ON A FAMILY OF HYPERBOLIC BRUNNIAN LINKS AND THEIR VOLUMES

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Abstract. An n-component link L is said to be *Brunnian* if it is non-trivial but every proper sublink of L is trivial. The simplest and best known example of a hyperbolic Brunnian link is the 3-component link known as "Borromean rings". For $n \geq 2$, we introduce an infinite family of n-component Brunnian links with positive integer parameters $Br(k_1, \dots, k_n)$ that generalize examples constructed by Debrunner in 1964. We are interested in hyperbolic invariants of 3-manifolds $S^3 \setminus Br(k_1, \dots, k_n)$ and we obtain upper bounds for their volumes. Our approach is based on Dehn fillings on cusped manifolds with volumes related to volumes of ideal right-angled hyperbolic antiprisms.

1. Introduction

If a link L in S^3 is nontrivial, yet every proper sublink of L is trivial, we say that L is *Brunnian* (or that L has the *Brunnian property*). Links with this property were studied by Milnor [18] who called them *almost trivial* links. In 1961 Debrunner [9] renamed them as *Brunnian* links, in honor of Hermann Brunn whose early contributions [8] to knot theory also included examples of such links. In recent decades, Brunnian links have been under investigation from several points of views. In particular, we refer to Bai [5], Bai and Ma [6], Bai and Wang [7], and Kanenobu [14, 15] for hyperbolic and satellite properties of Brunnian links, Lei, Wu and Zhang [17] for intersecting subgroups of Brunnian link groups, and Habiro and Meilhan [13] for finite-type invariants and Milnor invariants of Brunnian links. We would also like to mention molecular Borromean rings considered by Wang and Stoddart [22] and a possible physical realization of higher-order Brunnian structures studied by Baas [4].

The simplest and best known example of a 3-component Brunnian link is the 3-component link 6^3_2 in Rolfsen's notations [19] which is also known as the "Borromean rings". We denote it by $\mathcal{B}$. It is well-known from Thurston [21, Chapter 3] that link $\mathcal{B}$ is hyperbolic, the complement $S^3 \setminus \mathcal{B}$ can be decomposed into two copies of an ideal right-angled octahedron, and $\mathrm{vol}(S^3 \setminus \mathcal{B}) = 7.327724$, up to six digits.

We recall that a polyhedron in a hyperbolic 3-space $\mathbb{H}^3$ is said to be *ideal* if all of its vertices belong to $\partial \mathbb{H}^3$, and it is called *right-angled* if all of its dihedral angles are equal to $\pi/2$. By virtue of the Andreev theorem [3], every ideal right-angled hyperbolic polyhedron

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is determined by its 1-dimensional skeleton, up to an isometry of $\mathbb{H}^3$. An initial list of ideal right-angled hyperbolic polyhedra and their volumes was presented by Egorov and Vesnin [10, 11], and the upper volume bounds depending only of number of vertices were obtained by Alexandrov, Bogachev, Vesnin, and Egorov [2].

In the present chapter, we introduce for every $n \geq 2$, an infinite family of n-component Brunnian links which generalize examples constructed in 1961 by Debrunner [9]. We are interested in the hyperbolic structure on the complements of these links. By using decompositions of complements of fully augmented links into pairs of ideal right-angled polyhedra by a method from an appendix by Agol and Thurston in Lackenby [16], and the Dehn filling theorem, we provide upper bounds for volumes of such hyperbolic Brunnian links.

The chapter is organized as follows: In Section 2 we introduce a family of $(3n+2)$-component links L_n, for every $n \geq 2$, and demonstrate that their complement $S^3 \setminus L_n$ can be decomposed into four ideal right-angled antiprisms A_{2n}. Then we apply Thurston's formula from [21, Chapter 6] for volumes of antiprisms to obtain a formula for $\mathrm{vol}(S^3 \setminus L_n)$ (see Theorem 2.1). Next, applying Adams moves to $2n$ vertical components of L_n, we construct links L'_n with $3n$ components such that $\mathrm{vol}(S^3 \setminus L'_n) = \mathrm{vol}(S^3 \setminus L_n)$. In Section 3, we construct by Dehn fillings on $2n$ components of L'_n, a family of n-component links $Br(k_1, \ldots, k_n)$, depending on the filling parameters $k_1, \ldots, k_n$ (see Theorems 3.1 and 3.4). In particular, when all k_i are equal to 1, we get the Brunnian links from Debrunner [9], whose hyperbolicity was established by Bai [5].

2. Hyperbolic links and ideal right-angled antiprisms

Recall that a knot or a link $K \subset S^3$ is said to be *hyperbolic* if its complement $S^3 \setminus K$ admits a complete metric of constant curvature -1. Equivalently, the 3-manifold $S^3 \setminus K = \mathbb{H}^3/G$ is hyperbolic, where $\mathbb{H}^3$ is the hyperbolic 3-space and G is a discrete, torsion-free group of isometries, isomorphic to $\pi_1(S^3 \setminus K)$.

For $n \geq 2$, let us denote a link with $3n+2$ components by L_n, where each component is a circle and the components are linked in the same manner as shown in Figure 1 for the case $n = 3$.

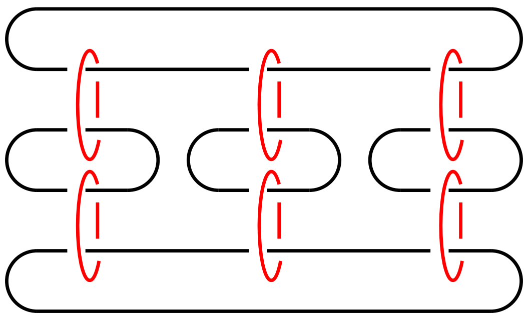

FIGURE 1. Link L_n with $3n+2$ components, case $n = 3$.

By using the computer program SnapPy [20], one can see that the 8-component link L_2 is hyperbolic and has $\mathrm{vol}(S^3 \setminus L_2) = 24.092184$, up to six digits. To demonstrate hyperbolicity of L_n for arbitrary $n \geq 2$ and find the volume formula for $S^3 \setminus L_n$, we shall use the approach from Lackenby [16] (see also Futer and Purcell [12]). In the terminology of Lackenby [16], the link L_n is augmented, with $2n$ "vertical" components (colored in red in Figure 1), and $S^3 \setminus L_n$ admits a decomposition into two ideal polyhedra P_n and P'_n with faces identified in pairs. The polyhedra P_n and P'_n are identical. By the construction each vertical component of L_n gives to a pair of triangles in P_n with a common vertex like a bowtie, as presented in Figure 2 where common vertices are red.

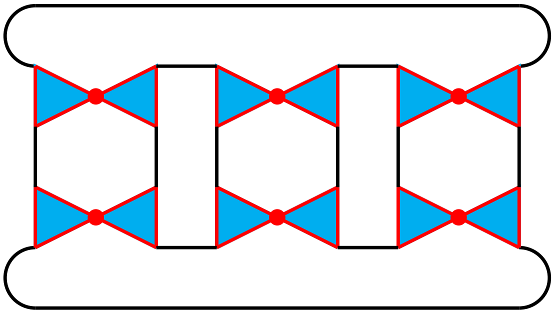

FIGURE 2. Associating bowties to vertical components of L_n, case $n = 3$.

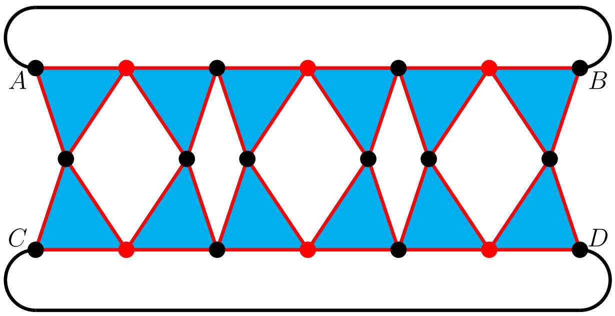

FIGURE 3. 1-skeleton of the polyhedron P_n, case $n = 3$.

At the last step, in order to obtain the polyhedron P_n, we shall compress black edges which connect two vertices of valence three, to obtain a new vertex of valence four. There are $4n$ such edges which give us $2n$ black vertices, as presented in Figure 3 (to complete the construction, it is necessary to identify the vertices A and B, as well as the vertices C and D). As a result, P_n is an ideal right-angled polyhedron with $6n$ vertices, where $2n$ red vertices correspond to bowties and $4n$ black vertices appeared after compressing black edges. Moreover, P_n has two $2n$-gonal faces as its top and bottom, $2n$ triangles incident to the top, $2n$ triangles incident to the bottom, and $4n$ quadrilaterals on the middle level. By cutting P_n along the middle line passing through quadrilaterals, we shall see that P_n can be decomposed into two identical ideal right-angled $2n$-gonal antiprisms A_{2n}. Recall that A_n has $2n+2$ faces, where two n-gonal faces can be considered as the top and the bottom, and $2n$ triangular faces on the lateral surface, see Figure 4 for the antprism A_4.

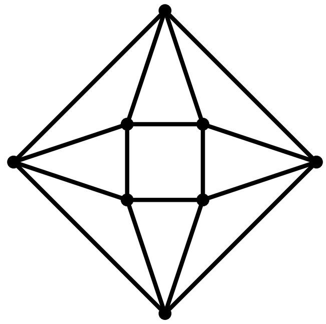

FIGURE 4. 1-skeleton of antiprism A_4.

Volumes of right-angled antiprisms are given by the following formula obtained by Thurston [21, Example 6.8.7], where an antiprism was named a *drum with triangles*,

$$\text{vol}(A_n) = 2n \left[\Lambda \left(\frac{\pi}{4} + \frac{\pi}{2n} \right) + \Lambda \left(\frac{\pi}{4} - \frac{\pi}{2n} \right) \right]. \tag{1}$$

Here, $\Lambda(\theta)$ is the Lobachevsky function defined in [21, Chapter 7] as

$$\Lambda(\theta) = - \int_0^{\theta} \log |2 \sin(t)| dt.$$

In particular, up to six digits, we have the following volumes of ideal right-angled antiprisms:

$$\begin{aligned} \text{vol}(A_3) &= 3.663863, \\ \text{vol}(A_4) &= 6.023046, \\ \text{vol}(A_5) &= 8.137885, \\ \text{vol}(A_6) &= 10.149416. \end{aligned}$$

Theorem 2.1. *For every $n \geq 2$, the following formula holds:*

$$\text{vol}(S^3 \setminus L_n) = 16n \left[\Lambda \left(\frac{\pi}{4} + \frac{\pi}{4n} \right) + \Lambda \left(\frac{\pi}{4} - \frac{\pi}{4n} \right) \right].$$

Proof. Indeed, from the above considerations we obtain

$$\text{vol}(S^3 \setminus L_n) = 2 \, \text{vol}(P_n) = 4 \, \text{vol}(A_{2n}),$$

so the assertion follows by formula (1). □

The following was proved by Adams [1, Corollary 5.1].

Theorem 2.2 (see [1])**.** *Let J be a link in S^3 such that $S^3 \setminus J$ is hyperbolic and J has a projection for which some part appears as in Figure 5(a). Let J' be the link obtained by replacing that part by the projection of J appearing in Figure 5(a) with the one appearing in Figure 5(b). Then $S^3 \setminus J'$ is hyperbolic with the same volume as $S^3 \setminus J$.*

We shall call the replacement presented in Figure 5 and its inverse the *Adams moves*. Let us apply Adams moves to each of $2n$ parts of related vertical components of the hyperbolic link L_n. The resulting link L'_n with $3n$ components is depicted in Figure 6 for the case $n = 3$.

Theorems 2.1 and 2.2 now yield the following result.

FIGURE 5. Replacing the part of the projection by Theorem 2.2.

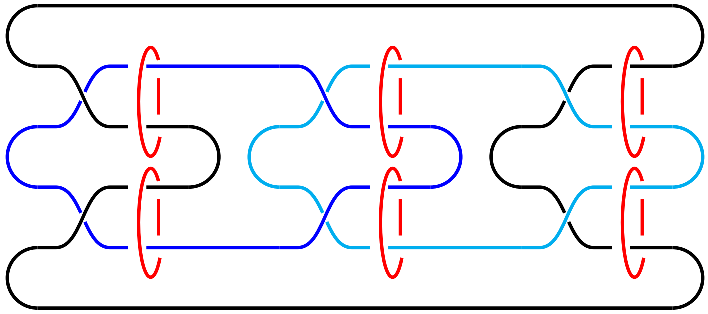

FIGURE 6. Link L'_n with $3n$ components, case $n = 3$.

Corollary 2.3. *For every $n \geq 2$, the following formula holds:*

$$\mathrm{vol}(S^3 \setminus L'_n) = 16n \left[\Lambda\left(\frac{\pi}{4} + \frac{\pi}{4n}\right) + \Lambda\left(\frac{\pi}{4} - \frac{\pi}{4n}\right)\right]. \tag{2}$$

3. A family of hyperbolic Brunnian links

Suppose that $\overline{M}$ is a compact orientable 3-manifold with ∂M a collection of tori, and that the interior $M \subset \overline{M}$ admits a complete hyperbolic structure. Then M is referred to as a *cusped manifold*. For any $n \geq 2$, the complement $S^3 \setminus L'_n$ is a cusped manifold by construction.

For every $n \geq 2$ and every positive integers $k_1, \dots, k_n$, we define an n-component link $Br(k_1, \dots, k_n)$ with a diagram having $2n$ blocks of twist regions as follows: Consider n top blocks consisting of $2k_i + 1$ positive half-twists and n bottom blocks consisting of $2k_i + 1$ negative half-twists for $i = 1, \dots, n$. A diagram of $Br(1, 2, 1)$ is presented in Figure 7. By Rolfsen [19], $Br(k_1, \dots, k_n)$ can be considered as a result of Dehn fillings with slopes $1/k_i$ and $-1/k_i$ for $i = 1, \dots, n$ on $2n$ cusps corresponding to vertical components of $S^3 \setminus L'_n$ in Figure 6. Thus, in total we get $2n$ twist regions with $2k_i + 1$ positive and negative half-twists, $i = 1, \dots, n$.

If all k_i are equal to some k, then $Br(k, k, \dots, k)$ admits a cyclic rotation symmetry of order n. The links $Br(1, 1, \dots, 1)$ appeared already in Debrunner [9, Fig. 2] and were denoted by L_F. It was demonstrated in [9] that L_F is unsplittable, but any proper sublink of L_F is completely splittable. The 5-component link presented in Rolfsen [19, p. 69] is link $Br(1, 1, 1, 1, 1)$ in our notation. By the same argument as in [9], the following result holds.

Theorem 3.1. *For every n, $Br(k_1, \dots, k_n)$ is an n-component Brunnian link.*

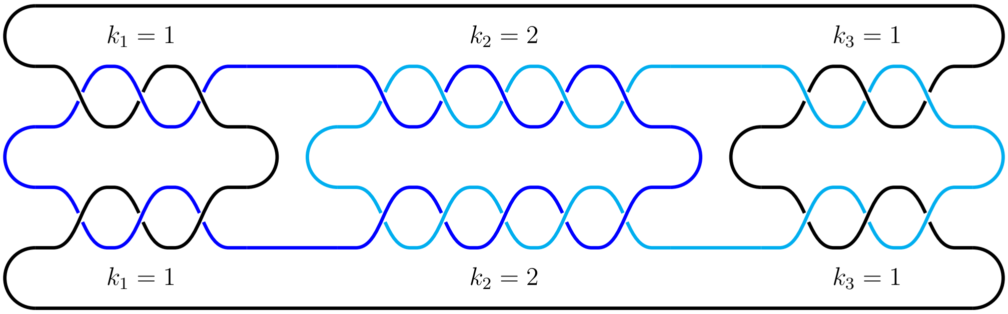

FIGURE 7. Link $Br(1, 2, 1)$.

Proof. It is easy to see that after removing one component of $Br(k_1, \dots, k_n)$, all other components can be transformed to trivial ones by a sequence of simplifying underpassing moves from (a) to (b) shown in Figure 8. □

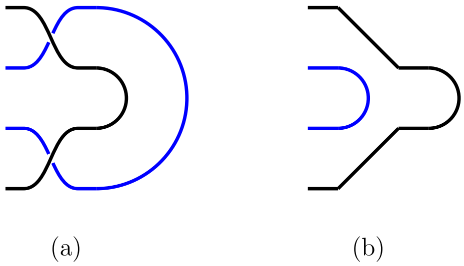

FIGURE 8. The simplifying underpassing move.

A practical method to check the hyperbolicity of Brunnian links was presented by Bai [5] who proved that for $n \geq 2$ with $k_i = 1$, $i = 1, \dots, n$, the link $Br(k_1, \dots, k_n)$ is hyperbolic [5, Theorem 1.3]. The smallest one is a 2-component link $Br(1, 1)$ with 12 crossings and $\mathrm{vol}(S^3 \setminus Br(1, 1)) = 12.528922$, up to six digits. Moreover, it can be recognized by using SnapPy [20] that $Br(1, 1) = L12n1180$, a non-alternating link with 12 components. At the same time, for $k_i \geq 3$, twisted regions in the diagram of $Br(k_1, \dots, k_n)$ have more than 6 crossings. Therefore, the following result from Futer and Purcell [12, Theorem 1.7] can be applied.

Theorem 3.2 (see [12])**.** *Let K be a link in S^3 with a prime, twist-reduced diagram $D(K)$. Suppose that every twist region of $D(K)$ contains at least 6 crossings and that each component of K passes through at least 7 twist regions (counted with their multiplicity). Then every non-trivial Dehn filling of all the components of K is hyperbolic.*

The relation between volumes of cusped manifold and hyperbolic manifolds obtained by Dehn filling is due to Gromov and Thurston [21, Theorem 6.5.6].

Theorem 3.3 (see [21]). *Suppose M is a complete hyperbolic manifold of finite volume and that $N \neq M$ is a complete hyperbolic manifold obtained topologically by replacing certain cusps of M by solid tori. Then* $\mathrm{vol}(N) < \mathrm{vol}(M)$.

Theorem 3.4. *For hyperbolic links $Br(k_1, \ldots, k_n)$, the following upper bound holds:*

$$\mathrm{vol}(S^3 \setminus B(k_1, \ldots, k_n)) < 16n \left[\Lambda\left(\frac{\pi}{4} + \frac{\pi}{4n}\right) + \Lambda\left(\frac{\pi}{4} - \frac{\pi}{4n}\right) \right].$$

Proof. Since $Br(k_1, \ldots, k_n)$ can be obtained by Dehn filling on $2n$ cusps of the hyperbolic link L'_n, the result follows by Corollary 2.3 and Theorem 3.3. □

Corollary 3.5. *For every $n \geq 2$, the value $\beta_n = 16n \left[\Lambda\left(\frac{\pi}{4} + \frac{\pi}{4n}\right) + \Lambda\left(\frac{\pi}{4} - \frac{\pi}{4n}\right)\right]$ is the limit point for volumes of hyperbolic Brunnian links with n components.*

We conclude the chapter by the following open problems concerning hyperbolic Brunnian links.

Problem 3.1. *What is the smallest volume hyperbolic Brunnian link with n components?*

It is well known that the link "Borromean rings" is arithmetic [21, Chapter 7].

Problem 3.2. *Which Brunnian links are arithmetic?*

Also, the following interesting problem was formulated by Bai and Ma [6, Problem 7.0.8].

Problem 3.3 (see [6]). *Let $B(n)$ be the number of Brunnian links with n or fewer crossings, $B_h(n)$ the number of hyperbolic Brunnian links with n or fewer crossings, and denote*

$$\limsup_{n\to\infty} \frac{B_h(n)}{B(n)} = a, \qquad \liminf_{n\to\infty} \frac{B_h(n)}{B(n)} = b.$$

Is then $a = b$, $b = 0$, or $a < 1$?

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