p < Q$, $-\Lambda < \lambda < \infty$ with $|\Lambda| \ll 1$, and every nonnegative function $f \in L^1_w(\Omega)\backslash\{0\}$, problem ($P$) has at least one weak solution u_η, which is positive on $W^{1,p}_0(\Omega) \cap L^\infty(\Omega)$. Moreover, when $-\Lambda < \lambda < 0, 1 < p < q$, problem ($P$) admits a unique weak solution.*

Theorem 1.2. *For every $1 < q < p < Q$, there exists $0 < \Lambda < \infty$ such that for every $\lambda \in (\frac{1}{p}, \Lambda)$, problem ($P$) has a sequence of nonnegative weak solutions $\{u_n\} \subset X$ and $u_n \to 0$ on X.*

The paper is structured as follows. In Section 2, we discuss some mathematical preliminaries and definitions. In Section 3, we consider an approximation problem and discuss the existence of solutions. In Section 4, we prove Theorem 1.1. In Section 5, we prove Theorem 1.2.

2. Preliminaries

In this section, we recall some general facts related to stratified Lie groups and the Marcinkiewicz space. For all other fundamental material used in this paper, we refer the reader to the comprehensive monograph by Papageorgiou et al. [22]. We begin by referring to the readers the article by Ghosh et al. [13, Definition 1,2] which covers the basic definitions about homogeneous and stratified Lie group. A working knowledge on these topics can be found in Garain–Ukhlov [11] and Hajlasz–Koskela [16].

Let Ω be an open subset of $\mathbb{G}$. The (horizontal) Sobolev space $W^{1,p}_{\mathbb{G}}(\Omega)$, $1 \leqslant p \leqslant \infty$, consists of the locally integrable functions $u : \Omega \to \mathbb{R}$ having the weak derivatives $X_i f$, $1 \leqslant i \leqslant N_1$, and the following finite norm

$$\|u\|_{W^{1,p}(\Omega)} := \|u\|_{L^p(\Omega)} + \|\nabla_{\mathbb{G}} u\|_{L^p(\Omega)}.$$

Here, $\nabla_{\mathbb{G}} u = (X_1 u, X_2 u, \dots, X_{N_1} u)$ is the horizontal subgradient of u.

The (horizontal) Sobolev space $W^{1,p}_{\mathbb{G},0}(\Omega)$ is the closure of $C^\infty_c(\Omega)$ in $W^{1,p}_{\mathbb{G}}(\Omega)$, equipped with the restriction $\|\cdot\|_{W^{1,p}_0(\Omega)}$ of $\|\cdot\|_{W^{1,p}(\Omega)}$. Then $W^{1,p}_{\mathbb{G},0}(\Omega)$ is a real separable and uniformly convex Banach space. The interested reader can consult e.g., Folland [8], Vodop'yanov [26], Vodop'yanov-Chernikov [27], and Xu et al. [28].

We shall refer to Capogna et al. [4, Theorem 2.3], Danielli [5, Theorem 2.8], Danielli [6, Theorem 2.2, 2.3], Folland [8, Theorem 5.15], and Hajlasz–Koskela [16, Theorem 8.1] for the following embedding result.

Lemma 2.1. *Let Ω be bounded on $\mathbb{G}$ and $1 \leqslant p \leqslant Q$. Then $W^{1,p}_{\mathbb{G},0}(\Omega)$ is continuously embedded in $L^q(\Omega)$, for every $1 \leqslant q \leqslant p^* := Qp/(Q-p)$. Moreover, the embedding is compact, for every $1 \leqslant q < p^*$.*

For every $1 < p < Q$, we equip the Sobolev space $W^{1,p}_{\mathbb{G},0}(\Omega)$ with the norm

$$\|u\| := \|u\|_{W^{1,p}_{\mathbb{G},0}(\Omega)} = \|\nabla_{\mathbb{G}} u\|_{L^p(\Omega)}.$$

Next, we recall the definition of the Marcinkiewicz space $\mathcal{M}^p(\Omega)$ and some results about this space.

Definition 2.1. For every $0 < s < \infty$, the Marcinkiewicz space $\mathcal{M}^s(\Omega)$ is the set of all measurable functions f, for which there exists $c > 0$ such that

$$\left|\left\{x : \left|f(x)\right| > t\right\}\right| \leqslant \frac{c^s}{t^s}.$$

The norm on this space is defined by

$$\|f\|_{L^s_w} := \inf\left\{c > 0 : t^s\left|\left\{x : \left|f(x)\right| > t\right\}\right| \leqslant c^s, \text{ for every } t > 0\right\}.$$

One should note that $\mathcal{M}^s(\Omega)$ is a quasinormed linear space, for every $0 < s < \infty$. For a proof of the following theorems, see Grafakos [15].

Theorem 2.1. *For every $0 < s < \infty$ and f in $\mathcal{M}^s(\Omega)$ such that the measure of Ω is finite, we have $\|f\|_{L^s_w(\Omega)} \leqslant \|f\|_{L^s(\Omega)}$. Hence $L^s(\Omega) \subset \mathcal{M}^s(\Omega)$. Moreover, the following inclusions hold, for every $0 < s < r$,*

$$L^r(\Omega) \subseteq \mathcal{M}^r(\Omega) \subseteq L^s(\Omega).$$

Next, we state the Hölder inequality for quasinorms on the Marcinkiewicz space.

Theorem 2.2. *Let X be a measurable space and $a_j \in \mathcal{M}^{s_j}(X)$, where $0 < s_j < \infty$ and $1 \leqslant j \leqslant k$. Let*

$$\frac{1}{s} = \frac{1}{s_1} + \frac{1}{s_2} + \cdots \frac{1}{s_k}.$$

Then

$$\|a_1 a_2 \cdots a_k\|_{L^s_w} \leqslant s^{\frac{-1}{s}} \Pi_{j=1}^k s_j^{\frac{1}{s_j}} \Pi_{j=1}^k \|a_j\|_{L^{s_j}_w}.$$

Next is a simple algebraic lemma due to Lucio [19, Lemma 2.1] that provides the estimates which will be necessary in the sequel.

Lemma 2.2. *For every $1 < p < \infty$, there exists a constant C which depends only on p, such that for every $\zeta, \rho \in \mathbb{R}^N$, we have*

$$\langle |\zeta|^{p-2}\zeta - |\rho|^{p-2}\rho, \zeta - \rho\rangle \geqslant C\big(|\zeta| + |\rho|\big)^{p-2}|\zeta - \rho|^2.$$

Now we shall give the definition of *weaksolution* of problem (P).

Definition 2.2. A function $u \in W_0^{1,p}(\Omega)$ is said to be a weak solution of problem (P), if $u > 0$ on Ω, so that for every $K \Subset \Omega$, there exists $\delta > 0$, depending on K, such that $u \geqslant \delta > 0$ on K and for every $\psi \in C_c^1(\Omega)$, we have

$$\int_\Omega |\nabla_{\mathbb{G}} u|^{p-2}\nabla_{\mathbb{G}} u \nabla_{\mathbb{G}} \psi \, dx = \int_\Omega f(x) u^{-\eta}\psi \, dx + \lambda \int_\Omega |u|^{q-2} u \psi \, dx. \tag{2.1}$$

Henceforth, by a solution we shall mean a weak solution, defined in the sense of Definition 2.2.

3. The approximation problem

In this section we shall study the following approximation problem:

$$\begin{aligned} -\Delta_{p,\mathbb{G}} u &= \frac{f_n}{(u^+ + \frac{1}{n})^\eta} + \lambda |u|^{q-2} u \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{3.1}$$

where $f_n(x) = \min\{f(x), n\}$, $f \in L^1_w(\Omega)$(or $\mathcal{M}^1(\Omega)$), and $0 < \eta < 1 < p < Q$. We shall subdivide it into two cases in order to prove the existence of solutions of problem (3.1).

Remark 3.1. From Theorem 2.1 we can infer that $f \in \mathcal{M}^r$, for every $r > 1$, which automatically implies that $f \in L^r$ for some r, and hence f is a fairly regular function. Thus, a natural upgrade of problem (P) is to consider irregular data, say, $f \in \mathcal{M}^1$.

3.1. Case 1: $\lambda < 0, 1 < p < q < p^*$

We begin by proving the following auxiliary lemma.

Lemma 3.1. *Let $1 < p < Q$ and suppose that $a \in L^\infty(\Omega) \setminus \{0\}$ is nonnegative on Ω. Then there exists at least one solution $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ of the following problem*

$$-\Delta_{p,\mathbb{G}} u = a + \lambda |u|^{q-2} u \quad \text{in } \Omega, \qquad u > 0 \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega. \tag{3.2}$$

Moreover, for every $K \Subset \Omega$, there exists a constant $\delta(K)$, such that $u \geqslant \delta(K) > 0$ on K.

Proof. We first prove the existence of solutions. To this end, we define the energy functional J : $W_0^{1,p}(\Omega) \cap L^\infty(\Omega) \to \mathbb{R}$ as follows

$$J(u) = \frac{1}{p} \int_\Omega |\nabla_{\mathbb{G}} u|^p \, dx - \int_\Omega a(u^+) \, dx - \frac{\lambda}{q} \int_\Omega (u^+)^q \, dx.$$

Since $a \in L^\infty(\Omega)$, we obtain by invoking the Sobolev embedding,

$$J(u) \geqslant \frac{1}{p} \|u\|^p - c_1 |\Omega|^{\frac{p-1}{p}} \|a\|_{L^\infty(\Omega)} \|u\| - \frac{\lambda}{q} \int_\Omega |u|^q \, dx,$$

and since $\lambda < 0$, we get the following estimate

$$J(u) \geqslant \frac{1}{p} \|u\|^p - c_1 |\Omega|^{\frac{p-1}{p}} \|a\|_{L^\infty(\Omega)} \|u\|. \tag{3.3}$$

Hence, we can conclude that J is coercive. It is a convex C^1 functional and it is weakly lower semicontinuous. Therefore, J has a minimizer, say u_0, that solves the following equation

$$-\Delta_{p,\mathbb{G}} u = a + \lambda |u|^{q-2} u \quad \text{in } \Omega.$$

We shall now prove that this solution u_0 is positive. Clearly, we have

$$\int_\Omega |\nabla_{\mathbb{G}} u|^{p-2} \nabla_{\mathbb{G}} u \nabla_{\mathbb{G}} \psi \, dx = \int_\Omega a\psi \, dx + \lambda \int_\Omega |u|^{q-2} u \psi \, dx, \tag{3.4}$$

for every $\psi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. In particular, on testing (3.4) with $\psi = u^-$, we obtain

$$0 \geqslant -\|u^-\|^p = \int_\Omega a u^- \, dx - \lambda \int_\Omega |u^-|^q \, dx > 0. \tag{3.5}$$

For a sufficiently small $\lambda < 0$, the right hand side of equation (3.5) is positive. Therefore $u^- = 0$ a.e. on Ω, and hence $u \geqslant 0$ a.e. on Ω. The positivity of u follows by the Strong Minimum Principle for nonnegative super solutions. One can refer to the consequence of Vodop'yanov [26, Theorem 5]. This completes the proof of Lemma 3.1. □

Remark 3.2. Note that the compact embedding from Lemma 2.1 also holds for our solution space.

Lemma 3.2. *For every $n \in \mathbb{N}$, problem (3.1) has a positive solution $u_n \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, and for every $K \Subset \Omega$, there exists a constant $\delta(K)$ such that $u_n \geqslant \delta(K) > 0$ on K. Moreover, $\|u_n\| \leqslant c_2$, for some $c_2 > 0$, independent of n.*

Proof. By Lemma 3.1, for a fixed $n \in \mathbb{N}$, and for any $g \in L^p(\Omega)$, there exists a u_n satisfying

$$\begin{aligned} -\Delta_{p,\mathbb{G}} u &= \frac{f_n}{(g^+ + \frac{1}{n})^\eta} + \lambda |u|^{q-2}u \quad \text{in } \Omega, \\ u = 0 \quad &\text{on } \partial\Omega. \end{aligned} \tag{3.6}$$

We define $F : L^p(\Omega) \to L^p(\Omega)$ so that $F(g) = I(u) = u$, where u is a solution of problem (3.6) and I is the inclusion map from $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ to $L^p(\Omega)$.

Claim. *F is continuous.*

Indeed, consider any $g \in L^p(\Omega)$. Then there exists a sequence $\{g_m\}$ in $L^p(\Omega)$ such that $g_m \to g$ in $L^p(\Omega)$. In order to show that F is continuous, we have to prove that $u_m := F(g_m) \to u =: F(g)$ in $L^p(\Omega)$. By recalling the properties of the map F, we have for every $\psi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$,

$$\int_\Omega |\nabla_{\mathbb{G}} u_m|^{p-2} \nabla_{\mathbb{G}} u_m \cdot \nabla_{\mathbb{G}} \psi \, dx = \int_\Omega \frac{f_n}{(g_m^+ + \frac{1}{n})^\eta} \psi \, dx + \lambda \int_\Omega |u_m|^{q-2} u_m \psi \, dx, \tag{3.7}$$

and

$$\int_\Omega |\nabla_{\mathbb{G}} u|^{p-2} \nabla_{\mathbb{G}} u \nabla_{\mathbb{G}} \psi \, dx = \int_\Omega \frac{f_n}{(g^+ + \frac{1}{n})^\eta} \psi \, dx + \lambda \int_\Omega |u|^{q-2} u \psi \, dx. \tag{3.8}$$

We define

$$g_{m,n} := \left(\left(g_m^+ + \frac{1}{n} \right)^{-\eta} - \left(g^+ + \frac{1}{n} \right)^{-\eta} \right).$$

Next, we choose $\psi = (u_m - u)$ in (3.7) and (3.8), then subtract the resulting equations and use the Sobolev embedding, to arrive at the following estimate

$$\begin{aligned} &\int_\Omega \big(|\nabla_{\mathbb{G}} u_m|^{p-2} \nabla_{\mathbb{G}} u_m - |\nabla_{\mathbb{G}} u|^{p-2} \nabla_{\mathbb{G}} u \big) \big(\nabla_{\mathbb{G}} (u_m - u) \big) \, dx \\ &\quad = \int_\Omega f_n \left(\left(g_m^+ + \frac{1}{n} \right)^{-\eta} - \left(g^+ + \frac{1}{n} \right)^{-\eta} \right) + \lambda \int_\Omega \big(|u_m|^{q-2} u_m - |u|^{q-2} u \big) (u_m - u) \, dx \\ &\quad \leqslant n \int_\Omega |g_{m,n}| |u_m - u| \, dx + \lambda \int_\Omega \big(|u_m|^{q-2} u_m - |u|^{q-2} u \big) (u_m - u) \, dx \end{aligned}$$

$$\leqslant n\|g_{m,n}\|_{L^{(p^*)'}(\Omega)}\|u_m - u\|_{L^{p^*}(\Omega)} + \lambda \int_\Omega \left(|u_m|^{q-2}u_m - |u|^{q-2}u\right)(u_m - u)\,dx$$
$$\leqslant c_3 n\|g_{m,n}\|_{L^{(p^*)'}(\Omega)}\|u_m - u\| + \lambda \int_\Omega \left(|u_m|^{q-2}u_m - |u|^{q-2}u\right)(u_m - u)\,dx, \tag{3.9}$$

where $c_3 > 0$ is the Sobolev constant. By using Lemma 2.2 in (3.9), we can conclude that

$$\|u_m - u\| \leqslant c_4 n\|g_{m,n}\|_{L^{(p^*)'}(\Omega)}^{\frac{1}{t-1}}, \tag{3.10}$$

where $t = p$ if $p \geqslant 2$, and $t = 2$ if $p < 2$. We note that $|g_{m,n}| \leqslant 2n^{\eta+1}$, and up to a subsequence, $g_{m,n} \to 0$, as $m \to \infty$. Thus, using the Lebesgue Dominated Convergence Theorem in (3.10), we can conclude that $u_m \to u$ in $L^p(\Omega)$. As a result, F is indeed continuous.

Claim. *F is compact.*

Indeed, on taking u as a test function in (3.6) we have,

$$\|u\|^p \leqslant \int_\Omega n^{\eta+1}u\,dx + \lambda \int_\Omega |u|^q\,dx.$$

Invoking the compact embedding and the fact that $\lambda < 0$, we obtain

$$\|u\|^p \leqslant C_5 n^{\eta+1}|\Omega|^{\frac{p-1}{p}}\|u\|.$$

Therefore,

$$\|u\| \leqslant C_6, \tag{3.11}$$

where $C_6 > 0$ is a constant, independent of choice of g. In order to show that F is compact, let $\{g_m\}$ be a bounded sequence in $L^p(\Omega)$. Then by (3.11), we have $\|F(g_m)\| \leqslant C_7$. Therefore, by the compact embedding of $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ in $L^p(\Omega)$, there is a subsequence which strongly converges in $L^p(\Omega)$. Hence, F is indeed compact.

We now apply the Schauder Fixed Point Theorem, which guarantees the existence of a fixed point u_n which solves problem (3.2). Positivity of the solution follows from Lemma 3.1.

Claim. *$\{u_n\}$ is uniformly bounded on $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.*

Indeed, by choosing $\psi = u_n$ as a test function in (3.6), we obtain

$$\|u_n\|^p \leqslant \int_\Omega f u_n^{1-\eta}\,dx + \lambda \int_\Omega |u_n|^q\,dx \leqslant \int_\Omega f u_n^{1-\eta}\,dx. \tag{3.12}$$

Let the domain be split into two regions in such a way that $\Omega = K \cup (\Omega \setminus K)$, where K is compact subset of Ω. We have the following estimate for $\int_\Omega f u_n^{1-\eta}\,dx$ near the boundary of $\Omega \setminus K$, which follows by the definition of the L^1_w- norm:

$$f(x)u^{1-\eta}(x)\left|\left\{x : \left|f(x)\right| \geqslant f(x)u^{1-\eta}(x)\right\}\right| < \|f\|_{1,w}, \tag{3.13}$$

where $A_f^u := |\{x : |f(x)| \geqslant f(x)u^{1-\eta}\}|$ is the measure of the set

$$\{x : |f(x)| \geqslant f(x)u^{1-\eta}\}.$$

On integrating (3.13) over $\Omega \setminus K$, we obtain

$$|A_f^u| \int_{\Omega\setminus K} f u^{1-\eta}\, dx \leqslant \int_{\Omega\setminus K} \|f\|_{1,w}\, dx,$$

which further implies

$$\int_{\Omega\setminus K} f u^{1-\eta}\, dx \leqslant C_9 \|f\|_{1,w}, \tag{3.14}$$

where the constant $C_9 > 0$ depends on the compact set K and the measure of Ω. Using (3.14) in (3.12), we can conclude that

$$\begin{aligned} \|u_n\|^p &\leqslant \int_\Omega f u_n^{1-\eta}\, dx \\ &\leqslant \int_K f u_n^{1-\eta}\, dx + \int_{\Omega\setminus K} f u_n^{1-\eta}\, dx \\ &\leqslant \int_K f u_n^{1-\eta}\, dx + C_9 \|f\|_{1,w}. \end{aligned} \tag{3.15}$$

Since $\{u_n\} \subset L^\infty(\Omega)$, it is bounded on a fixed compact set $K \subset \Omega$. Furthermore, by Lemma 3.1, we have $u_n \geqslant \delta(K) > 0$. Using the positivity of u_n on compact subset K in (3.15), we have

$$\|u_n\|^p \leqslant C_{10} \int_K f\, dx + C_9 \|f\|_{1,w}, \tag{3.16}$$

so by applying Hölder's inequality in (3.16), we get

$$\|u_n\|^p \leqslant C_{10} |K| \|f\|_{1,w} + C_9 \|f\|_{1,w}, \tag{3.17}$$

therefore, we can conclude that

$$\|u_n\| \leqslant C_{11},$$

thereby proving our claim that $\{u_n\}$ is uniformly bounded on $W_0^{1,p}(\Omega)$. Finally, we have that $\{u_n\}$ is uniformly bounded on $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. This completes the proof of Lemma 3.2. $\square$

3.2. *Case 2:* $\lambda > 0$, $1 < q < p < p^*$

We shall first prove the following auxiliary lemma.

Lemma 3.3. *Let $1 < q < p < Q$ and suppose that $b(x) \in L^\infty(\Omega)\backslash\{0\}$ is nonnegative on Ω. Then there exists at least one solution $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ of the following problem*

$$-\Delta_{p,\mathbb{G}} u = b + \lambda |u|^{q-2}u \quad \text{in } \Omega, \qquad u > 0 \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega. \tag{3.18}$$

Moreover, for every $\omega \subset \Omega$, there exists a constant $\gamma(\omega)$ such that $u \geqslant \gamma(\omega) > 0$ on K.

Proof. We first define the energy functional I in the following way, to establish the existence of solutions of (3.18). Let $I : W_0^{1,p}(\Omega) \cap L^\infty(\Omega) \to \mathbb{R}$ be defined as

$$I(u) = \frac{1}{p}\int_\Omega |\nabla_{\mathbb{G}} u|^p\, dx - \int_\Omega bu\, dx - \frac{\lambda}{q}\int_\Omega \left|u^q\right| dx.$$

Since $b \in L^\infty(\Omega)$, using the Sobolev embedding, we have

$$I(u) \geqslant \frac{1}{p}\|u\|^p - C_{12}|\Omega|^{\frac{p-1}{p}}\|b\|_{L^\infty(\Omega)}\|u\| - \frac{\lambda}{q}C_{13}\|u\|^q,$$

where C_{12} and C_{13} are the Sobolev constants. Since $\lambda > 0$, we have the following estimate

$$I(u) \geqslant \|u\|^q\left(\frac{1}{p}\|u\|^{p-q} - C_{12}|\Omega|^{\frac{p-1}{p}}\|b\|_{L^\infty(\Omega)}\|u\|^{1-q} - \frac{\lambda}{q}C_{13}\right), \tag{3.19}$$

and since $1 < q < p$, we can conclude that I is coercive. In a similar manner as in Lemma 3.1, we can conclude that I has a minimizer u which solves the equation

$$-\Delta_{p,\mathbb{G}} u = b + \lambda |u|^{q-2}u \quad \text{in } \Omega,\, u|_{\partial\Omega} = 0.$$

We now prove that $u > 0$ a.e. in Ω. for every $\psi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. We have

$$\int_\Omega |\nabla_{\mathbb{G}} u|^{p-2}\nabla_{\mathbb{G}} u \nabla_{\mathbb{G}} \psi\, dx = \int_\Omega a\psi\, dx + \lambda \int_\Omega |u|^{q-2}u\psi\, dx \tag{3.20}$$

Taking $\psi = u^-$ as the test function in (3.4), we get

$$0 \geqslant -\left\|u^-\right\|^p = \int_\Omega bu^-\, dx + \lambda \int_\Omega \left|u^-\right|^{q-2}u^-\, dx,$$

and since $\lambda > 0$, the right hand side of the above equation is positive, which is a contradiction. Therefore $u^- = 0$, hence $u \geqslant 0$ a.e. in Ω. The positivity of u follows by the Strong Minimum Principle for nonnegative super solutions. This completes the proof of Lemma 3.3. □

Lemma 3.4. *For every $n \in \mathbb{N}$, problem (3.1) has a positive solution $u_n \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, and for every $\omega \Subset \Omega$, there exists a constant $\gamma(\omega)$ such that $u_n \geqslant \gamma(\omega) > 0$ on ω. Moreover, $\|u_n\| \leqslant C_{14}$, for some $C_{14} > 0$ which is independent of n.*

Proof. By Lemma 3.3, for every $n \in \mathbb{N}$ and $h \in L^p(\Omega)$, there exists u_n such that

$$-\Delta_{p,\mathbb{G}} u = \frac{f_n}{(h^+ + \frac{1}{n})^\eta} + \lambda |u|^{q-2} u \quad \text{in } \Omega, u|_{\partial\Omega} = 0. \tag{3.21}$$

Define $F : L^p(\Omega) \to L^p(\Omega)$ so that $F(h) = I(u) = u$, u is a solution of (3.21), where I is the inclusion map from $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ to $L^p(\Omega)$.

Claim. *F is continuous.*

The proof that F is continuous is very similar to that of Lemma 3.2.

Claim. *F is compact.*

Indeed, taking u as a test function in (3.21), we have

$$\|u\|^p \leqslant \int_\Omega n^{\eta+1} u \, dx + \lambda \int_\Omega |u|^q \, dx. \tag{3.22}$$

Then by using the compact embedding on the right side of (3.22), we obtain

$$\|u\|^p \leqslant C_{15} n^{\eta+1} |\Omega|^{\frac{p-1}{p}} \|u\| + \lambda C_{16} \|u\|^q. \tag{3.23}$$

Suppose that$u$ were unbounded. Then on dividing $\|u\|^q$ in (3.23), we would have

$$\|u\|^{p-q} \leqslant C_{15} n^{\eta+1} |\Omega|^{\frac{p-1}{p}} \|u\|^{1-q} + \lambda C_{16},$$

and since $q < p$, we arrive at a contradiction. Therefore,

$$\|u\| \leqslant C_{17}, \tag{3.24}$$

where $C_{17} > 0$ is a constant independent of the choice of h. Working along the similar lines as in the proof of Lemma 3.2, we can conclude that F is a compact operator. Hence for every $n \in \mathbb{N}$, problem (3.21) admits at least one solution u_n.

Claim. *$\{u_n\}$ is uniformly bounded on $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.*

Indeed, by choosing $\psi = u_n$ as a test function in (3.21), we get

$$\|u_n\|^p \leqslant \int_\Omega f u_n^{1-\eta} \, dx + \lambda \int_\Omega |u_n|^q \, dx \tag{3.25}$$

and by using the Sobolev embedding in (3.25), we have

$$\|u_n\|^p \leqslant \int_\Omega f u_n^{1-\eta}\, dx + C_{18}\lambda \|u_n\|^q,$$

where $C_{18} > 0$ is the Sobolev constant. Now, by using the idea of splitting of the domain as in the proof of Lemma 3.2, we arrive at a similar situation as in (3.7). Thus, we have the following estimate

$$\|u_n\|^p \leqslant C_{19}\|f\|_{1,w} + C_{18}\lambda \|u_n\|^q. \tag{3.26}$$

Now suppose that $\|u_n\| \to \infty$, as $n \to \infty$. Since $q < p$, on dividing (3.26) by $\|u_n\|^q$, we arrive at a contradiction. Therefore, we can conclude that $\{u_n\}$ is uniformly bounded on $W_0^{1,p}(\Omega)$, which further implies that $\{u_n\}$ is bounded on $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. This completes the proof of Lemma 3.4. □

Remark 3.3. One can observe from Lemmas 3.2 and 3.4 that $\{u_n\}$ is uniformly bounded on $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. So the compact embedding of $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ in $L^p(\Omega)$ guarantees that up to a subsequence $u_n \to u_\eta$, as $n \to \infty$, where u_η is the pointwise limit of u_n in $L^p(\Omega)$.

4. Proof of Theorem 1.1

In this section we shall prove the strong convergence of the sequence of solutions of the approximation problems (3.6) and (3.21), where the limit of these solutions gives the solution of our main problem (P) for both of these cases.

Lemma 4.1. *Let $\{u_n\}$ be a sequence of solutions of the approximation problem (3.6), given by Lemma 3.2. By Remark 3.3, if u_∞ is the pointwise limit of $\{u_n\}$, then up to a subsequence,*

$$u_n \to u_\infty \quad \text{strongly in } W_0^{1,p}(\Omega) \cap L^\infty(\Omega). \tag{4.1}$$

Proof. Since $\{u_n\}$ is the solution of (3.6), we can conclude from Remark 3.3 that $u_n \rightharpoonup u_\infty$ in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ and $u_n \to u_\infty$ in $L^s(\Omega)$for1 $\leqslant s < p^*$. On testing the weak formulation of problem (3.6) with $\psi = u_n$, we get

$$\|u_n\|^p = \int_\Omega \frac{f_n}{(u_n + \frac{1}{n})^\eta} u_n\, dx + \lambda \int_\Omega |u_n|^q\, dx,$$

and letting $n \to \infty$, we have

$$\lim_{n\to\infty} \|u_n\|^p = \int_\Omega f u_\infty^{1-\eta}\, dx + \lambda \int_\Omega u_\infty^q\, dx, \tag{4.2}$$

so since $u_n \to u_\infty$ in $L^s(\Omega)$, u_∞ satisfies the weak formulation. Thus, we have the following

$$\int_\Omega |\nabla_{\mathbb{G}} u_\infty|^{p-2} \nabla_{\mathbb{G}} u_\infty \cdot \nabla_{\mathbb{G}} \psi\, dx = \int_\Omega f u_\infty^{-\eta} \psi\, dx + \lambda \int_\Omega |u_\infty|^{q-2} u_\infty \psi\, dx. \tag{4.3}$$

Choosing $\psi = u_\eta$ as a test function in (4.3), we obtain

$$\|u_\infty\|^p = \int_\Omega f u_\infty^{1-\eta}\, dx + \lambda \int_\Omega u_\infty^q\, dx, \tag{4.4}$$

thus, by virtue of (4.2) and (4.4) along with the *uniform convexity* of $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, we can conclude that

$$\lim_{n\to\infty} \|u_n\|^p = \|u_\infty\|^p.$$

Therefore, (4.1) follows. This completes the proof of Lemma 4.1. □

Remark 4.1. With a similar argument as in the proof of Lemma 4.1, one can prove the strong convergence of the solutions of $\{u_n\}$ of (3.21).

Next, we shall prove the existence of solutions of problem (P) for both the sublinear and superlinear cases and an additional restriction on λ.

Proof of Theorem 1.1. For every $n \in \mathbb{N}$, Lemmas 3.2 and 3.4 guarantee the existence of $u_n \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ which obeys

$$\int_\Omega |\nabla_{\mathbb{G}} u_n|^{p-2} \nabla_{\mathbb{G}} u_n \nabla_{\mathbb{G}} \psi\, dx = \int_\Omega \frac{f_n}{(u_n + \frac{1}{n})^\eta} \psi\, dx + \lambda \int_\Omega |u_n|^{q-2} u_n \psi\, dx,$$
$$\text{for every } \psi \in C_c^1(\Omega). \tag{4.5}$$

Using the strong convergence derived in Lemma 4.1, we have up to a subsequence, $\nabla_{p,\mathbb{G}} u_n \to \nabla_{p,\mathbb{G}} u_\infty$ as $n \to \infty$ pointwise a.e. in Ω. Then by passing to the limit, we have

$$\lim_{n\to\infty} \int_\Omega |\nabla_{\mathbb{G}} u_n|^{p-2} \nabla_{\mathbb{G}} u_n \cdot \nabla_{\mathbb{G}} \psi\, dx = \int_\Omega |\nabla_{\mathbb{G}} u_\infty|^{p-2} \nabla_{\mathbb{G}} u_\infty \cdot \nabla_{\mathbb{G}} \psi\, dx. \tag{4.6}$$

Suppose that supp $\psi = K$, where $K \subset \Omega$ is compact. Then by Lemma 3.2 (or 3.4), there exists a constant $\delta(K)$ such that $u_n \geqslant \delta(K) > 0$ in K. Thus, $u_\infty \geqslant \delta(K) > 0$ on K, and we also have

$$\frac{f_n}{u_n^\eta} \psi \leqslant \frac{f}{\delta(K)^\eta} \|\psi\|_{L^\infty(\Omega)}.$$

Now as mentioned in Remark 3.3, we can use the pointwise convergence of $u_n \to u_\infty$ a.e. in Ω and the Lebesgue Dominated Convergence Theorem to conclude

$$\lim_{n\to\infty} \left(\int_\Omega \frac{f_n}{(u_n + \frac{1}{n})^\eta} \psi\, dx + \lambda \int_\Omega |u_n|^{q-2} u_n \psi\, dx \right) = \int_\Omega \frac{f}{(u_\infty)^\eta} \psi\, dx + \lambda \int_\Omega |u_\infty|^{q-2} u_\infty \psi\, dx \tag{4.7}$$

and invoke (4.6) and (4.7) in (4.5) to conclude that u_∞ is a weak solution of problem (P), for both consired cases, i.e. $\lambda < 0$, $1 < p < q$ and $\lambda > 0$, $1 < q < p$.

We shall prove the uniqueness of solutions for the case when $\lambda < 0, 1 < p < q$. Suppose the obtained solution is not unique, i.e., let u_1, u_2 be two solutions of (P). Then choosing $\psi = (u_1 - u_2)^+$ as a test function in (2.1), we get

$$\begin{aligned}
&\int_\Omega |\nabla_{\mathbb{G}} u_1|^{p-2} \nabla_{\mathbb{G}} u_1 \nabla_{\mathbb{G}} (u_1 - u_2)^+ \, dx \\
&\quad = \int_\Omega f u_1^{-\eta} (u_1 - u_2)^+ \, dx + \lambda \int_\Omega |u_1|^{q-2} u_1 (u_1 - u_2)^+ \, dx, \qquad (4.8) \\
&\int_\Omega |\nabla_{\mathbb{G}} u_2|^{p-2} \nabla_{\mathbb{G}} u_2 \nabla_{\mathbb{G}} (u_1 - u_2)^+ \, dx \\
&\quad = \int_\Omega f u_2^{-\eta} (u_1 - u_2)^+ \, dx + \lambda \int_\Omega |u_2|^{q-2} u_2 (u_1 - u_2)^+ \, dx. \qquad (4.9)
\end{aligned}$$

On subtracting (4.9) from (4.8), we obtain

$$\begin{aligned}
&\int_\Omega \left(|\nabla_{\mathbb{G}} u_1|^{p-2} \nabla_{\mathbb{G}} u_1 - |\nabla_{\mathbb{G}} u_2|^{p-2} \nabla_{\mathbb{G}} u_2\right) \cdot \left(\nabla_{\mathbb{G}} (u_1 - u_2)^+\right) dx \\
&\quad = \int_\Omega f \left(u_1^{-\eta} - u_2^{-\eta}\right)(u_1 - u_2)^+ \, dx + \lambda \int_\Omega \left(|u_1|^{q-2} u_1 - |u_2|^{q-2} u_2\right)(u_1 - u_2)^+ \, dx \leqslant 0. \qquad (4.10)
\end{aligned}$$

Thus, by Lemma 2.2, we have $u_1 \leqslant u_2$ a.e. in Ω. Furthermore, on subtracting (4.8) from (4.9) with $\psi = (u_2 - u_1)^+$ as a test function, we obtain by a similar argument that $u_2 \leqslant u_1$ a.e. in Ω. Therefore, the uniqueness follows. This completes the proof of Theorem 1.1. □

Remark 4.2. Consider the following problem

$$\begin{cases} -\Delta_{p,\mathbb{G}} u = f(x)|u|^{-\eta-1} u + \lambda |u|^{q-2} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \qquad \text{(Q)}$$

A careful observation reveals that the solution of problem (P) is also a solution of problem (Q). This implies that $-u$ is also a solution of problem (Q). Therefore, problem (Q) has at least two solutions. One possible direction to investigate at this stage is under what conditions does problem (Q) have infinitely many solutions – this will lead us to the next section.

5. Proof of Theorem 1.2

It will be shown in this section that there exist infinitely many solutions of problem (Q), for every $\lambda > 0, 1 < q < p$, using the variational methods. Here, the idea is to use the Symmetric Mountain Pass Theorem in some restricted domain of Ω to guarantee the existence of distinct infinitely many weak solutions. In order to use the symmetric mountain pass geometry, we shall refer to Kajikiya [17, Definition 1.1] for the definition of genus. Further, we recall the set,

$$\Gamma_k = \left\{S_k \subset X : S_k \text{ is closed and symmetric, } 0 \notin S_k, \text{ and } \gamma(S_k) \geqslant k\right\},$$

where X is a Banach space and $S \subset X$. The set Γ_k is the collection of closed and symmetric set around 0, with genus at least k.

Remark 5.1. According to Rabinowitz [24], roughly speaking, genus is a tool for measuring the size of symmetric sets.

Definition 5.1. Let X be any Banach space, $J \in C^1(X, \mathbb{R})$, and $c \in \mathbb{R}$. The function J is said to satisfy the $(PS)_c$ condition if every sequence $(u_n) \subset X$ such that $J(u_n) \to c$ and $J'(u_n) \to 0$, as $n \to \infty$, has a convergent subsequence.

As a consequence, if there exists a $(PS)_c$ subsequence, still denoted by (u_n), which converges strongly on $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, then J is said to satisfy the PS condition.

Here, the following version of the Symmetric Mountain Pass Theorem by Kajikiya [17] will be considered.

Theorem 5.1. *Let X be an infinite-dimensional Banach space and $\hat{J}$ a C^1 functional on X that satisfies the following conditions*

(i) $\hat{J}$ is even, bounded below, $\hat{J}(0) = 0$, and $\hat{J}$ satisfies the $(PS)_c$ condition.

(ii) For every $k \in \mathbb{N}$, there exists $S_k \in \Gamma_k$ such that $\sup_{u \in S_k} \hat{J}(u) < 0$.

Then for every $k \in \mathbb{N}$, $c_k = \inf_{S \in \Gamma_k} \sup_{u \in S} \hat{J}(u) < 0$ is a critical value of $\hat{J}$.

For more details on various versions of the Mountain Pass Theorem, we refer to Youssef [29].

To apply Theorem 5.1, we shall follow the technique from Ghosh–Choudhuri [12], for our problem (P) for the case $1 < q < p$. On modifying (P) as follows

$$\begin{cases} -\Delta_{p,\mathbb{G}} u = f(x)\frac{\operatorname{sign}(u)}{|u|^\eta} + \lambda |u|^{q-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.1}$$

a careful observation reveals that the weak solution of problem (5.1) is also a weak solution of problem (Q). We use the cutoff technique to guarantee the existence of infinitely many solutions of (5.1) under suitable assumptions on u given in Ghosh et al. [12], which are as follows:

(a) There exists $\delta > 0$ such that for every $x \in \Omega$, $|u| \leqslant \delta$.

(b) There exist $r > 0$ and $\beta \in (1-\eta, 2)$ such that, for every x in Ω and $|u| < r$, we have $|u|^q \leqslant \beta |u|^q$.

By choosing i so that $0 < i < \frac{1}{2}\min\{\delta, r\}$, the cutoff function is defined as $\phi : \mathbb{R} \to \mathbb{R}^+$ so that $0 \leqslant \phi(t) \leqslant 1$ and

$$\phi(t) = \begin{cases} 1, & \text{if } |t| \leqslant i \\ \phi & \text{is decreasing, if } i \leqslant t \leqslant 2i \\ 0, & \text{if } |t| \geqslant 2i. \end{cases}$$

Indeed, at first, we have to prove the existence of solutions of the following cutoff problem

$$-\Delta_{p,\mathbb{G}} u = f(x)\frac{\operatorname{sign}(u)}{|u|^\eta} + \lambda |u|^{q-2} u \phi(u), \tag{5.2}$$

with $\|u\|_{L^\infty} \leqslant i$. By Theorem 1.1, the solution of problem (P) is positive. We only need to show that u is bounded by i.

The energy functional $\hat{J} : W_0^{1,p}(\Omega) \cap L^\infty(\Omega) \to \mathbb{R}$ associated to problem (5.2) is defined as

$$\hat{J}(u) = \frac{1}{p}\int_\Omega |\nabla_{\mathbb{G}} u|^p \, dx - \frac{1}{1-\eta}\int_\Omega f|u|^{1-\eta} \, dx - \lambda \int_\Omega G(u) \, dx, \tag{5.3}$$

where $G(t) = \int_0^t |s|^{q-2} s\phi(s) \, ds$. We now prove the following auxiliary lemma.

Lemma 5.1. *For every* $0 < \eta < 1$*, let us define*

$$\Lambda = \inf\{\lambda > 0 : \textit{problem (P) has no weak solutions}\}.$$

Then $0 \leqslant \Lambda < \infty$.

Proof. Invoking Garain–Ukhlov [10], let λ_0 be the eigenvalue of the operator $(-\Delta_{p,\mathbb{G}})$ in Ω and let ξ be the associated eigenfunction. Then

$$\begin{aligned} &(-\Delta_{p,\mathbb{G}})\xi = \lambda_0 \xi \quad \text{in } \Omega, \\ &\xi > 0, \\ &\xi = 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{5.4}$$

Using $\xi \geqslant 0$ as a test function in the weak formulation of (5.4), we have

$$\lambda_0 \int_\Omega u\xi \, dx = \int_\Omega (\Delta_{p,\mathbb{G}}\xi) u \, dx = \int_\Omega \left(\frac{f(x)}{|u|^\eta} + \lambda |u|^{q-2} u\right) \xi \, dx. \tag{5.5}$$

Since solution exists for every λ, we can choose a sufficiently large $\lambda = \Lambda_1 > 0$ for which $|t|^{q-2}t + \Lambda_1 t^{-\eta} > 2\lambda_0 t$, for every $t > 0$. This is a contradiction to (5.5). Hence we can conclude that $\Lambda < \infty$. This completes the proof of Lemma 5.1. □

We now prove the following two lemmas necessary to apply the Symmetric Mountain Pass Theorem.

Lemma 5.2. *The functional* $\hat{J}$ *satisfies* $(PS)_c$ *condition and is bounded below in* $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

Proof. By using the definition of ϕ in (5.3), we have

$$\hat{J}(u) \geqslant \frac{1}{p}\|u\|^p - \frac{1}{1-\eta}\int_\Omega f|u|^{1-\eta} \, dx - \lambda C_{20}. \tag{5.6}$$

Let $K \Subset \Omega$, then by using the same idea of splitting domain like in the proof of Lemma 3.2, we have from (3.14)

$$\begin{aligned} \int_\Omega f u^{1-\eta} \, dx &= \int_K f u^{1-\eta} \, dx + \int_{\Omega\setminus K} f u^{1-\eta} \, dx \\ &\leqslant \int_K f u^{1-\eta} \, dx + C_{21}\|f\|_{1,w} \end{aligned} \tag{5.7}$$

and since $u \in L^\infty(\Omega)$, using the Hölder inequality, we get

$$\int_K f u^{1-\eta} \leqslant C_{22} \|f\|_{1,w}, \tag{5.8}$$

hence by invoking (5.7) and (5.8) in (5.6), we can conclude that

$$\hat{J}(u) \geqslant \frac{1}{p}\|u\|^p - \frac{C_{23}}{1-\eta}\|f\|_{1,w} - \lambda C_{20}.$$

Thus, $\hat{J}$ is coercive and bounded below in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

Suppose now that $\{u_n\}$ is a Palais–Smale sequence for $\hat{J}$. Then $\{u_n\}$ is bounded due to the coerciveness of $\hat{J}$. Therefore, up to a subsequence, $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. So we have, for every $\psi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$,

$$\int_\Omega |\nabla_{\mathbb{G}} u_n|^{p-2} \nabla_{\mathbb{G}} u_n \nabla_{\mathbb{G}} \psi \, dx \to \int_\Omega |\nabla_{\mathbb{G}} u|^{p-2} \nabla_{\mathbb{G}} u \nabla_{\mathbb{G}} \psi \, dx,$$

this using the compact embedding, we can conclude that $u_n \to u$ in $L^p(\Omega)$. Finally, by Lemma 4.1 we can conclude that $\|u_n\| \to \|u\|$. This completes the proof of Lemma 5.2. □

Lemma 5.3. *For every $k \in \mathbb{N}$, there exists a symmetric, closed subset $S_k \subset W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ with $0 \notin S_k$, such that $\gamma(S_k) \geqslant k$, and for every $\frac{1}{p} < \lambda < \Lambda$, we have $\sup_{u \in S_k} \hat{J}(u) < 0$.*

Proof. Let us define $X := W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, and let X_m denote any finite-dimensional subspace of X such that $\dim(X_m) = m$. By the equivalence of norms on a finite-dimensional space, we have $\|u\| \leqslant M\|u\|_{L^p(\Omega)}$, for every $u \in X_m$. To prove that there exists constant $R > 0$ such that

$$\frac{1}{p}\int_\Omega |u|^p \, dx \geqslant \int_{\{|u| \geqslant i\}} |u|^p \, dx, \quad \text{for every } u \in X_m, \|u\| \leqslant R, \tag{5.9}$$

we use the method of contradiction. Let $\{u_n\}$ be a sequence in $X_m \setminus \{0\}$ such that $u_n \to 0$ in X and

$$\frac{1}{p}\int_\Omega |u_n|^p \, dx < \int_{\{|u_n| \geqslant i\}} |u_n|^p \, dx.$$

Let us define $w_n = \frac{u_n}{\|u_n\|_{L^p(\Omega)}}$. Then we have

$$\frac{1}{p} < \int_{\{|u_n| \geqslant i\}} |w_n|^p \, dx. \tag{5.10}$$

Since X_m is finite-dimensional, we have up to a subsequence, that $w_n \to w$ in X, which further implies that $w_n \to w$ also in $L^p(\Omega)$, as $n \to \infty$. Since $u_n \to 0$, we have

$$\left|\left\{x \in \Omega : |u_n| > i\right\}\right| \to 0 \quad \text{as } n \to \infty$$

which contradicts (5.10).

Now, since $q < p$, we have $\lim_{t\to 0}\frac{|t|^{q-2}t}{t^p} = \infty$. Therefore, we can choose $0 < i \leqslant 1$ such that

$$G(u) = |u|^q \geqslant 2M^p u^p,$$

so for every $u \in X_m \setminus \{0\}$ such that $\|u\| \leqslant R$, using (5.9), we obtain

$$\begin{aligned}
\hat{J}(u) &\leqslant \frac{1}{p}\int_\Omega |\nabla_{\mathbb{G}} u|^p\,dx - \frac{1}{1-\eta}\int_\Omega f|u|^{1-\eta}\,dx - \lambda\int_{\{|u|\leqslant i\}} G(u)\,dx \\
&\leqslant \frac{1}{p}\|u\|^p - \frac{1}{1-\eta}\int_\Omega f|u|^{1-\eta}\,dx - 2M^p\lambda\int_{\{|u|\leqslant i\}} |u|^p\,dx \\
&= \frac{1}{p}\|u\|^p - \frac{1}{1-\eta}\int_\Omega f|u|^{1-\eta}\,dx - 2M^p\lambda\left(\int_\Omega |u|^p\,dx - \int_{\{|u|>i\}} |u|^p\,dx\right) \\
&\leqslant \frac{1}{p}\|u\|^p - \frac{1}{1-\eta}\int_\Omega f|u|^{1-\eta}\,dx - M^p\lambda\int_\Omega |u|^p\,dx \\
&\leqslant M^p\left(\frac{1}{p} - \lambda\right)\|u\|^p_{L^p(\Omega)} - \frac{1}{1-\eta}\int_\Omega f|u|^{1-\eta}\,dx.
\end{aligned}$$

For every $\frac{1}{p} < \lambda < \Lambda$, we have $\hat{J}(u) < 0$, so se now choose $0 < r \leqslant R$ and $S_k = \{u \in X_k : \|u\| = r\}$. This yields $\Gamma_k \neq \phi$. Since S_k is symmetric and closed with $\gamma(S_k) \geqslant k$, it follows that $\sup_{u\in S_k}\hat{J}(u) < 0$. This completes the proof of Lemma 5.3. □

Finally, we can prove our second main result.

Proof of Theorem 1.2. By the definition of $\hat{J}$ and the way we defined our cutoff function ϕ, we have that $\hat{J}$ is even and $\hat{J}(0) = 0$. By Theorem 5.1 and Lemmas 5.1 and 5.2, we can conclude that $\hat{J}$ has sequence of critical points $\{u_n\}$ such that $\hat{J}(u_n) < 0$ and $\hat{J}(u_n) \to 0^-$. The positivity and boundedness of $\{u_n\}$ follow from Lemma 3.4.

Moreover, by the definition of $\hat{J}$,

$$\begin{aligned}
\frac{1}{\beta}\langle \hat{J}'(u_n), u_n\rangle - \hat{J}(u_n) &= \frac{1}{\beta}\left[\|u_n\|^p - \int_\Omega\left(f\frac{\operatorname{sign}(u_n)u_n}{|u_n|^\eta} + \lambda|u_n|^q\phi(u_n)\right)dx\right] \\
&\quad - \left[\frac{1}{p}\|u_n\|^p - \frac{1}{1-\eta}\int_\Omega \big(f|u|^{1-\eta} - \lambda G(u)\big)\,dx,\right] \\
&= \left(\frac{1}{\beta} - \frac{1}{p}\right)\|u_n\|^p - \left(\frac{1}{\beta} - \frac{1}{1-\eta}\right)\int_\Omega f|u|^{1-\eta}\,dx \\
&\quad + \frac{\lambda}{\beta}\int_\Omega \big(\beta G(u_n) - |u_n|^{q-2}u_n\phi(u_n)\big)\,dx \\
&\geqslant \left(\frac{1}{\beta} - \frac{1}{p}\right)\|u_n\|^p + \left(\frac{1}{1-\eta} - \frac{1}{\beta}\right)\int_\Omega f|u|^{1-\eta}\,dx
\end{aligned}$$

$$\geqslant \left(\frac{1}{\beta} - \frac{1}{p}\right) \|u_n\|^p,$$

so by a simple observation,

$$\frac{1}{\beta}\langle \hat{J}'(u_n), u_n\rangle - \hat{J}(u_n) = o_n(1),$$

which implies

$$\left(\frac{1}{\beta} - \frac{1}{p}\right) \|u_n\|^p \leqslant o_n(1),$$

thus, $u_n \to 0$ in X. Now, using the fact that $u_n \in L^\infty(\Omega)$, we apply the Moser iteration technique to conclude that $\|u_n\|_{L^\infty(\Omega)} \leqslant i$, as $n \to \infty$. Therefore, problem (5.2) has infinitely many solutions, which further guarantees that problem (P) has infinitely many solutions for every $\frac{1}{p} < \lambda < \Lambda$. This completes the proof of Theorem 1.2. □

Remark 5.2. We have proved the existence and uniqueness of solutions for the superlinear case in Theorem 1.1 for sufficiently small $\lambda < 0$, hence infinitely many solutions of problem (P) will cease to exists. However, for a larger magnitude of $-\lambda$, even the existence of solutions cannot be established, and therefore we have not discussed this case in this paper.

Remark 5.3. As an illustrative example, we consider the singular semilinear Dirichlet boundary value problem studied by Perera–Silva [23]:

$$\begin{cases} -\Delta_p u = a(x)u^{-\eta} + \lambda f(x, u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.11}$$

They considered the case when $\eta > 0$, whereas we consider the case when $\eta \in (0, 1)$. However, our analysis implies that problem (5.11) has infinitely many solutions, which is an improvement over the result in [23] for the case when $\eta \in (0, 1)$.

Acknowledgements

Sahu acknowledges the funding received from the National Board for Higher Mathematics (NBHM) and Department of Atomic Energy (DAE) of India [02011/47/2021/NBHM(R.P.)/R&D II/2615]. Repovš acknowledges the funding received from the Slovenian Research and Innovation Agency [P1-0292, J1-4031, J1-4001, N1-0278]. We thank the referee for comments and suggestions.

References

[1] R. Aris, *The Mathematical Theory of Diffusion and Reaction in Permeable Catalysts, Volumes I and II*, Oxford University Press, New York, 1975.

[2] R. Arora, A. Fiscella, T. Mukherjee and P. Winkert, On double phase Kirchhoff problems with singular nonlinearity, *Adv. Nonlinear Anal.* **12** (2023), 20220312.
[3] R. Biswas, S. Goyal and K. Sreenadh, Modified quasilinear equations with strongly singular and critical exponential nonlinearity, *Adv. Nonlinear Anal.* **13** (2024), 20240019.
[4] L. Capogna, D. Danielli and N. Garofalo, An embedding theorem and the Harnack inequality for nonlinear subelliptic equations, *Comm. in Part. Diff. Equ.* **18**(9–10) (1993), 1765–1794. doi:10.1080/03605309308820992.
[5] D. Danielli, A compact embedding theorem for a class of degenerate Sobolev spaces, *Rend. Sem. Mat. Univ. Politec. Torino* **49**(3) (1991), 399–420.
[6] D. Danielli, Regularity at the boundary for solutions of nonlinear subelliptic equations, *Indiana Univ. Math. J.* **44**(1) (1995), 269–286. doi:10.1512/iumj.1995.44.1988.
[7] H.V. Emden and A.W. Shaker, Multigrid methods for a semilinear PDE in the theory of pseudoplastic fluids, in: *The Sixth Copper Mountain Conference on Multigrid Methods, Part 1*, Calhoun, 1993, pp. 231–242.
[8] G.B. Folland, Subelliptic estimates and function spaces on nilpotent Lie groups, *Ark. Mat.* **13**(2) (1975), 161–207. doi:10.1007/BF02386204.
[9] W. Fulks and J.S. Maybee, A singular non-linear equation, *Osaka Math. J.* **12** (1960), 1–19.
[10] P. Garain and A. Ukhlov, On (p, q)-eigenvalues of subelliptic operators on nilpotent Lie groups, 2021, arXiv preprint arXiv:2110.07263.
[11] P. Garain and A. Ukhlov, Singular subelliptic equations and Sobolev inequalities on Carnot groups, *Analysis and Mathematical Physics* **12** (2022), 67. doi:10.1007/s13324-022-00676-8.
[12] S. Ghosh and D. Choudhuri, Existence of infinitely many solutions for a nonlocal elliptic PDE involving singularity, *Positivity* **24**(2) (2020), 463–479. doi:10.1007/s11117-019-00690-4.
[13] S. Ghosh, V. Kumar and M. Ruzhansky, Compact embeddings, eigenvalue problems, and subelliptic Brezis–Nirenberg equations involving singularity on stratified Lie groups, *Math. Ann.* (2023). doi:10.1007/s00208-023-02609-7.
[14] J. Giacomoni, I. Schindler and P. Takáč, Sobolev versus Hölder local minimizers and existence of multiple solutions for a singular quasilinear equation, *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5)* **6**(1) (2007), 117–158.
[15] L. Grafakos, *Classical Fourier Analysis*, Vol. 2, Springer, New York, 2008.
[16] P. Hajlasz and P. Koskela, Sobolev met Poincaré, *Mem. Amer. Math. Soc.* **145** (2000), 688.
[17] R. Kajikiya, A critical point theorem related to the symmetric mountain pass lemma and its applications to elliptic equations, *J. Funct. Anal.* **225**(2) (2005), 352–370. doi:10.1016/j.jfa.2005.04.005.
[18] A.C. Lazer and P.J. McKenna, On a singular nonlinear elliptic boundary-value problem, *Proc. Amer. Math. Soc.* **111**(3) (1991), 721–730. doi:10.1090/S0002-9939-1991-1037213-9.
[19] D. Lucio, Comparison theorems for some quasilinear degenerate elliptic operators and applications to symmetry and monotonicity results, *Annales de l'Institut Henri Poincaré C, Analyse non linéaire* **15** (1998), 4.
[20] G. Molica Bisci and A. Ortega, Subcritical nonlocal problems with mixed boundary conditions, *Bull. Math. Sci.* **14** (2024), 2350011.
[21] A. Nachman and A. Callegari, A nonlinear singular boundary value problem in the theory of pseudoplastic fluids, *SIAM J. Appl. Math.* **38**(2) (1980), 275–281. doi:10.1137/0138024.
[22] N.S. Papageorgiou, V.D. Rădulescu and D.D. Repovš, *Nonlinear Analysis – Theory and Applications*, Springer, Cham, 2019.
[23] K. Perera and E.A.B. Silva, Existence and multiplicity of positive solutions for singular quasilinear problems, *J. Math. Anal. Appl.* **323** (2006), 1238–1252. doi:10.1016/j.jmaa.2005.11.014.
[24] P.H. Rabinowitz, *Minimax Methods in Critical Point Theory with Applications to Differential Equations*, CBMS Regional Conference Series in Mathematics, Vol. 65, Amer. Math. Soc., Providence, RI, 1986.
[25] A. Sbai and Y.E. Hadfi, Degenerate elliptic problem with a singular nonlinearity, *Complex Variables and Elliptic Equations* **68**(5) (2023), 701–718. doi:10.1080/17476933.2021.2014458.
[26] S.K. Vodop'yanov, Weighted Sobolev spaces and the boundary behavior of solutions of degenerate hypoelliptic equations, *Siberian Math. J.* **36**(2) (1995), 246–264, (in Russian, translated from Sibirsk. Mat. Zh., 36(2), 278–300, 1995.). doi:10.1007/BF02110147.
[27] S.K. Vodop'yanov and V.M. Chernikov, Sobolev spaces and hypoelliptic equations, *Trudy Inst. Mat.* **29** (1995), 7–62, (in Russian).
[28] C.J. Xu, Subelliptic variational problems, *Bull. Soc. Math. France* **118**(2) (1990), 147–169. doi:10.24033/bsmf.2141.
[29] J. Youssef, *The Mountain Pass Theorem: Variants*, Generalizations and Some Applications., Vol. 95, Cambridge University Press, Cambridge, 2003.
[30] J. Zuo, D. Choudhuri and D.D. Repovš, On critical variable-order Kirchhoff type problems with variable singular exponent, *J. Math. Anal. Appl.* **514**(1) (2022), 126264. doi:10.1016/j.jmaa.2022.126264.