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Codimensions of identities of solvable Lie superalgebras

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Abstract. We study identities of Lie superalgebras over a field of characteristic zero. We construct a series of examples of finite-dimensional solvable Lie superalgebras with non-nilpotent commutator subalgebra for which the PI-exponent of codimension growth exists and is an integer number.

Keywords: identities, codimensions, Lie superalgebras, PI-exponent.

§ 1. Introduction

We study identities of Lie superalgebras over a field F of characteristic zero. The existence of a non-trivial identity of an algebra plays a important role in the study of its properties and structure. For example, if A is an associative finitely generated PI-algebra, then its Gelfand–Kirillov dimension $\operatorname{Gkdim}(A)$ is finite, and the Jacobson radical $J(A)$ is nilpotent. Moreover, if A is simple, then $\dim A < \infty$. If A and B are two finite-dimensional simple algebras (not necessarily associative) over an algebraically closed field, then they are isomorphic if and only if A and B satisfy the same polynomial identities.

Analysis of numerical invariants is one of the fundamental directions in the study of identity relations. One of the most significant numerical invariants that characterize the quantity of identities of algebra A is the sequence $c_n(A)$, $n = 1, 2, \dots$, called the codimension sequence. In the general case, the sequence $\{c_n(A)\}$ has an overexponential growth. For example, if A is a free assocative algebra of countable rank, then $c_n(A) = n!$. For a free Lie algebra, we have $c_n(A) = (n-1)!$. Even if a Lie algebra L satisfies the sufficiently strong identity $[[x_1, x_2, x_3], [y_1, y_2, y_3]] \equiv 0$, its codimension sequence $\{c_n(L)\}$ grows like $\sqrt{n!}$ (see [1]). Nevertheless, for a wide class of algebras, the codimension sequence is exponentially bounded. So, for any associative PI-algebra A, there is a constant a such that $c_n(A) < a^n$ for all $n \geqslant 1$ (see [2], and also [3]). If A is an arbitrary finite-dimensional algebra, $\dim A = d$, then $c_n(A) \leqslant d^{n+1}$ (see [4] or [5]). If L is an infinite-dimensional simple Lie algebra of Cartan type or a Virasoro algebra, then $c_n(A) < a^n$ (see [6]). A similar restriction holds also for any affine Kac–Moody algebra (see [7]). If L is a Lie superalgebra with nilpotent commutator subalgebra, $(L^2)^{t+1} = 0$, then the codimension sequence $\{c_n(L)\}$ grows asymptotically not faster than $(2t)^n$ (see [8]). For

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any Novikov algebra, A the codimension sequence is also exponentially bounded, $c_n(A) \leqslant 4^n$ (see [9]).

In the 1980s, S. Amitsur posed a conjecture that the limit of the sequence $\{\sqrt[n]{c_n(A)}\}$ exists and is a non-negative integer for any associative PI-algebra A. This conjecture was confirmed in [10], [11], and the limit

$$\exp(A) = \lim_{n\to\infty} \sqrt[n]{c_n(A)}, \tag{1}$$

was called the PI-exponent of algebra A. Later on, the existence and integrality of the limit (1) was proved for any finite-dimensional Lie algebra [12], Jordan algebra [13], and some other algebras. It turned out that in the case of finite-dimensional associative, Lie, or Jordan algebra over an algebraically closed field, the PI-exponent of A is equal to $\dim A$ if and only if A is simple.

If A is graded by a group G, one can also study, along with usual identities, G-graded identities of A and their numerical invariants. Graded identities form a more precise charateristic than ordinary identities. For example, if $G = \mathbb{Z}_2$, then any multilinear identity of degree n is equivalent to the system of 2^n graded identities. Therefore, it is reasonable in the Lie superalgebra case to consider both graded and non-graded identities.

It turned out that, in the super Lie case, the situation differs significantly from the ordinary Lie or associative case. In [14]–[16], examples of finite-dimensional Lie superalgebras are given for which graded and ordinary PI-exponent exist but they are not integer numbers. It was also shown that the PI-exponent of a simple Lie superalgebra L can be less than $\dim L$.

In the above mentioned examples, the finite-dimensional Lie superalgebras are not solvable. So, the natural question arises: Is it true that graded and non-graded exponents exist for any finite-dimensional solvable Lie superalgebra L? If the commutator subalgebra of L is nilpotent, then the answer is affirmative (see [8]). On the other hand, for solvable Lie superalgebra $L = L_0 \oplus L_1$ with non-zero odd component L_1, its ideal L^2 can be non-nilpotent. In [17], a series of finite-dimensional solvable Lie superalgebras $S(t)$, $t \geqslant 2$, with non-nilpotent commutator subalgebras was constructed. It was also shown that $\exp(S(2)) = \exp^{\mathrm{gr}}(S(2)) = 4$. In the present paper, we prove existence and integrality of graded PI-exponent for any superalgebra $S(t)$, $t \geqslant 3$. We also compute the value of this exponent.

All the necessary information about polynomial identities and their numerical invariants can be found in the monographs [18]–[20].

§ 2. Preliminaries

Let F be a field of characteristic zero and let $F\{X,Y\}$ be an absolutely free algebra over F with two infinite sets of generators X and Y. The algebra $F\{X,Y\}$ can be naturally endowed with $\mathbb{Z}_2$-grading $F\{X,Y\} = F\{X,Y\}_0 \oplus F\{X,Y\}_1$ if we define all generators from X as even and all from Y as odd. If $L = L_0 \oplus L_1$ is some $\mathbb{Z}_2$-graded algebra over F, then a non-associative polynomial $f = f(x_1,\dots,x_m, y_1,\dots,y_n) \in F\{X,Y\}$ is called a graded identity of algebra L if $f = f(a_1,\dots,a_m, b_1,\dots,b_n) = 0$ for any $a_1,\dots,a_m \in L_0$, $b_1,\dots,b_n \in L_1$. The set of all identities $\mathrm{Id}^{\mathrm{gr}}(L)$ forms a graded ideal of $F\{X,Y\}$ invariant under all endomorphisms of $F\{X,Y\}$ preserving grading, that is, it is a T-ideal.

Denote by $P_{k,m}$ the subspace of all multilinear polynomials of degree $n = k+m$ of $x_1,\dots,x_k \in X$, $y_1,\dots,y_m \in Y$. It is well-known that the family of all subspaces $P_{r,m} \cap \mathrm{Id}^{\mathrm{gr}}(L)$, $k,m \geqslant 1$, uniquely defines $\mathrm{Id}^{\mathrm{gr}}(L)$ as a T-ideal. Let also

$$P_{k,n-k}(L) = \frac{P_{k,n-k}}{P_{k,n-k} \cap \mathrm{Id}^{\mathrm{gr}}(L)}.$$

In this case, the value

$$c_{k,n-k}(L) = \dim P_{k,n-k}(L)$$

is called the partial $(k,n-k)$-graded codimension, whereas the value

$$c_n^{\mathrm{gr}}(L) = \sum_{k=0}^{n} \binom{n}{k} c_{k,n-k}(L)$$

is called the nth graded codimension of L.

As in the non-graded case, the sequence of graded codimensions of a finite-dimensional algebra L is exponentially bounded (see [4]). This implies the existence of the limits

$$\overline{\exp}^{\mathrm{gr}}(L) = \limsup_{n\to\infty} \sqrt[n]{c_n^{\mathrm{gr}}(L)}, \qquad \underline{\exp}^{\mathrm{gr}}(L) = \liminf_{n\to\infty} \sqrt[n]{c_n^{\mathrm{gr}}(L)},$$

which are called the upper and the lower graded PI-exponents of L, respectively. If the ordinary limit

$$\exp^{\mathrm{gr}}(L) = \lim_{n\to\infty} \sqrt[n]{c_n^{\mathrm{gr}}(L)},$$

exists, it is called the (ordinary) graded PI-exponent of L.

Representation theory of symmetric groups is the main tool in the study of numerical characteristics of polynomial relations. The permutation group S_n acts naturally on multilinear expressions

$$\sigma \circ f(z_1,\dots,z_n) = f(z_{\sigma(1)},\dots,z_{\sigma(n)}).$$

Let us recall some elements of the symmetric groups required in what follows. All the required details of the representation theory of permutation groups can be found in [21].

Denote by $R = FS_m$ the group algebra of group S_m. Recall the construction of minimal left ideals of R. Let $\lambda \vdash m$ be a partition of m, that is, an ordered set of integers $(\lambda_1,\dots,\lambda_k)$ such that $\lambda_1 \geqslant \cdots \geqslant \lambda_k > 0$, $\lambda_1 + \cdots + \lambda_k = m$. To this partition there corresponds the so-called Young diagram, that is, the tableau consisting of m cells, where λ_1 cells stay in the first row, λ_2 cells stay in the second row, etc. In this case, the Young tableau T_λ is the Young diagram D_λ filled up by integers $1,\dots,m$.

Given a Young tableau T_λ in FS_m, one can construct two subgroups R_{T_λ} and C_{T_λ} in S_m. The first one is called the row stabilizer and consists of those $\sigma \in S_m$ which move integers only within rows of T_λ. The second one is called the column stabilizer and consists of permutations which move numbers $1,2,\dots,m$ only within columns of T_λ. Given Young tableau T_λ, one can associate with it the element

$$e_{T_\lambda} = \Bigl(\sum_{\sigma\in R_{T_\lambda}} \sigma\Bigr)\Bigl(\sum_{\tau\in C_{T_\lambda}} (-1)^\tau \tau\Bigr), \tag{2}$$

of group ring called the Young symmetrizer. It is well-known that Young symmetrizer is quasi-idempotent, that is, $e_{T_\lambda}^2 = \gamma e_{T_\lambda}$ where $\gamma \in \mathbb{Q}$ is a non-zero scalar. Moreover, the left ideal Re_{T_λ} is minimal. Its character is denoted by χ_λ. Any irreducible left R-module M is isomorphic to some Re_{T_λ}. In this case, its character $\chi(M)$ is equal to χ_λ. Recall also that Re_{T_λ} and Re_{T_μ} are isomorphic as FS_m-modules if and only if $\lambda = \mu$.

Any finite-dimensional S_m-module M can be decomposed into a direct sum of irreducible components $M = M_1 \oplus \cdots \oplus M_t$. In this case, the expression

$$\chi(M) = \sum_{\lambda \vdash m} m_\lambda \chi_\lambda \tag{3}$$

means that among $M_1, \ldots, M_t$ there are exactly m_λ summands with character χ_λ. The sum of multiplicities m_λ in decomposition (3) (that is, the number t) is called the length of the module M.

When we study identities of $\mathbb{Z}_2$-graded algebras, we need to use the action of the direct product of two symmetric groups on multilinear components of the direct product of two symmetric groups. The group $S_k \times S_{n-k}$ acts on the space $P_{k,n-k}$. The intersection $P_{k,n-k} \cap \mathrm{Id}^{\mathrm{gr}}(L)$ is invariant under this action for any algebra $L_0 \oplus L_1$. Hence $P_{k,n-k}(L)$ is also an $(S_k \times S_{n-k})$-module. Any irreducible $S_k \times S_{n-k}$-module is isomorphic to the tensor product $M \otimes N$ of irreducible S_k- and S_{n-k}-modules, respectively. The character of this module is denoted by $\chi_{\lambda,\mu}$, where $\chi_\lambda = \chi(M)$, $\chi_\mu = \chi(N)$. In this notation, the decomposition of $P_{k,n-k}(L)$ into irreducible components has the form

$$\chi_{k,n-k}(L) = \chi(P_{k,n-k}(L)) = \sum_{\substack{\lambda \vdash k \\ \mu \vdash n-k}} m_{\lambda,\mu} \chi_{\lambda,\mu}, \tag{4}$$

where $m_{\lambda,\mu}$ is the multiplicity of $\chi_{\lambda,\mu}$ in the decomposition of $\chi_n(L)$. Hence

$$c_{k,n-k}(L) = \sum_{\substack{\lambda \vdash k \\ \mu \vdash n-k}} m_{\lambda,\mu} d_\lambda d_\mu, \tag{5}$$

where d_λ and d_μ are, respectively, the dimensions of irreducible S_k- and S_{n-k} representations with characters χ_λ and χ_μ, respectively.

There is another important series of numerical invariants for estimating the growth of codimensions. The value $l_{k,n-k}(L)$

$$l_{k,n-k}(L) = \sum_{\substack{\lambda \vdash k \\ \mu \vdash n-k}} m_{\lambda,\mu},$$

is called the partial colength of L. Here, $m_{\lambda,\mu}$ is an integer on the right-hand side of (4). The total sum

$$l_n^{\mathrm{gr}}(L) = \sum_{k=0}^{n} l_{k,n-k}(L) = \sum_{k=0}^{n} \sum_{\substack{\lambda \vdash k \\ \mu \vdash n-k}} m_{\lambda,\mu}$$

is called the graded colength.

An important role is played by the estimate of colength obtained in [22].

Lemma 1 (see [22; Theorem 1]). *Let $L=L_0\oplus L_1$ be a finite-dimensional $\mathbb{Z}_2$-graded algebra*, $\dim L=d$. *Then*

$$l_n^{\mathrm{gr}}(L)\leqslant d(n+1)^{d^2+d+1}.$$

§ 3. Upper estimates for codimension growth

In this section, we obtain an upper bound estimate for codimension growth of Lie superalgebras close to finite-dimensional. We shall need a technical statement related to the choice of generators in $(S_k\times S_{n-k})$-submodules in $P_{k,n-k}$.

Lemma 2. *Let M be an irreducible $(S_k\times S_{n-k})$-submodule in $P_{k,n-k}$ with character $\chi_{\lambda,\mu}$, $\lambda=(\lambda_1,\dots,\lambda_p)\vdash k$, $\mu=(\mu_1,\dots,\mu_q)\vdash(n-k)$. Then there exist $0\neq f=f(x_1,\dots,x_k,y_1,\dots,y_{n-k})\in M$ and decompositions $\{x_1,\dots,x_k\}=X_1\cup\dots\cup X_{\lambda_1}$, $\{y_1,\dots,y_{n-k}\}=Y_1\cup\dots\cup Y_{\mu_1}$ into disjoint subsets such that f is skew symmetric on each of the subsets $X_1,\dots,X_{\lambda_1},Y_1,\dots,Y_{\mu_1}$. Here, the cardinality $|X_i|$ of each X_i, $1\leqslant i\leqslant\lambda_1$ is equal to the height of the ith column of Young diagram D_λ, whereas the cardinality of each $|Y_j|$, $1\leqslant j\leqslant\mu_1$ is equal to the height of the jth column of the diagram D_μ.*

Proof. By the hypotheses of the lemma, M is isomorphic to $FS_ke_{T_\lambda}\otimes FS_{n-k}e_{T_\mu}$, where $\lambda\vdash k,\mu\vdash(n-k)$. In particular, M is generated, as an $F[S_k\times S_{n-k}]$-module, by elements of type $(e_{T_\lambda}\otimes e_{T_\mu})h$, where $h=h(x_1,\dots,x_k,y_1,\dots,y_{n-k})$ is a multilinear polynomial. Denote $h'=e_{T_\lambda}h$. If e_{T_λ} has the form (2), and then we take

$$h''(x_1,\dots,x_k,y_1,\dots,y_{n-k})=\biggl(\sum_{\sigma\in C_{T_\lambda}}(-1)^\sigma\sigma\biggr)h'.$$

Let $X_1\subseteq\{x_1,\dots,x_k\}$ consist of all x_i such that indices i are in the first column of the tableau T_λ, $X_2\subseteq\{x_1,\dots,x_k\}$ consists of all x_i such that indices i are in the second column of T_λ, and so on. Then $\{x_1,\dots,x_k\}=X_1\cup\dots\cup X_{\lambda_1}$, and h'' is skew symmetric on each of the sets $X_1,\dots,X_{\lambda_1}$. Besides, $h''\neq0$, since $e_{T_\lambda}^2\neq0$, and

$$\biggl(\sum_{\rho\in R_{T_\lambda}}\rho\biggr)h''=e_{T_\lambda}h'=e_{T_\lambda}^2h.$$

Next, we set

$$f=\biggl(\sum_{\tau\in C_{T_\mu}}(-1)^\tau\tau\biggr)h''$$

and decompose $\{y_1,\dots,y_{n-k}\}$ into the union $Y_1\cup\dots\cup Y_{\mu_1}$ according to the distribution of the indices y_i's among the columns of T_μ. We have $f\neq0$, and $Y_1,\dots,Y_{\mu_1}$ satisfy all the required conditions. This completes the proof of Lemma 2.

Recall that any ideal of a Lie superalgebra is by definition homogeneous in $\mathbb{Z}_2$-grading. For an upper bound of codimension growth, we need the following observation.

Lemma 3. *Let $L=L_0\oplus L_1$ be a Lie superalgebra and $I_0\oplus I_1$ be its nilpotent ideal of L of finite codimension, $I^{m+1}=0$. Let also $d_0=\dim(L_0/I_0)$, $d_1=\dim(L_1/I_1)$.*

If $\lambda = (\lambda_1, \dots, \lambda_p) \vdash k$, $\mu = (\mu_1, \dots, \mu_q) \vdash (n-k)$ *are two partitions such that* $m_{\lambda,\mu} \neq 0$ *in the decomposition* (4) *for* L, *then* $\lambda_{d_0+1} + \cdots + \lambda_p \leqslant m$ *and* $\mu_{d_1+1} + \cdots + \mu_q \leqslant m$.

Proof. We fix a basis $u_1, u_2, \dots$ of L_0 such that $u_1, \dots, u_{d_0}$ are linearly independent modulo I_0, whereas all the remaining u_i lie in I_0. Similarly, we choose a basis $v_1, v_2, \dots$ in L_1 such that $v_1, \dots, v_{d_1}$ are linearly independent modulo I_1 and $v_j \in I_1$, $j > d_1$.

Consider an irreducible $S_k \times S_{n-k}$-submodule in $P_{k,n-k}$ with character $\chi_{\lambda,\mu}$ and take in M a generator $f = f(x_1, \dots, x_k, y_1, \dots, y_{n-k})$ and distributions $X_1, \dots, X_{\lambda_1}$, $Y_1, \dots, Y_{\mu_1}$ constructed in Lemma 2. Suppose that $\lambda_{d_0+1} + \cdots + \lambda_p \geqslant m+1$. In order to check whether f is an identity of L or not, it is sufficient to replace variables with elements of fixed bases of corresponding parity. Let exactly t first columns of the diagram D_λ have the height strictly greater than d_0, that is, $|X_1|, \dots, |X_t| > d_0$, $|X_{t+1}| \leqslant d_0$. If we substitute instead of variables from one of the sets X_i, $1 \leqslant i \leqslant t$, more than d_0 basis vectors u_j with $j \leqslant d_0$, we get zero value of f due to skew symmetry. Otherwise, we need to substitute at least

$$N = (|X_1| - d_0) + \cdots + (|X_t| - d_0)$$

basis elements from I. But since

$$N = \lambda_{d_0+1} + \cdots + \lambda_p \geqslant m+1,$$

and $I^{m+1} = 0$, we again obtain zero value for f. Analogously, $f \equiv 0$, provided that $\mu_{d_1+1} + \cdots + \mu_q \geqslant m+1$. Since the inequality $m_{\lambda,\mu} \neq 0$ implies that f is not an identity of L, the proof of Lemma 3 is completed.

We now estimate the dimensions of irreducible components in the decomposition of $P_{k,n-k}(L)$.

Lemma 4. *Let* $\lambda = (\lambda_1, \dots, \lambda_p) \vdash n$ *be a partition of* n *such that* $p \geqslant d+1$ *and* $\lambda_{d+1} + \cdots + \lambda_p \leqslant m$. *Then, given,* p *and* m, *the inequality* $d_\lambda \leqslant n^m d^n$ *holds.*

Proof. Consider a partition $\nu = (\lambda_1, \dots, \lambda_d)$ of the integer $n' = \lambda_1 + \cdots + \lambda_d$. Hence $n - n' \leqslant m$, and, by Lemma 6.2.4 in [20], $d_\lambda \leqslant n^m d_\nu$ and by Corollary 4.4.7 in [20], we have $d_\nu \leqslant d^{n'}$.

Proposition 1. *Let* $L_0 \oplus L_1$ *be a finite-dimensional Lie superalgebra* $\dim L = d$ *and let* $I = I_0 \oplus I_1$ *be a nilpotent ideal in* L, $I^{m+1} = 0$, $\dim(L_0/I_0) = d_0$, $\dim(L_1/I_1) = d_1$. *Then there exists a polynomial* $\varphi(n)$ *depending only on* m, d, d_0 *and* d_1 *such that*

$$c_{k,n-k}(L) \leqslant \varphi(n) d_0^k d_1^{n-k} \quad \textit{for all} \quad 0 \leqslant k \leqslant n. \tag{6}$$

In particular,

$$c_n^{\mathrm{gr}}(L) \leqslant \varphi(n)(d_0 + d_1)^n. \tag{7}$$

Proof. Consider expression (5) for $c_{k,n-k}(L)$. Since all multiplicities $m_{\lambda,\mu}$ are bounded from above by $l_n^{\mathrm{gr}}(L)$, we have, by Lemma 1,

$$c_{k,n-k}(L) = d(n+1)^{d^2+d+1} \sum_{\substack{\lambda \vdash k \\ \mu \vdash n-k}} d_\lambda d_\mu. \tag{8}$$

Skew symmetry considerations applied in the proof of Lemma 3 allow us to claim that the height of partitions λ and μ (that is, the height of corresponding Young diagram) does not exceed d. Clearly, the number of such partitions is $< n^d$. Hence, applying Lemma 3 and 4, we deduce from (8) the bound (6) for some polynomial $\varphi(n)$. Now inequality (7) follows from (6) and the definition of the graded codimension. This completes the proof of the proposition.

§ 4. Lie superalgebras of the series $S(t)$

In this section, we define an infinite series of finite-dimensional solvable Lie superalgebras with non-nilpotent commutator subalgebra. We will use the following agreements. If A is a Lie superalgebra, then we denote the product of elements of A by an ordinary commutator bracket $[x,y]$. If A is an associative algebra, then $[x,y]=xy-yx$. If $A=A_0\oplus A_1$ is an associative algebra with $\mathbb{Z}_2$-grading and x and y are homogeneous elements from A, then

$$[x,y]=xy-(-1)^{|x|\,|y|}yx,$$

where $|x|$ is the parity of x, that is, 0 or 1. We agree to omit the brackets in the case of left-normed arrangement, that is, $[x_1,\dots,x_{k+1}]=[[x_1,\dots,x_k],x_{k+1}]$ for all $k\geqslant 2$.

First, let R be an arbitrary associative algebra with involution $*\colon R\to R$. Consider the associative algebra $Q=M_2(R)$,

$$Q=\left\{\begin{pmatrix} A & B\\ C & D\end{pmatrix}\;\middle|\; A,B,C,D\in R\right\},$$

and endow Q with the $\mathbb{Z}_2$-grading $Q=Q_0\oplus Q_1$ by setting

$$Q_0=\left\{\begin{pmatrix} A & 0\\ 0 & D\end{pmatrix}\right\},\qquad Q_1=\left\{\begin{pmatrix} 0 & B\\ C & 0\end{pmatrix}\right\}.$$

It is well-known that Q with the product $[\,\cdot\,,\cdot\,]$ is a Lie superalgebra. Given an associative algebra R with involution, we denote by R^+ and R^- the subspaces of symmetric and skew elements of R, respectively:

$$R^+=\{x\in R\mid x^*=x\},\qquad R^-=\{x\in R\mid x^*=-x\}.$$

Hence the subspace

$$L=\left\{\begin{pmatrix} x & y\\ z & -x^*\end{pmatrix}\;\middle|\; x\in R,\ y\in R^+,\ z\in R^-\right\} \tag{9}$$

is also a Lie superalgebra with the same product as in R where

$$L_0=\left\{\begin{pmatrix} x & 0\\ 0 & -x^*\end{pmatrix}\right\},\qquad L_1=\left\{\begin{pmatrix} 0 & y\\ z & 0\end{pmatrix}\right\}.$$

Remark 1. In fact, one of the series of simple Lie superalgebras, namely $p(t)$, is constructed in this way (see, for example, [23]).

Remark 2. For the Lie superalgebra L constructed above, the following conditions are equivalent:

1) L is solvable,
2) L_0 is a solvable Lie algebra,
3) R is Lie solvable,
4) the maximal semisimple subalgebra of R is commutative.

Thus, the proposed construction gives us a wide class of finite-dimensional solvable Lie superalgebras with non-nilpotent (as a rule) commutator subalgebra. As an example, we can take the algebra of upper triangular $(t\times t)$-matrices $R=UT_t(F)$, the finite-dimensional incidence algebra, or any associative subalgebra in $UT_t(F)$. We restrict ourselves to the case $R=\mathrm{UT}_t(F)$.

Recall the description of involutions on $\mathrm{UT}_t(F)$. One of them $\circ\colon R\to R$ is the reflection along the secondary diagonal. That is, $e_{ij}^{\circ}=e_{t+1-j,t+1-i}$ (e_{ij} are the matrix units). It is defined for all $t\geqslant 2$. We will call it *orthogonal*. Another one if defined only for even t. Let $t=2m$. We set

$$D=\begin{pmatrix} E & 0\\ 0 & -E\end{pmatrix},$$

where E is the identity $(m\times m)$-matrix. The map $s\colon R\to R$,

$$X^s=DX^{\circ}D^{-1},$$

is also an involution on R. It is said to be *symplectic*. If we write X as

$$X=\begin{pmatrix} U & V\\ 0 & W\end{pmatrix},$$

then

$$X^s=\begin{pmatrix} W^{\circ} & -V^{\circ}\\ 0 & U^{\circ}\end{pmatrix}.$$

Proposition 2 (see [24; Proposition 2.5]). *Any involution on* $\mathrm{UT}_t(F)$ *is equivalent to* $\circ$ *or* s.

Definition. A *Lie superalgebra* $(S(t),*)$, $t\geqslant 2$, is algebra (9), where $R=\mathrm{UT}_t(F)$ and $*=\circ$ or s, is the orthogonal or symplectic involution on R.

Sometimes we will denote both $(S(t),\circ)$ and $(S(t),s)$ just by $S(t)$. We need multiplication formulas in L:

$$\left[\begin{pmatrix} A & 0\\ 0 & -A^*\end{pmatrix},\begin{pmatrix} 0 & B\\ 0 & 0\end{pmatrix}\right]=\begin{pmatrix} 0 & AB+BA^*\\ 0 & 0\end{pmatrix}, \tag{10}$$

$$\left[\begin{pmatrix} A & 0\\ 0 & -A^*\end{pmatrix},\begin{pmatrix} 0 & 0\\ C & 0\end{pmatrix}\right]=\begin{pmatrix} 0 & 0\\ -A^*C-CA & 0\end{pmatrix}, \tag{11}$$

$$\left[\begin{pmatrix} A & 0\\ 0 & -A^*\end{pmatrix},\begin{pmatrix} B & 0\\ 0 & -B^*\end{pmatrix}\right]=\begin{pmatrix} AB-BA & 0\\ 0 & -(AB-BA)^*\end{pmatrix}, \tag{12}$$

$$\left[\begin{pmatrix} 0 & B\\ 0 & 0\end{pmatrix},\begin{pmatrix} 0 & 0\\ C & 0\end{pmatrix}\right]=\begin{pmatrix} BC & 0\\ 0 & CB\end{pmatrix}. \tag{13}$$

Let us introduce a few more notation. First, we note that both involutions $\circ$ and s act similarly on diagonal matrix units: $e_{ii}^* = e_{t+1-i,t+1-i}$. Now, for even $t = 2m \geqslant 2$ or for odd $t = 2m+1 \geqslant 3$, we denote

$$X_i = \begin{pmatrix} e_{ii} - e_{ii}^* & 0 \\ 0 & e_{ii} - e_{ii}^* \end{pmatrix}, \qquad Y_i = \begin{pmatrix} 0 & e_{ii} + e_{ii}^* \\ 0 & 0 \end{pmatrix}, \qquad Z_i = \begin{pmatrix} 0 & 0 \\ e_{ii} - e_{ii}^* & 0 \end{pmatrix}$$

for all $i = 1, \dots, m$, and

$$E_{ij} = \begin{pmatrix} e_{ij} & 0 \\ 0 & -e_{ij}^* \end{pmatrix}, \quad 1 \leqslant i < j \leqslant t, \qquad I = \begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix}, \qquad Y_0 = \begin{pmatrix} 0 & E \\ 0 & 0 \end{pmatrix},$$

where E is the identity $(t \times t)$-matrix. The following relations follow from multiplication formulas (10)–(13):

$$[X_i.Y_j] = [X_i, Z_j] = [X_i, X_j] = 0, \quad [Y_i, Z_j] = \delta_{ij} Z_i, \qquad 1 \leqslant i, j \leqslant m, \tag{14}$$

where δ_{ij} is the Kronecker delta. We also have

$$\begin{gathered} [E_{ik}, E_{kj}] = E_{ij}, \qquad 1 \leqslant i < k < j \leqslant 2m, \\ [E_{k,k+1}, X_{k+1}] = -[E_{k,k+1}, X_k] = E_{k,k+1}, \\ [E_{k,k+1}, X_j] = 0, \quad j \neq k, k+1, \qquad [I, E_{ij}] = 0, \qquad [I, Y_0] = 2Y_0. \end{gathered} \tag{15}$$

To conclude this section, we give a lower bound for the PI-exponent.

Proposition 3. *Let L be a Lie superalgebrra of type $S(t)$. Then $\overline{\exp}^{\,\mathrm{gr}}(L) \leqslant 2t$ for even t or $\overline{\exp}^{\,\mathrm{gr}}(L) \leqslant 2t-1$ for odd t.*

Proof. We first note that, in addition to $\mathbb{Z}_2$-grading, the algebra L is also endowed with $\mathbb{Z}$-grading of type $L = L^{(0)} \oplus \dots \oplus L^{(t-1)}$. The initial algebra R has $\mathbb{Z}$-grading $R = R^{(0)} \oplus \dots \oplus R^{(t-1)}$, where

$$R^{(k)} = \mathrm{Span}\{e_{ij} \mid j - i = k\}.$$

Now, if we put

$$L^{(k)} = \left\{ \begin{pmatrix} A & B \\ C & -A^* \end{pmatrix} \,\middle|\, A \in R^{(k)},\, B \in R^+ \cap R^{(k)},\, C \in R^- \cap R^{(k)} \right\},$$

then the multiplication rules (10)–(13) show that $L = L^{(0)} \oplus \dots \oplus L^{(t-1)}$ is the required $\mathbb{Z}$-decomposition. All the subspaces $L^{(j)}$ are homogeneous in $\mathbb{Z}_2$-grading, hence $L^{(1)} \oplus \dots \oplus L^{(t-1)}$ is an ideal of L of codimension $2t$. Since this ideal is nilpotent, Proposition 1 completes the proof for even t.

Now let $t = 2m+1$. In order to apply Proposition 1 again, it is enough to show that $I = \langle b \rangle + L^{(1)} \oplus \dots \oplus L^{(t-1)}$ is a nilpotent ideal of L, where

$$b = \begin{pmatrix} 0 & E_{m+1,m+1} \\ 0 & 0 \end{pmatrix} \in L_1 \cap L^{(0)}.$$

First, we need to check that $[a,b] \in I$ if a is an even or odd element from $L^{(0)}$. If a is even, then $[a,b] = \alpha b$, $\alpha \in F$, as follows from (10) and the definition of $L^{(0)}$. If

$$a = \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix}, \tag{16}$$

then $[ab] = 0$, since the product of any two elements of type (16) is zero. On the other hand, if

$$a = \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix},$$

then c is a diagonal matrix with zero entry at the $(m+1)$th position. Hence $[a, b] = 0$ according to (13).

Now let us prove that $I^{4t} = 0$. Let $a = [b_1, \ldots, b_{4t}]$ be a left-normed commutator of elements which are homogeneous both in $\mathbb{Z}_2$- and in $\mathbb{Z}$-gradings. If, among b_i's, there appear at least t factors from $L^{(1)} \oplus \cdots \oplus L^{(t-1)}$, then $a = 0$, as follows from $\mathbb{Z}$-grading arguments. On the other hand, if the number of such factors is smaller than t, then b appears at least three times in a row, since $I^{(0)} = \langle b \rangle$. In this case, we also have $a = 0$, since $(\operatorname{ad} b)^3 = 0$, and since $\operatorname{ad} x$ is the operator of the right-hand side multiplication on x. We have

$$\dim(L/I) = 2m + 1 + 2m = 2t - 1,$$

and now the required result follows again from Proposition 1. This proves Proposition 3.

§ 5. Exponents of superalgebras of series $S(t)$

For a lower bound of codimension growth, we need to consider multialternating polynomials. It will be convenient to use the following agreement. If some expression depends on skew symmetric set of arguments, then instead of alternating sum we will mark these arguments from above by some common symbol (line, tilde, etc.). For example,

$$\widetilde{x_1} \cdots \widetilde{x_n} = \sum_{\sigma \in S_n} x_{\sigma(1)} \cdots x_{\sigma(n)}$$

is the standard polynomial in an associative algebra,

$$x\bar{y}z\bar{t} = xyzt - xtzy,$$

and

$$\bar{x}\,\bar{\bar{y}}z\bar{\bar{x}}\,\bar{y} = x\bar{\bar{y}}z\bar{\bar{x}}y - y\bar{\bar{y}}z\bar{\bar{x}}x = xyzxy - xxzyy - yyzxx + yxzyx.$$

We first consider superalgebras $S(t)$ with even t.

Lemma 5. *Let $S(t)$ be a Lie superalgebra defined by orthogonal or symplectic involution $*$ and $t = 2m$. Then $\underline{\exp}^{\mathrm{gr}}(L) \geqslant 4m$.*

Proof. In the algebra of upper triangular matrices UT_{2m}, we have the following relation:

$$\bigl[[e_{12}, \bar{e}_{11}], \ldots, [e_{m,m+1}, \bar{e}_{mm}]\bigr] = \bigl[[e_{12}, e_{11}], \ldots, [e_{m,m+1}, e_{mm}]\bigr] = (-1)^m e_{1,m+1}.$$

It follows that

$$\begin{aligned} a_1 &= \bigl[[E_{12}, \overline{E}_{11}], \ldots, [E_{m,m+1}, \overline{E}_{mm}]\bigr] \\ &= \bigl[[E_{12}, E_{11}], \ldots, [E_{m,m+1}, E_{mm}]\bigr] = (-1)^m E_{1,m+1}. \end{aligned} \tag{17}$$

The expression a_1 contains an alternating set of even elements $E_{11}, \dots, E_{mm}$. Let us complicate its construction by adding m-alternating odd set. Since $[Y_i, Z_i] = X_i$, $[Y_i, Z_j] = 0$ if $i \neq j$ (see (14)) and $[E_{k,k+1}, X_{k+1}] = [E_{k,k+1}, E_{k+1,k+1}]$ (see (15)), we have

$$\begin{aligned}&\bigl[[E_{12}, [\widetilde{Y}_1, Z_1]], \dots, [E_{m,m+1}, [\widetilde{Y}_m, Z_m]]\bigr]\\ &\quad= \bigl[[E_{12}, [Y_1, Z_1]], \dots, [E_{m,m+1}, [Y_m, Z_m]]\bigr]\\ &\quad= (-1)^m [E_{12}, \dots, E_{m,m+1}] = (-1)^m E_{1,m+1}.\end{aligned}$$

Now let us double the number of alternating odd elements. We put

$$a_2 = \bigl[[E_{12}, [\widetilde{Y}_1, Z_1], [Y_1, \widetilde{Z}_1]], \dots, [E_{m,m+1}, [\widetilde{Y}_m, Z_m], [Y_m, \widetilde{Z}_m]]\bigr].$$

Since $[Z_i, Z_j] = 0$ if $i \neq j$, we can omit an alternation in a_2 not changing the value of whole expression, that is

$$a_2 = \bigl[[E_{12}, [Y_1, Z_1], [Y_1, Z_1]], \dots, [E_{m,m+1}, [Y_m, Z_m], [Y_m, Z_m]]\bigr].$$

Finally, we put

$$\begin{aligned}a_3 = \bigl[&[E_{12}, \overline{E}_{11}, [\widetilde{Y}_1, Z_1], [Y_1, \widetilde{Z}_1]], \dots, [E_{m,m+1}, \overline{E}_{mm}, [\widetilde{Y}_m, Z_m], [Y_m, \widetilde{Z}_m]],\\ &[E_{m+1,m+2}, \overline{E}_{m+1,m+1}], \dots, [E_{2m-1,2m}, \overline{E}_{2m-1,2m-1}], [Y_0, \overline{I}]\bigr].\end{aligned}$$

It follows from the multiplication formulas (14), (15) that one can omit both alternations in a_3 preserving the value. In particular,

$$a_3 = \bigl[E_{1,2m-1}, [Y_0, I]\bigr] = 2[E_{1,2m-1}, Y_0] = 2\begin{pmatrix} 0 & e_{1,2m-1} \pm e_{2,2m} \\ 0 & 0 \end{pmatrix}, \tag{18}$$

where the plus or minus sign on the right-hand side of (18) depends on the choice of the involution $*$.

The construction of the element a_3 allows us to replicate skew symmetric sets of even factors $\{E_{i,i}, I\}$ as well as odd factors $\{Y_i, Z_i\}$ of A. Namely, we set

$$\begin{aligned}A_i^{(0)} &= \bigl[E_{i,i+1}, \bigl[Y_i^{(1)}, Z_i^{(0)}\bigr]\bigr],\\ A_i^{(1)} &= \bigl[A_i^{0}, \bigl[Y_i^{(2)}, Z_i^{(1)}\bigr]\bigr],\\ &\cdots\cdots\cdots\cdots\cdots\cdots\\ A_i^{(p)} &= \bigl[A_i^{p-1}, \bigl[Y_i^{(p+1)}, Z_i^{(p)}\bigr]\bigr].\end{aligned}$$

Here, all $Y_i^{(j)}$ are copies of the element Y_i. We use the upper index only for further indication of the alternation set in which it will be included. A similar remark holds also for $Z_i^{(j)}$.

Further, we set

$$A_i^{(p,1)} = \bigl[A_i^{(p)}, E_{ii}^{(1)}\bigr], \quad \dots, \quad A_i^{(p,q)} = \bigl[A_i^{(p,q-1)}, E_{ii}^{(q)}\bigr],$$

and, for $j = m+1, \dots, 2m-1$, we define

$$A_j^{(0)} = E_{j,j+1},$$
$$A_j^{(1)} = \bigl[A_j^{(0)}, E_{jj}^{(1)}\bigr],$$
$$\dots\dots\dots\dots\dots\dots$$
$$A_j^{(q)} = \bigl[A_j^{(q-1)}, E_{jj}^{(q)}\bigr].$$

Finally,

$$A_{2m}^{(1)} = [Y_0, I^{(1)}], \quad \dots, \quad A_{2m}^{(q)} = \bigl[A_{2m}^{(q-1)}, I^{(q)}\bigr].$$

Now let

$$W^{(p,q)} = \bigl[A_1^{(p,q)}, \dots, A_m^{(p,q)}, A_{m+1}^{(q)}, \dots, A_{2m}^{(q)}\bigr]$$

for all $p, q \geqslant 1$. Note that for computing the value of the product $W^{(p,q)}$ it is useful to remember that the right-hand multiplication by E_{ii} commutes with the right-hand multiplication by $[Y_i, Z_i] = X_i$.

The commutator $W(p,q)$ depends on

- p sets of odd elements $Y_1^{(i)}, \dots, Y_m^{(i)}$, $Z_1^{(i)}, \dots, Y_m^{(i)}$, $1 \leqslant i \leqslant p$, of size $2m$;
- q sets of even elements $E_{11}^{(j)}, \dots, E_{2m-1,2m-1}^{(j)}$, $I^{(j)}$, $1 \leqslant j \leqslant q$, of size $2m$;
- and also on $4m$ factors $E_{12}, \dots, E_{2m-1,2m}$, Y_0, $Z_1^{(0)}, \dots, Z_m^{(0)}$, $Y_1^{(p+1)}, \dots, Y_m^{(p+1)}$ outside these sets.

Applying to $W^{(p,q)}$ the alternation on the sets of order $2m$, we get the expression

$$\widetilde{W}^{(p,q)} = \mathrm{Alt}_1^{(0)} \cdots \mathrm{Alt}_q^{(0)} \mathrm{Alt}_1^{(1)} \cdots \mathrm{Alt}_p^{(1)} (W^{(p,q)}).$$

Here, $\mathrm{Alt}_j^{(0)}$ is the alternation on $E_{11}^{(j)}$, $E_{2m-1,2m-1}^{(j)}$ and $I^{(j)}$, whereas $\mathrm{Alt}_i^{(1)}$ is the alternation on $Y_1^{(i)}, \dots, Y_m^{(i)}$, $Z_1^{(i)}, \dots, Z_m^{(i)}$.

As in computing expressions a_1, a_2 and a_3, alternation in $\widetilde{W}^{(p,q)}$ does not play any role, that is,

$$\widetilde{W}^{(p,q)} = W^{(p,q)} = \pm 2^q [E_{1,2m-1}, Y_0] \neq 0. \tag{19}$$

Now we construct $\widetilde{w}^{(p,q)}$ in $F\{X, Y\}$ using the same procedure as for the product $\widetilde{W}^{(p,q)}$, only changing $E_{12}, \dots, E_{2m-1,2m}$ by the even generators $x_{12}, \dots, x_{2m-1,2m}$, changing $E_{11}^{(j)}, \dots, E_{2m-1,2m-1}^{(j)}, I^{(j)}$ by the even generators $x_1^{(j)}, \dots, x_{2m,2m}^{(j)}$, changing $Y_1^{(i)}, \dots, Y_m^{(i)}$ by odd $y_1^{(i)}, \dots, y_m^{(i)}$, changing $Z_1^{(i)}, \dots, Z_m^{(i)}$ by odd $z_1^{(i)}, \dots, z_m^{(i)}$, and replacing Y_0 with odd y_0.

The element $\widetilde{w}^{(p,q)}$ includes q skew symmetric sets of even variables $X^{(j)} = \{x_1^{(j)}, \dots, x_{2m}^{(j)}\}$, $1 \leqslant j \leqslant q$, and p skew symmetric sets of odd variables $Y^{(i)} = \{y_1^{(i)}, \dots, y_m^{(i)}, z_1^{(i)}, \dots, z_m^{(i)}\}$. In addition to these variables, $\widetilde{w}^{(p,q)}$ contains $4m$ variables $x_{12}, \dots, x_{2m-1,2m}$, y_0, $y_1^{(p+1)}, \dots, y_m^{(p+1)}$, $z_1^{(0)}, \dots, z_m^{(0)}$ not participating in alternations.

Fixing now $n = 2mp + 2mq + 4m$ and $k = 2mq + 2m - 1$, we have $n - k = 2mp + 2m + 1$. The subgroup $H = S_{2mq} \times S_{2mp}$ of $S_k \times S_{n-k}$ acts on the space $P_{k,n-k}$. The left factor S_{2mq} acts on $\overline{X} = X^{(1)} \cup \dots \cup X^{(q)}$, whereas S_{2mp} acts on $\overline{Y} = Y^{(1)} \cup \dots \cup Y^{(p)}$. Relation (11) means that $\widetilde{w}(p,q)$ is not an identity of L. Moreover, $\varphi(\widetilde{w}(p,q)) \neq 0$ for the evaluation φ such that $\varphi(\overline{X}) \subseteq V_0$, $\varphi(\overline{Y}) \subseteq (V_1)$, where $V_0 = L_0 \cap L^{(0)}$, $V_1 = L_1 \cap L^{(0)}$ are subspaces of dimension $2m$. It follows from the structure of essential idempotent (see (2)) and skew symmetry of $\widetilde{w}(p,q)$ that the decomposition of the FH-submodule in $P_{k,n-k}$ generated by $\widetilde{w}(p,q)$ involves only irreducible components with character $\chi_{\lambda,\mu}$, where

$$\lambda = (q^{2m}) = (\underbrace{q,\dots,q}_{2m}), \qquad \mu = (p^{2m}) = (\underbrace{p,\dots,p}_{2m})$$

are two rectangular partitions. Hence $c_{k,n-k}(L) \geqslant \deg \chi_{\lambda,\mu} = d_\lambda d_\mu$.

It is well-known that the dimension of an irreducible representation with rectangular Young diagram is exponential, where the ratio of exponent is the height of the diagram. For example, by Lemma 5.10.1 in [20], for $\nu = s^d \vdash N = sd$ for all s large enough,

$$d_\nu > N^{-d(d-1)/2} d^N$$

provided that d is fixed. In our case, for $k = N + 2m - 1$, $N = 2mq$, we have

$$d_\lambda > \frac{1}{N^{m(m-1)}} (2m)^{k-2m+1} > \frac{1}{n^{m(m-1)}} \frac{(2m)^k}{(2m)^{2m-1}}.$$

Similarly,

$$d_\mu > \frac{1}{n^{m(m-1)}} \frac{(2m)^{n-k}}{(2m)^{2m+1}}.$$

Hence we have proved the inequality

$$c_{k,n-k} > \frac{1}{n^{2m(m-1)}(2m)^{2m}} (2m)^n \tag{20}$$

for $k = 2mq + 2m - 1$, $n - k = 2mp + 2m + 1$.

To obtain an analogous lower bound estimate for $c_{k,n-k}$ for arbitrary k and $n-k$ large enough, we note that

$$\big[\widetilde{W}^{(p,q)}, \underbrace{E_{11},\dots,E_{11}}_{r}\big] \neq 0, \qquad \big[\widetilde{W}^{(p,q)}, \underbrace{[Y_1,Z_1],\dots,[Y_1,Z_1]}_{r}\big] \neq 0$$

in L for any $r \geqslant 1$. Hence the polynomial

$$[\widetilde{w}^{(p,q)}, x_1, \dots, x_i, y_1, \dots, y_j] \tag{21}$$

is not a graded identity of L.

Now, for an arbitrary pair k, n, we can find $0 \leqslant i, j \leqslant 2m - 1$, p and q such that $k = k_0 + i$, $n = n_0 + j$, where $k_0 = 2mq + 2m - 1$, $n_0 - k_0 = 2mp + 2m + 1$. Proceeding with polynomial (21) as for $\widetilde{w}^{(p,q)}$, we obtain the lower bound

$$\begin{aligned}\dim P_{k,n-k}(L) &\geqslant d_\lambda d_\mu \geqslant \frac{(2m)^{k_0}}{n_0^{m(m-1)}(2m)^{2m-1}} \frac{(2m)^{n_0-k_0}}{n_0^{m(m-1)}(2m)^{2m+1}} \\ &\geqslant \frac{(2m)^{n_0}}{n^{2m(m-1)}(2m)^{4m}} = \frac{(2m)^n}{n^{2m(m-1)}(2m)^{4m+j}}.\end{aligned}$$

Hence, taking into account (20), we get the restriction

$$c_{k,n-k}(L) \geqslant \frac{(2m)^n}{n^{2m(m-1)(2m)^{8m}}}$$

for all k and $n-k$ large enough. Hence

$$c_n^{\mathrm{gr}}(L) \geqslant \frac{(2m)^n}{n^{2m(m-1)(2m)^{8m}}} \sum_{k=C+1}^{n-C} \binom{n}{k}, \tag{22}$$

where C is some constant depending only on m. Since the sum of the binomial coefficients is 2^n, we obtain from (22) the estimate

$$\underline{\exp}^{\mathrm{gr}}(L) \geqslant 4m$$

for the lower limit, thereby completing the proof of Lemma 5.

Now consider the case of odd t.

Lemma 6. *Let $t = 2m+1$ and $l = s(t) = (S(t), \circ)$. Then $\underline{\exp}^{\mathrm{gr}}(L) \geqslant 2t-1 = 4m+1$.*

Proof. The proof in this case largely repeats that of Lemma 5, we omit the repetitive details. The values $A_i^{(p)}$ and $A_i^{(p,q)}$, $1 \leqslant i \leqslant m$, remain the same under the above notation. Also, $A_j^{(1)}, \ldots, A_j^{(q)}$, $m+1 \leqslant j \leqslant 2m-1$ do not change. The elements $A_{2m}^{(1)}, \ldots, A_{2m}^{(q)}$ are defined inductively: $A_{2m}^{(1)} = E_{2m,2m+1}^{(1)}$, $\ldots, A_{2m}^{(q)} = [A_{2m}^{(q-1)}, E_{2m,2m}^{(q)}]$, and $A_{2m+1}^{(q)}$ is defined as $A_{2m}^{(q)}$ in Lemma 5. In the expression for $W^{(p,q)}$, we need to replace the last factor $A_{2m}^{(q)}$ by $A_{2m+1}^{(q)}$.

The modified element $\widetilde{w}^{(p,q)}$ depends on q skew symmetric sets of even variables of order $2m+1$, depends on p skew symmetric of odd sets variables of order $2m$, and has total degree $n = 2mp+(2m+1)q+4m+1$. The lower bounds for d_λ and d_μ are slightly different:

$$d_\lambda > \frac{1}{n^{m(2m+1)}} \frac{(2m+1)^k}{(2m+1)^{2m}}, \qquad d_\mu > \frac{1}{n^{m(2m+1)}} \frac{(2m)^{n-k}}{(2m)^{2m+1}},$$

where $\lambda = (q^{2m+1})$, $\mu = (p^{2m})$; for $c_{k,n-k}(L)$, we have

$$c_{k,n-k}(L) \geqslant \frac{(2m+1)^k (2m)^{n-k}}{n^{2m(2m+1)} (2m+1)^{8(m+1)}}.$$

Therefore, the lower bound estimate for graded codimension takes the form

$$c_n^{\mathrm{gr}}(L) \geqslant \frac{1}{n^{2m(2m+1)} (2m+1)^{8(m+1)}} \sum_{k=C+1}^{n-C} \binom{n}{k} (2m+1)^k (2m)^{n-k},$$

which implies

$$\underline{\exp}^{\mathrm{gr}}(L) \geqslant 4m+1,$$

This completes the proof of Lemma 6.

The main result of the paper is now immediate from Lemmas 5, 6, and Proposition 3.

Theorem 1. *Let $L = (S(t), *)$ be a Lie superalgebra of type $S(t)$, where $*$ is the orthogonal or symplectic involution. Then the graded PI-exponent of L exists, and*

- $\exp^{\mathrm{gr}}(L) = 2t$ *if t is even*;
- $\exp^{\mathrm{gr}}(L) = 2t - 1$ *if t is odd.*

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