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ORIGINAL ARTICLE

Generalized noncooperative Schrödinger–Kirchhoff–type systems in $\mathbb{R}^N$

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Abstract

We consider a class of noncooperative Schrödinger–Kirchhof–type system, which involves a general variable exponent elliptic operator with critical growth. Under certain suitable conditions on the nonlinearities, we establish the existence of infinitely many solutions for the problem by using the limit index theory, a version of concentration–compactness principle for weighted-variable exponent Sobolev spaces and the principle of symmetric criticality of Krawcewicz and Marzantowicz.

KEYWORDS

concentration–compactness principle, critical points theory, critical Sobolev exponents, generalized capillary operator, limit index theory, p-Laplacian, $p(x)$-Laplacian, Palais–Smale condition, Schrödinger–Kirchhoff–type problems, weighted exponent spaces

1 | INTRODUCTION

The purpose of this paper is to investigate the multiplicity of solutions for the noncooperative Schrödinger–Kirchhoff–type systems involving a general variable exponent elliptic operator and critical nonlinearity in $\mathbb{R}^N$:

$$\begin{cases} K(\mathcal{B}(u))(\operatorname{div}(\mathcal{A}_1(\nabla u)) - b(x)\mathcal{A}_2(u)) = |u|^{r(x)-2}u + \lambda(x)\dfrac{\partial \mathcal{F}}{\partial u}(x,u,v) & \text{in } \mathbb{R}^N, \\ K(\mathcal{B}(v))(-\operatorname{div}(\mathcal{A}_1(\nabla v)) + b(x)\mathcal{A}_2(v)) = |v|^{r(x)-2}v + \lambda(x)\dfrac{\partial \mathcal{F}}{\partial v}(x,u,v) & \text{in } \mathbb{R}^N, \\ u, v \in W_b^{1,p(x)}(\mathbb{R}^N) \cap W_b^{1,h(x)}(\mathbb{R}^N), \end{cases} \tag{1.1}$$

where $N \geq 2$, λ is a continuous, radially symmetric function on $\mathbb{R}^N$, $b \in L^\infty(\mathbb{R}^N)$ satisfies $b_0 := \operatorname{ess\,inf}\{\mathrm{b(x)} : \mathrm{x} \in \mathbb{R}^\mathrm{N}\} > 0$, $\nabla \mathcal{F} = (\frac{\partial \mathcal{F}}{\partial u}, \frac{\partial \mathcal{F}}{\partial v})$ is the gradient of a C^1 function $\mathcal{F} : \mathbb{R}^N \times \mathbb{R}^2 \to \mathbb{R}^+$, the functions p and q are log-Holder continuous, radially symmetric on $\mathbb{R}^N$, and satisfy the following inequality,

$$1 < p^- \leq p(x) \leq p^+ < q^- \leq q(x) \leq q^+ < N, \tag{1.2}$$

for all $x \in \mathbb{R}^N$, and the function s is continuous, radially symmetric on $\mathbb{R}^N$, and satisfies the following inequality,

$$h^- \leq h(x) \leq r^- \leq r(x) \leq r^+ \leq h^*(x) < \infty, \tag{1.3}$$

for all $x \in \mathbb{R}^N$, where $p^- := \operatorname{ess\,inf}\{p(x) : x \in \mathbb{R}^N\}$, $p^+ := \operatorname{ess\,sup}\{p(x) : x \in \mathbb{R}^N\}$, and analogously for q^-, q^+, h^-, h^+, r^-, and r^+, with $h(x) = (1 - \mathcal{H}(\kappa_\star^3))p(x) + \mathcal{H}(\kappa_\star^3)q(x)$, where $\kappa_\star^3$ is given by condition (H_{a_2}) below, and

$$h^*(x) = \begin{cases} \frac{Nh(x)}{N-h(x)} & \text{if } h(x) < N, \\ +\infty & \text{if } h(x) \geq N, \end{cases}$$

for all $x \in \mathbb{R}^N$, where $\mathcal{H} : \mathbb{R}_0^+ \to \{0,1\}$ is given by

$$\mathcal{H}(\kappa_\star^3) = \begin{cases} 1 & \text{if } \kappa_\star^3 > 0, \\ 0 & \text{if } \kappa_\star^3 < 0. \end{cases}$$

Furthermore, we assume that the set C_h defined as $\{x \in \mathbb{R}^N \mid r(x) = h^*(x)\}$ is not empty.

The operators $\mathcal{A}_i : X \to \mathbb{R}$, where i can be either 1 or 2, and the operator $\mathcal{B} : X \to \mathbb{R}$, are defined as follows,

$$\mathcal{A}_i(u) = a_i(|u|^{p(x)})|u|^{p(x)-2}u \text{ and } \mathcal{B}(u) = \int_{\mathbb{R}^N} \frac{1}{p(x)} \big(A_1(|\nabla u|^{p(x)}) + b(x)A_2(|u|^{p(x)})\big)dx,$$

where X is the following Banach space:

$$X := W_b^{1,p(x)}(\mathbb{R}^N) \cap W_b^{1,h(x)}(\mathbb{R}^N).$$

Function $A_i(.)$ is defined as $A_i(t) = \int_0^t a_i(k)dk$ and function $a_i(.)$ is from condition (H_{a_1}) below.

In this paper, we shall consider the function $a_i : \mathbb{R}^+ \to \mathbb{R}^+$, which satisfies the following assumptions for either $i = 1$ or $i = 2$:

(H_{a_1}) $a_i(.)$ is of class C^1.

(H_{a_2}) There exist positive constants $\kappa_i^0, \kappa_i^1, \kappa_i^2$, and $\kappa_\star^3$, for $i = 1$ or 2, such that

$$\kappa_i^0 + \mathcal{H}(\kappa_\star^3)\kappa_i^2 t^{\frac{q(x)-p(x)}{p(x)}} \leq a_i(t) \leq \kappa_i^1 + \kappa_\star^3 t^{\frac{q(x)-p(x)}{p(x)}}, \text{ for a.e. } x \in \mathbb{R}^N \text{ and all } t \geq 0.$$

(H_{a_3}) There exists a positive constant $c > 0$ such that

$$\min\left\{a_i(t^{p(x)})t^{p(x)-2},\ a_i(t^{p(x)})t^{p(x)-2} + t\frac{\partial(a_i(t^{p(x)})t^{p(x)-2})}{\partial t}\right\} \geq ct^{p(x)-2}, \text{ for a.e. } x \in \mathbb{R}^N \text{ and all } t > 0.$$

(H_{a_4}) There exist positive constants γ, α_i (for $i = 1$ or 2), and a positive function ϑ satisfying condition (F_2) below, such that

$$A_i(t) \geq \frac{1}{\alpha_i}a_i(t)t \text{ with } h^+ < \vartheta(x) < r^- \text{ and } \frac{q^+}{p^+} \leq \frac{\alpha_i}{\gamma} < \frac{\vartheta^-}{p^+}, \text{ for all } t \geq 0,$$

where γ_i satisfies condition (K_2) below.

Next, let $K : \mathbb{R}_0^+ \to \mathbb{R}^+$ be a nondecreasing and continuous Kirchhoff function such that

(K_1) there exists $\mathfrak{K}_0 > 0$ such that

$$K(t) \geq \mathfrak{K}_0 = K(0), \text{ for all } t \in \mathbb{R}_0^+;$$

(K_2) there exists $\gamma \in (\frac{q^+}{r^-}, 1]$ such that

$$\widehat{K}(t) \geq \gamma K(t)t, \text{ for all } t \in \mathbb{R}_0^+, \text{ where } \widehat{K}(t) := \int_0^t K(s)ds.$$

There are many functions satisfying conditions $(K_1)-(K_2)$, for example, $K(t) = \mathfrak{K}_0 + \mathfrak{K}_1 t^{\frac{1}{\gamma}}$, for $\gamma \leq 1$, $\mathfrak{K}_0 > 0$, and $\mathfrak{K}_1 \geq 0$.

In recent years, increasing attention has been paid to the study of differential and partial differential equations involving variable exponent. The interest in studying such problems was stimulated by their many physical applications. For example, they have been applied in nonlinear elasticity problems, electrorheological fluids, image processing, flow in porous media, and elsewhere, see, for example, Chen et al. [9], Diening et al. [17, 18], Halsey [26], Rădulescu and Repovš [46], Ružička [47, 48], and the references therein.

We shall illustrate the degree of generality of the kind of problems studied here, with adequate hypotheses on functions a_1 and a_2, by exhibiting some examples of problems, which are also interesting from the mathematical point of view and have a wide range of applications in physics and other fields.

Example 1. Considering $a_1 \equiv 1$ and $a_2 \equiv 1$, we see that a_1 and a_2 satisfy conditions $(H_{a_1}), (H_{a_2})$, and (H_{a_3}) for $\kappa_i^0 = \kappa_i^1 = 1$, $\kappa_i^2 > 0$ (where $i = 1$ or 2), and $\kappa_\star^3 = 0$. In this particular case, we are investigating the following problem:

$$\begin{cases} K(\mathcal{B}(u))\big(\Delta_{p(x)}u - b(x)|u|^{p(x)-2}u\big) = |u|^{r(x)-2}u + \lambda(x)\frac{\partial \mathcal{F}}{\partial u}(x,u,v) & \text{in } \mathbb{R}^N, \\ K(\mathcal{B}(v))\big(-\Delta_{p(x)}v + b(x)|v|^{p(x)-2}v\big) = |v|^{r(x)-2}v + \lambda(x)\frac{\partial \mathcal{F}}{\partial v}(x,u,v) & \text{in } \mathbb{R}^N, \end{cases}$$

where

$$\mathcal{B}(u) = \int_{\mathbb{R}^N} \frac{1}{p(x)}\big(|\nabla u|^{p(x)} + b(x)|u|^{p(x)}\big)dx$$

and

$$\mathcal{B}(v) = \int_{\mathbb{R}^N} \frac{1}{p(x)}\big(|\nabla v|^{p(x)} + b(x)|u|^{p(x)}\big)dx.$$

The operator $\Delta_{p(x)}u := \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ is known as the $p(x)$-Laplacian, which coincides with the usual p-Laplacian when $p(x) = p$, and with the Laplacian when $p(x) = 2$.

Example 2. Considering the functions

$$a_1(t) = 1 + t^{\frac{q(x)-p(x)}{p(x)}} \text{ and } a_2(t) = 1 + t^{\frac{q(x)-p(x)}{p(x)}}.$$

We observe that both a_1 and a_2 satisfy conditions $(H_{a_1}), (H_{a_2})$, and (H_{a_3}), with $\kappa_i^0 = \kappa_i^1 = \kappa_i^2 = \kappa_\star^3 = 1$ (for $i = 1$ or 2). In this case, we are investigating the following noncooperative $p\&q$-Laplacian system:

$$\begin{cases} K(\mathcal{B}(u))\big(\Delta_{p(x)}u + \Delta_{q(x)}u - b(x)(|u|^{p(x)-2}u + |u|^{q(x)-2}u)\big) = |u|^{r(x)-2}u + \lambda(x)\frac{\partial \mathcal{F}}{\partial u}(x,u,v) & \text{in } \mathbb{R}^N, \\ K(\mathcal{B}(v))\big(-\Delta_{p(x)}v - \Delta_{q(x)}v + b(x)(|v|^{p(x)-2}v + |v|^{q(x)-2}v)\big) = |v|^{r(x)-2}v + \lambda(x)\frac{\partial \mathcal{F}}{\partial v}(x,u,v) & \text{in } \mathbb{R}^N, \end{cases}$$

where

$$\mathcal{B}(u)=\int_{\mathbb{R}^N}\left(\frac{1}{p(x)}\left(|\nabla u|^{p(x)}+b(x)|u|^{p(x)}\right)+\frac{1}{q(x)}\left(|\nabla u|^{q(x)}+b(x)|u|^{q(x)}\right)\right)dx$$

and

$$\mathcal{B}(v)=\int_{\mathbb{R}^N}\left(\frac{1}{p(x)}\left(|\nabla v|^{p(x)}+b(x)|v|^{p(x)}\right)+\frac{1}{q(x)}\left(|\nabla v|^{q(x)}+b(x)|v|^{q(x)}\right)\right)dx.$$

This class of systems arises in various applications, such as reaction–diffusion systems described by

$$\frac{\partial u}{\partial t}=\operatorname{div}(a_1(\nabla u)\nabla u)+P(x,u),\ \text{where}\ a_1(\nabla u)=|\nabla u|^{p(x)-2}+|\nabla u|^{q(x)-2}, \tag{1.4}$$

where the reaction term $P(x,u)$ is a polynomial of u with variable coefficients. Such systems have wide applications in physics and related sciences, including plasma physics, biophysics, and chemical reaction design. In these applications, the function u represents concentration, the first term on the right-hand side of (1.4) accounts for diffusion with a diffusion coefficient $a_1(\nabla u)$, and the second term represents the reaction, which is related to source and loss processes, typically in chemical and biological applications. For further details, interested readers can refer to works by Mahshid and Razani [42], He and Li [27], and the references therein.

We continue with other examples that are also interesting from the mathematical point of view.

Example 3. Considering $a_1(t)=1+\frac{t}{\sqrt{1+t^2}}$ and $a_2\equiv 1$, we can observe that both a_1 and a_2 satisfy conditions $(H_{a_1}),(H_{a_2})$, and (H_{a_3}), for $\kappa_1^0=\kappa_2^0=\kappa_2^1=1, \kappa_1^1=2$, $\kappa_\star^3=0$, $\kappa_1^2>0$, and $\kappa_2^2>0$. In this scenario, we are studying the following problem:

$$\begin{cases} K(\mathcal{B}(u))\left(\operatorname{div}\left(\left(1+\frac{|\nabla u|^{p(x)}}{\sqrt{1+|\nabla u|^{2p(x)}}}\right)|\nabla u|^{p(x)-2}\nabla u\right)-b(x)|u|^{p(x)-2}u\right)=|u|^{r(x)-2}u+\lambda(x)\frac{\partial\mathcal{F}}{\partial u}(x,u,v)\ \text{in}\ \mathbb{R}^N,\\ K(\mathcal{B}(v))\left(-\operatorname{div}\left(\left(1+\frac{|\nabla v|^{p(x)}}{\sqrt{1+|\nabla v|^{2p(x)}}}\right)|\nabla v|^{p(x)-2}\nabla v\right)+b(x)|v|^{p(x)-2}v\right)=|v|^{r(x)-2}v+\lambda(x)\frac{\partial\mathcal{F}}{\partial v}(x,u,v)\ \text{in}\ \mathbb{R}^N,\end{cases}$$

where

$$\mathcal{B}(u)=\int_{\mathbb{R}^N}\frac{1}{p(x)}\left(|\nabla u|^{p(x)}+\sqrt{1+|\nabla u|^{2p(x)}}+b(x)|u|^{p(x)}\right)dx$$

and

$$\mathcal{B}(v)=\int_{\mathbb{R}^N}\frac{1}{p(x)}\left(|\nabla v|^{p(x)}+\sqrt{1+|\nabla v|^{2p(x)}}+b(x)|v|^{p(x)}\right)dx.$$

The operator

$$\operatorname{div}\left(\left(1+\frac{|\nabla u|^{p(x)}}{\sqrt{1+|\nabla u|^{2p(x)}}}\right)|\nabla u|^{p(x)-2}\nabla u\right)$$

is known as the $p(x)$-Laplacian-like operator or the generalized capillary operator, which has applications in various fields such as industry, biomedicine, and pharmaceuticals. For further details, you can refer to Ni and Serrin [43].

Example 4. Considering $a_1(t)=1+t^{\frac{q(x)-p(x)}{p(x)}}+\frac{1}{(1+t)^{\frac{p(x)-2}{p(x)}}}$ and $a_2(t)=1+t^{\frac{q(x)-p(x)}{p(x)}}$, we have that a_1 and a_2 satisfy conditions $(H_{a_1}),(H_{a_2})$, and (H_{a_3}) with $\kappa_1^0=\kappa_2^0=\kappa_2^1=1$, $\kappa_1^1=2$ and $\kappa_\star^3=\kappa_1^2=\kappa_2^2=1$. In this case, we are studying problem

$$\begin{cases} K(\mathcal{B}(u))\left(\Delta_{p(x)}u+\Delta_{q(x)}u+\operatorname{div}\left(\dfrac{|\nabla u|^{p(x)-2}\nabla u}{(1+|\nabla u|^{p(x)})^{\frac{p(x)-2}{p(x)}}}\right)-b(x)\left(|u|^{p(x)-2}u+|u|^{q(x)-2}u\right)\right) \\ \qquad = |u|^{r(x)-2}u+\lambda(x)\dfrac{\partial\mathcal{F}}{\partial u}(x,u,v) \quad \text{in } \mathbb{R}^N, \\ K(\mathcal{B}(v))\left(-\Delta_{p(x)}v-\Delta_{q(x)}v-\operatorname{div}\left(\dfrac{|\nabla v|^{p(x)-2}\nabla v}{(1+|\nabla v|^{p(x)})^{\frac{p(x)-2}{p(x)}}}\right)+b(x)\left(|v|^{p(x)-2}v+|v|^{q(x)-2}v\right)\right) \\ \qquad = |v|^{r(x)-2}v+\lambda(x)\dfrac{\partial\mathcal{F}}{\partial v}(x,u,v) \quad \text{in } \mathbb{R}^N, \end{cases}$$

where

$$\mathcal{B}(u)=\int_{\mathbb{R}^N}\left(\frac{1}{p(x)}\left(|\nabla u|^{p(x)}+b(x)|u|^{p(x)}\right)+\frac{1}{q(x)}\left(|\nabla u|^{q(x)}+b(x)|u|^{q(x)}\right)+\frac{1}{2}\left(1+|\nabla u|^{p(x)}\right)^{\frac{2}{p(x)}}\right)dx$$

and

$$\mathcal{B}(v)=\int_{\mathbb{R}^N}\left(\frac{1}{p(x)}\left(|\nabla v|^{p(x)}+b(x)|v|^{p(x)}\right)+\frac{1}{q(x)}\left(|\nabla v|^{q(x)}+b(x)|v|^{q(x)}\right)+\frac{1}{2}\left(1+|\nabla v|^{p(x)}\right)^{\frac{2}{p(x)}}\right)dx.$$

Moreover, the class of systems (1.1) can include either a single model of the divergence operators mentioned above, as in Examples I–IV, or two different models in each equation for divergence operators simultaneously, depending on the studied phenomenon. Also, every equation in this class can be either degenerate or nondegenerate.

In the case of a single equation, system (1.1) is related to a model that was first proposed by Kirchhoff in 1883. This model represents the stationary version of the Kirchhoff equation, which can be written as:

$$\rho\frac{\partial^2 u}{\partial t^2}-\left(\frac{\rho_0}{h}+\frac{E}{2L}\int_0^L\left|\frac{\partial u(x)}{\partial x}\right|^2dx\right)\frac{\partial^2 u}{\partial x^2}=0. \tag{1.5}$$

This equation extends the classical D'Alembert wave equation by considering the small vertical vibrations of a stretched elastic string with variable tension and fixed ends. One distinctive feature of Equation (1.5) is the presence of a nonlocal coefficient:

$$\frac{\rho_0}{h}+\frac{E}{2L}\int_0^L\left|\frac{\partial u}{\partial x}\right|^2dx.$$

This coefficient depends on the average value:

$$\frac{1}{2L}\int_0^L\left|\frac{\partial u}{\partial x}\right|^2dx.$$

As a result, the equation is no longer a pointwise equation, and this nonlocal aspect distinguishes it from the classical wave equation.

The parameters in Equation (1.5) have the following meanings: $u=u(x,t)$ represents the transverse string displacement at the spatial coordinate x and time t, E is the Young modulus of the material, also known as the elastic modulus, which measures the string's resistance to elastic deformation, ρ is the mass density, L is the length of the string, h is the area of cross-section, and ρ_0 is the initial tension (for more details see Kirchhoff [30]).

Almost one century later, Jacques-Louis Lions [40] returned to the equation and proposed a general Kirchhoff equation in arbitrary dimension with an external force term, which was written as

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - \left(a + b \int_\Omega |\nabla u|^2 dx\right) \Delta u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega; \end{cases} \tag{1.6}$$

this problem is often referred to as a nonlocal problem because it involves an integral over the domain Ω. This integral component introduces mathematical complexities that make the study of such problems particularly interesting. The nonlocal problem serves as a model for various physical and biological systems in which the variable u represents a process dependent on its own average, such as population density. For further references on this subject, interested readers can explore the works of Arosio and Pannizi [3], Cavalcanti et al. [8], Chipot and Lovat [14], and Corrêa and Nascimento [15], along with the references provided therein.

On one hand, it is widely acknowledged that the class of elliptic problems with constant critical exponents in bounded or unbounded domains holds a significant place in the literature. This class of problems was first introduced in the seminal paper by Brezis and Nirenberg [6], which primarily focused on Laplacian equations. Subsequently, various extensions of the results presented in [6] have been explored in many directions. A notable feature of elliptic equations involving critical growth is the issue of a lack of compactness, which is closely tied to the variational approach. To address this lack of compactness, P. L. Lions [41] developed a method employing the concentration–compactness principle (CCP) to establish that a minimizing sequence or a Palais–Smale (PS) sequence is precompact. Following this development, a variable exponent version of P. L. Lions' CCP for bounded domains was independently formulated by Bonder and Silva [5], Fu [23], while the version for unbounded domains was introduced by Fu [25]. Subsequently, numerous researchers have employed these results to investigate critical elliptic problems involving variable exponents, as evidenced by the works of Alves et al. [1, 2], Chems Eddine et al. [10, 11, 13], Hurtado et al. [29], Liang et al. [34–37], and Fu and Zhang [24, 50].

On the other hand, over the past few decades, there has been significant interest among researchers in studying elliptic problems that lead to indefinite functionals. For instance, in the work by Benci [4], it was assumed that X is a Hilbert space, and f satisfies the PS condition, and has the form

$$f(u) = \frac{1}{2}\langle L(u), u\rangle + \Phi(u), \quad u \in H,$$

where L is a bounded self-adjoint operator and Φ' is compact. Nevertheless, the solution spaces are not necessarily Hilbert spaces. To overcome this difficulty, in [33], Li introduced a limit index theory and applied it to estimate the number of solutions for the following noncooperative p-Laplacian elliptic system with Dirichlet boundary conditions

$$\begin{cases} \Delta_p u = \frac{\partial F}{\partial u}(x,u,v) & \text{in } \Omega, \\ -\Delta_p v = \frac{\partial F}{\partial v}(x,u,v) & \text{in } \Omega, \\ u = 0, \quad v = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.7}$$

Following that, Huang and Li [28] studied the following noncooperative p-Laplacian elliptic system in the unbounded domain of $\mathbb{R}^N$ by using the principle of symmetric criticality and the limit index theory

$$\begin{cases} \Delta_p u - |u|^{p-2}u = \frac{\partial F}{\partial u}(x,u,v) & \text{in } \mathbb{R}^N, \\ -\Delta_p v + |v|^{p-2}v = \frac{\partial F}{\partial v}(x,u,v) & \text{in } \mathbb{R}^N, \\ u, v \in W^{1,p}(\mathbb{R}^N), \end{cases}$$

where $1 < p < N$, and extended some results of Li [33]. Next, Cai and Li [7] dealt with the case when the corresponding functional of (1.7) may not be locally Lipschitz continuous in Banach spaces. Lin and Li [38], studied problem (1.7) with critical exponents of the form

$$\begin{cases} \Delta_p u = |u|^{p^*-2}u + \frac{\partial F}{\partial u}(x,u,v) & \text{in } \Omega, \\ -\Delta_p v = |v|^{p^*-2}v + \frac{\partial F}{\partial v}(x,u,v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.8}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $1 < p, q < N$, $p^* = \frac{Np}{N-p}$, and $q^* = \frac{Nq}{N-q}$, and established the existence of multiple solutions for problem (1.8) without using CCP. Some similar results for the noncooperative $p(x)$-Laplacian elliptic problems were obtained by Liang et al. [35, 36]. Recently, Chems Eddine [12] extended these results to the problem (1.1) when $K \equiv 1$, λ is a real number, and the functions p and q are Lipschitz continuous.

Our objective in this paper is to study the existence and multiplicity of solutions for a class of the generalized noncooperative Schrödinger–Kirchhoff–type systems with critical nonlinearity involving a general variable exponent elliptic operator in $\mathbb{R}^N$. More precisely, our main results of this work extend, complement, and complete several works, in particular Chems Eddine [12], Fang and Zhang [22], Huang and Li [28], Li [33], Liang and Zhang [36], and some papers listed therein.

As we shall see in the next sections, there are three main difficulties in our situation. First, the energy functional corresponding to problem (1.1) is strongly indefinite. Here, we mean strongly indefinitely that a functional is unbounded from below and from above on any subspace of finite codimension. Hence, we cannot apply the Mountain pass theorem for the energy functional. The second difficulty in solving problem (1.1) is the lack of compactness, which can be illustrated by the fact that the embedding of $W_b^{1,h(x)}(\mathbb{R}^N) \hookrightarrow L^{h^*(x)}(\mathbb{R}^N)$ is no longer compact. The third difficulty is that problem (1.1) involves nonlocal terms $K(\mathcal{B}(u))$ and $K(\mathcal{B}(v))$, which prevent us from applying the methods as before. To overcome these difficulties, we use the limit index theory developed by Li [33], the principle of concentration–compactness (Theorem 2.7), the CCP at infinity (Theorem 2.8) for the weighted-variable exponent Sobolev spaces $W_b^{1,h(x)}(\mathbb{R}^N)$, and the principle of symmetric criticality of Krawcewicz and Marzantowicz [32].

Throughout this paper, we shall assume that $\mathcal{F}$ satisfies the following conditions:

(F_1) $\mathcal{F} \in C^1(\mathbb{R}^N \times \mathbb{R}^2, \mathbb{R}^+)$ and it satisfies

$$\left|\frac{\partial \mathcal{F}}{\partial \eta}(x,\eta,\xi)\right| + \left|\frac{\partial \mathcal{F}}{\partial \xi}(x,\eta,\xi)\right| \le f_1(x)|\eta|^{\ell(x)-1} + f_2(x)|\xi|^{\ell(x)-1},$$

where $\ell \in \mathcal{M}(\mathbb{R}^N)$, $q^+ < \ell(x) < h^*(x)$ for all $x \in \mathbb{R}^N$, and $0 \le f_1, f_2 \in L^{l(x)} \cap L^\infty(\mathbb{R}^N)$ with $l(x) = h^*(x)/(h^*(x) - \ell(x))$.

(F_2) There exists $\vartheta(x)$ such that $h^+ < \vartheta(x)$ and $0 < \vartheta(x)\mathcal{F}(x,\eta,\xi) \le \eta\frac{\partial \mathcal{F}}{\partial \eta}(x,\eta,\xi) + \xi\frac{\partial \mathcal{F}}{\partial \xi}(x,\eta,\xi)$, for all $(x,\eta,\xi) \in (\mathbb{R}^N \times \mathbb{R}^2)$.

(F_3) $\eta\frac{\partial \mathcal{F}}{\partial \eta}(x,\eta,\xi) \ge 0$ for all $(x,\eta,\xi) \in \mathbb{R}^N \times \mathbb{R}^2$.

(F_4) $\mathcal{F}$ is even in (η,ξ): $\mathcal{F}(x,\eta,\xi) = \mathcal{F}(x,-\eta,-\xi)$ for all $(x,\eta,\xi) \in \mathbb{R}^N \times \mathbb{R}^2$.

(F_5) $\mathcal{F}(x,\eta,\xi) = \mathcal{F}(|x|,\eta,\xi)$ for all $(x,\eta,\xi) \in \mathbb{R}^N \times \mathbb{R}^2$.

The main result of this paper is as follows.

Theorem 1.1. *Assume that conditions $(H_{a_1}) - (H_{a_4})$, $(K_1) - (K_2)$, and $(F_1) - (F_5)$ are satisfied. Then, there exists a constant $\lambda_\star > 0$, such that if $\lambda(x)$ satisfies the following condition,*

$$0 < \lambda^- := \inf_{x \in \mathbb{R}^N} \lambda(x) \le \lambda^+ := \|\lambda\|_{L^\infty(\mathbb{R}^N)} \le \lambda_\star,$$

then problem (1.1) possesses infinitely many weak solutions in $X \times X$.

The paper is organized as follows. In Section 2.1, we briefly present some properties of the generalized weighted Sobolev spaces with variable exponents. In addition, we introduce the principle of concentration–compactness and the CCP at infinity in the generalized weighted-variable exponent Sobolev spaces. In Section 2.2, we mainly introduce the limit index theory due to Li [33]. In Section 3, we provide proof for the main results, after we have verified the PS condition at some special energy levels, by using the CCP.

2 | PRELIMINARIES AND BASIC NOTATIONS

In this section, we introduce some definitions and results, which will be used in the next section. Throughout this paper, we employ the following notation and conventions: We use $\to$ to denote strong convergence, $\rightharpoonup$ for weak convergence, and $\overset{*}{\rightharpoonup}$ for weak-* convergence. For any given $\rho > 0$ and $x \in \Omega$, $B_\rho(x)$ represents the ball with radius ρ centered at x. The duality pairing between X' and X is represented by $\langle \cdot, \cdot \rangle$. C and c denote a positive constants and can be determined based on specific conditions.

2.1 | Generalized weighted variable Sobolev spaces and the principle of concentration–compactness

First, we shall introduce some fundamental results from the theory of Lebesgue–Sobolev spaces with variable exponents. The details can be found in Diening et al. [18], Fan and Zhao [21], and Kováčik and Rákosní [31]. Let $\mathcal{M}(\mathbb{R}^N)$ be the set of all measurable real functions on $\mathbb{R}^N$. We define

$$C_+(\mathbb{R}^N) = \left\{p \in C(\mathbb{R}^N) \,:\, \operatorname{ess\,inf}_{x\in\mathbb{R}^N} p(x) > 1\right\}.$$

Additionally, we denote by $C_+^{\log}(\mathbb{R}^N)$ the set of functions $p \in C_+(\mathbb{R}^N)$ that satisfy the log-Holder continuity condition

$$\sup\left\{ |p(x) - p(y)| \log \frac{1}{|x-y|} \,:\, x, y \in \mathbb{R}^N, 0 < |x-y| < \frac{1}{2} \right\} < \infty.$$

For any $p \in C_+(\mathbb{R}^N)$, we define

$$p^- := \operatorname{ess\,inf}_{x\in\mathbb{R}^N} p(x) \quad \text{ and } p^+ := \operatorname{ess\,sup}_{x\in\mathbb{R}^N} p(x).$$

For any $p \in C_+(\mathbb{R}^N)$, we define the variable exponent Lebesgue space as

$$L^{p(x)}(\mathbb{R}^N) = \left\{ u \in \mathcal{M}(\mathbb{R}^N) \,:\, \int_{\mathbb{R}^N} |u(x)|^{p(x)} dx < \infty \right\},$$

endowed with the Luxemburg norm

$$|u|_{p(x)} := |u|_{L^{p(x)}(\mathbb{R}^N)} = \inf\left\{ \tau > 0 \,:\, \int_{\mathbb{R}^N} \left|\frac{u(x)}{\tau}\right|^{p(x)} dx \le 1 \right\}.$$

Let $b \in \mathcal{M}(\mathbb{R}^N)$, and $b(x) > 0$ for a.e. $x \in \mathbb{R}^N$. Define the weighted variable exponent Lebesgue space $L_b^{p(x)}(\mathbb{R}^N)$ by

$$L_b^{p(x)}(\mathbb{R}^N) = \left\{ u \in \mathcal{M}(\mathbb{R}^N) \,:\, \int_{\mathbb{R}^N} b(x)|u(x)|^{p(x)} dx < \infty \right\},$$

with the norm

$$|u|_{b,p} := |u|_{L_b^{p(x)}(\mathbb{R}^N)} = \inf\left\{ \tau > 0 \,:\, \int_{\mathbb{R}^N} b(x)\left|\frac{u(x)}{\tau}\right|^{p(x)} dx \le 1 \right\}.$$

From now on, we shall assume that $w \in L^{\infty}(\mathbb{R}^N)$ with $b_0 := \operatorname{ess\,inf}_{x\in\mathbb{R}^N} b(x) > 0$. Then obviously $L_b^{p(x)}(\mathbb{R}^N)$ is a Banach space (see Cruz-Uribe et al. [16] for details), and the norms $|u|_{b,p}$ and $|u|_p$ are equivalent in $L_b^p(\mathbb{R}^N)$.

On the other hand, the variable exponent Sobolev space $W^{1,p(x)}(\mathbb{R}^N)$ is defined by

$$W^{1,p(x)}(\mathbb{R}^N) = \{u \in L^{p(x)}(\mathbb{R}^N) : |\nabla u| \in L^{p(x)}(\mathbb{R}^N)\},$$

and is endowed with the norm

$$\|u\|_{1,p(x)} := \|u\|_{W^{1,p(x)}(\mathbb{R}^N)} = |u|_{p(x)} + |\nabla u|_{p(x)}, \quad \text{for all } u \in W^{1,p(x)}(\mathbb{R}^N).$$

Next, we define the weighted-variable exponent Sobolev space $W_b^{1,p(x)}(\mathbb{R}^N)$ as follows:

$$W_b^{1,p(x)}(\mathbb{R}^N) = \left\{u \in L_b^{p(x)}(\mathbb{R}^N) : |\nabla u| \in L_b^{p(x)}(\mathbb{R}^N)\right\}.$$

This space is equipped with the norm

$$\|u\|_{b,p} := \inf\left\{\tau > 0 : \int_{\mathbb{R}^N} \left|\frac{\nabla u(x)}{\tau}\right|^{p(x)} + b(x)\left|\frac{u(x)}{\tau}\right|^{p(x)} dx \le 1\right\}, \text{ for all } u \in W_b^{1,p(x)}(\mathbb{R}^N).$$

It is worth noting that the norms $\|u\|_{b,p}$ and $\|u\|_{1,p(x)}$ are equivalent in the space $W_b^{1,p(x)}(\mathbb{R}^N)$. Moreover, if $p^- > 1$, then the spaces $L^{p(x)}(\mathbb{R}^N)$, $W^{1,p(x)}(\mathbb{R}^N)$, and $W_b^{1,p(x)}(\mathbb{R}^N)$ are separable, reflexive, and uniformly convex Banach spaces.

We now present some essential facts that will be utilized later.

Proposition 2.1 (see [18, 21]). *The conjugate space of $L^{p(x)}(\mathbb{R}^N)$ is $L^{p'(x)}(\mathbb{R}^N)$, where*

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1.$$

Furthermore, for any $(u, v) \in L^{p(x)}(\mathbb{R}^N) \times L^{p'(x)}(\mathbb{R}^N)$, we have the following Hölder-type inequality:

$$|\int_{\mathbb{R}^N} uv dx| \le \left(\frac{1}{p^-} + \frac{1}{(p')^-}\right)|u|_{p(x)}|v|_{p'(x)} \le 2|u|_{p(x)}|v|_{p'(x)}.$$

Proposition 2.2 (see [18, 21]). *Denote $\rho_p(u) = \int_{\mathbb{R}^N} |u|^{p(x)} dx$, for all $u \in L^{p(x)}(\mathbb{R}^N)$. We have*

$$\min\left\{|u|_{p(x)}^{p^-}, |u|_{p(x)}^{p^+}\right\} \le \rho_p(u) \le \max\left\{|u|_{p(x)}^{p^-}, |u|_{p(x)}^{p^+}\right\},$$

and the following implications are true:

(i) $|u|_{p(x)} < 1$ *(resp.* $= 1, > 1$*)* $\Leftrightarrow \rho_p(u) < 1$ *(resp.* $= 1, > 1$*),*
(ii) $|u|_{p(x)} > 1 \Rightarrow |u|_{p(x)}^{p^-} \le \rho_p(u) \le |u|_{p(x)}^{p^+}$,
(iii) $|u|_{p(x)} < 1 \Rightarrow |u|_{p(x)}^{p^+} \le \rho_p(u) \le |u|_{p(x)}^{p^-}$.

Additionally, in particular, for any sequence $\{u_n\} \subset L^{p(x)}(\mathbb{R}^N)$,

$$|u_n|_{p(x)} \to 0 \text{ if and only if } \rho_p(u_n) \to, 0$$

and

$$\{u_n\} \text{ is bounded in } L^{p(x)}(\mathbb{R}^N) \text{ if and only if } \rho_p(u_n) \text{ is bounded in } \mathbb{R}.$$

According to Proposition 2.2, we can derive the following inequalities:

$$\|u\|_{b,p}^{p^-} \le \int_{\mathbb{R}^N} \left(|\nabla u(x)|^{p(x)} + b(x)|u(x)|^{p(x)}\right)dx \le \|u\|_{b,p}^{p^+} \quad \text{for} \quad \|u\|_{b,p} \ge 1. \tag{2.1}$$

$$\|u\|_{b,p}^{p^+} \le \int_{\mathbb{R}^N} \left(|\nabla u(x)|^{p(x)} + b(x)|u(x)|^{p(x)}\right)dx \le \|u\|_{b,p}^{p^-} \quad \text{for} \quad \|u\|_{b,p} \le 1. \tag{2.2}$$

Moreover, for any sequence $\{u_n\} \subset W_b^{1,p(x)}(\mathbb{R}^N)$,

$$\text{when } \|u_n\|_{b,p} \to 0, \text{ it is equivalent to } \int_{\mathbb{R}^N} \left(|\nabla u_n(x)|^{p(x)} + b(x)|u_n(x)|^{p(x)}\right)dx \to 0,$$

and

$$\text{when } \{u_n\} \text{ is bounded in } W_b^{1,p(x)}(\mathbb{R}^N), \text{ it is equivalent to } \int_{\mathbb{R}^N} \left(|\nabla u_n|^{p(x)} + b(x)|u_n|^{p(x)}\right)dx \text{ is bounded in } \mathbb{R}.$$

Proposition 2.3 (see [19]). *Let p and q be measurable functions such that $p \in L^\infty(\mathbb{R}^N)$ and $1 \le p(x), q(x) \le \infty$ for a.e. $x \in \mathbb{R}^N$. If $u \in L^{q(x)}(\mathbb{R}^N)$, $u \ne 0$, then the following inequalities hold:*

$$\text{If} \quad |u|_{p(x)q(x)} \le 1, \quad \text{then} \quad |u|_{p(x)q(x)}^{p^-} \le \left||u|^{p(x)}\right|_{q(x)} \le |u|_{p(x)q(x)}^{p^+}.$$

$$\text{If} \quad |u|_{p(x)q(x)} \ge 1, \quad \text{then} \quad |u|_{p(x)q(x)}^{p^+} \le \left||u|^{p(x)}\right|_{q(x)} \le |u|_{p(x)q(x)}^{p^-}.$$

In particular, if $p(x) = p$ is constant, then $||u|^p|_{q(x)} = |u|_{pq(x)}^p$.

Proposition 2.4 (see [18, 19]). *Let $p \in C_+^{log}(\mathbb{R}^N)$ be such that $p^+ < N$ and let $q \in C(\mathbb{R}^N)$ satisfy $1 < q(x) < p^*(x)$ for each $x \in \mathbb{R}^N$, then there exists a continuous and compact embedding $W_b^{1,p(x)}(\mathbb{R}^N) \hookrightarrow L^{q(x)}(\mathbb{R}^N)$.*

Proposition 2.5 (see [18, 19]). *Let $p \in C_+^{log}(\mathbb{R}^N)$. Then, there exists a positive constant C^* such that*

$$|u|_{p^*(x)} \le C^* \|u\|_{b,p}, \qquad \textit{for all } u \in W_b^{1,p(x)}(\mathbb{R}^N).$$

In the upcoming discussions, we shall work with the product space denoted as

$$Y := \left(W_b^{1,p(x)}(\mathbb{R}^N) \cap W_b^{1,h(x)}(\mathbb{R}^N)\right) \times \left(W_b^{1,p(x)}(\mathbb{R}^N) \cap W_b^{1,h(x)}(\mathbb{R}^N)\right).$$

This space is endowed with the norm

$$\|(u,v)\|_Y := \max\left\{\|u\|_{b,h}, \|v\|_{b,h}\right\}, \qquad \text{for all } (u,v) \in Y,$$

where $\|u\|_{b,h} := \|u\|_{b,p} + \mathcal{H}(\kappa_\star^3)\|u\|_{b,q}$ represents the norm of $W_b^{1,p(x)}(\mathbb{R}^N) \cap W_b^{1,h(x)}(\mathbb{R}^N)$. The space Y^* corresponds to the dual space of Y and is endowed with the standard dual norm.

Definition 2.6. Consider a Banach space Y. An element $(u, v) \in Y$ is said to be a weak solution of the system (1.1) if

$$\begin{aligned}
&-K(\mathcal{B}(u)) \int_{\mathbb{R}^N} (\mathcal{A}_1(\nabla u).\nabla \tilde{u} + b(x)\mathcal{A}_2(u)\tilde{u})dx - \int_{\mathbb{R}^N} |u|^{r(x)-2}u\tilde{u}\, dx \\
&+K(\mathcal{B}(v)) \int_{\mathbb{R}^N} (\mathcal{A}_1(\nabla v).\nabla \tilde{v} + b(x)\mathcal{A}_2(v)\tilde{v})dx - \int_{\mathbb{R}^N} |v|^{r(x)-2}v\tilde{v}\, dx \\
&- \int_{\mathbb{R}^N} \lambda(x)\frac{\partial \mathcal{F}}{\partial u}(x,u,v)\tilde{u}\, dx - \int_{\mathbb{R}^N} \lambda(x)\frac{\partial \mathcal{F}}{\partial v}(x,u,v)\tilde{v}\, dx = 0,
\end{aligned}$$

for all $(\tilde{u},\tilde{v}) \in Y = \left(W_b^{1,p(x)}(\mathbb{R}^N) \cap W_b^{1,h(x)}(\mathbb{R}^N)\right) \times \left(W_b^{1,p(x)}(\mathbb{R}^N) \cap W_b^{1,h(x)}(\mathbb{R}^N)\right)$.

The energy functional $\tilde{E}_\lambda : Y \longrightarrow \mathbb{R}$ associated with problem (1.1) is given by,

$$\tilde{E}_\lambda(u,v) = -\hat{K}(\mathcal{B}(u(x))) + \hat{K}(\mathcal{B}(v(x))) - \int_{\mathbb{R}^N} \frac{1}{r(x)}|u|^{r(x)}dx - \int_{\mathbb{R}^N} \frac{1}{r(x)}|v|^{r(x)}dx - \int_{\mathbb{R}^N} \lambda(x)\mathcal{F}(x,u,v)dx,$$

for each (u,v) in Y.

Through standard calculus, one can establish that, under the above assumptions, the energy functional $\tilde{E}_\lambda : Y \to \mathbb{R}^N$ associated with problem (1.1) is well-defined and belongs to $C^1(Y,\mathbb{R})$. Its derivative, denoted as $\tilde{E}'_\lambda(u,v)$, satisfies

$$\begin{aligned}\left\langle \tilde{E}'_\lambda(u,v),(\tilde{u},\tilde{v})\right\rangle = &-K(\mathcal{B}(u)) \int_{\mathbb{R}^N} (\mathcal{A}_1(\nabla u).\nabla\tilde{u} + b(x)\mathcal{A}_2(u)\tilde{u})\,dx - \int_{\mathbb{R}^N} |u|^{r(x)-2}u\tilde{u}\,dx \\ &+ K(\mathcal{B}(v)) \int_{\mathbb{R}^N} (\mathcal{A}_1(\nabla v).\nabla\tilde{v} + b(x)\mathcal{A}_2(v)\tilde{v})\,dx - \int_{\mathbb{R}^N} |v|^{r(x)-2}v\tilde{v}\,dx \\ &- \int_{\mathbb{R}^N} \lambda(x)\frac{\partial\mathcal{F}}{\partial u}(x,u,v)\tilde{u}\,dx - \int_{\mathbb{R}^N} \lambda(x)\frac{\partial\mathcal{F}}{\partial v}(x,u,v)\tilde{v}\,dx,\end{aligned}$$

for all $(\tilde{u},\tilde{v}) \in Y$. Consequently, the critical points of the functional $\tilde{E}_\lambda$ correspond to weak solutions of the system (1.1).

To establish our existence result, we need to address the loss of compactness in the inclusion $W_b^{1,p(x)}(\mathbb{R}^N) \hookrightarrow L^{p^\star(x)}(\mathbb{R}^N)$. As a consequence, we can no longer expect the PS condition to hold uniformly. However, we can prove a local PS condition that will hold for $E_\lambda(u,v)$ below a certain value of energy, by using the principle of concentration–compactness for the weighted-variable exponent Sobolev space $W_b^{1,p(x)}(\mathbb{R}^N)$. For the reader's convenience, we state this result in order to prove Theorem 1.1, see Fu and Zhang [25, Theorem 2.2] for the proof.

Let us recall that $\mathcal{M}_B(\mathbb{R}^N)$ denotes the space of finite nonnegative Borel measures on $\mathbb{R}^N$. For any $\nu \in \mathcal{M}_B(\mathbb{R}^N)$, we have $\nu(\mathbb{R}^N) = \|\nu\|$. We say that $\nu_n \overset{*}{\rightharpoonup} \nu$ weakly-* in $\mathcal{M}_B(\mathbb{R}^N)$ if, as $n \to \infty$, we have $(\nu_n,\xi) \to (\nu,\xi)$ for all $\xi \in C_0(\mathbb{R}^N)$. Therefore, just as in Fu and Zhang [25, Theorem 2.2], we can readily deduce the following.

Theorem 2.7. *Consider $h \in C_+^{log}(\mathbb{R}^N)$ and $q \in C(\mathbb{R}^N)$ such that*

$$1 < \inf_{x\in\mathbb{R}^N} p(x) \le \sup_{x\in\mathbb{R}^N} h(x) < N \quad \text{and} \quad 1 \le r(x) \le h^*(x) \quad \text{for all } x \text{ in } \mathbb{R}^N.$$

Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence that weakly converges in $W_b^{1,h(x)}(\mathbb{R}^N)$ to u, and such that $\|u_n\|_{b,h} \le 1$. The sequence satisfies the following conditions as $n \to \infty$:

(1) $|\nabla u_n|^{h(x)} + b(x)|u_n|^{h(x)} \overset{*}{\rightharpoonup} \mu$ in $\mathcal{M}_B(\mathbb{R}^N)$,
(2) $|u_n|^{r(x)} \overset{*}{\rightharpoonup} \nu$ in $\mathcal{M}_B(\mathbb{R}^N)$,

as $n \to \infty$. Additionally, suppose that $C_h = \{x \in \mathbb{R}^N : r(x) = h^*(x)\}$ is nonempty. Then, for some countable index set I, we have

$$\mu = |\nabla u|^{h(x)} + b(x)|u|^{h(x)} + \sum_{i\in I}\mu_i\delta_{x_i} + \tilde{\mu}, \quad \mu(C_h) \le 1; \tag{2.3}$$

$$\nu = |u|^{r(x)} + \sum_{i\in I}\nu_i\delta_{x_i}, \quad \nu(C_h) \le S; \tag{2.4}$$

with

$$S = \sup\left\{\int_{\mathbb{R}^N}|u|^{r(x)}dx : u \in W_b^{1,h(x)}, \|u\|_{b,h} \le 1\right\}, \{x_i\}_{i\in I} \subset C_h,$$

and $\{\mu_i\}, \{\nu_i\} \subset [0, \infty)$, δ_{x_i} *is the Dirac mass at* $x_i \in C_h$, *and* $\tilde{\mu} \in \mathcal{M}_B(\mathbb{R}^N)$ *is a nonatomic nonnegative measure. The atoms and the regular part satisfy the generalized Sobolev inequality*

$$\nu(C_h) \le 2^{(h^+ r^+)/h^-} C^* \max\left\{\mu(C_h)^{r^+/h^-}, \mu(C_h)^{r^-/h^+}\right\}$$

and

$$\nu_i \le C^* \max\left\{\mu_i^{r^+/h^-}, \mu_i^{r^-/h^+}\right\}.$$

Theorem 2.7 does not account for the potential mass loss at infinity within a weakly convergent sequence. Subsequently, Theorem 2.8 quantifies this occurrence. In a manner reminiscent of Fu and Zhang [25, Theorem 2.5], we deduce the following outcomes.

Theorem 2.8. *Assuming* $C_h = \{x \in \mathbb{R}^N : r(x) = h^*(x)\}$, *let* $\{u_n\}_{n\in\mathbb{N}}$ *be a sequence that weakly converges in* $W_b^{1,h(x)}(\mathbb{R}^N)$ *to u, and such that*

(1) $|\nabla u_n|^{h(x)} + b(x)|u_n|^{h(x)} \overset{*}{\rightharpoonup} \mu$ *in* $\mathcal{M}_B(\mathbb{R}^N)$,
(2) $|u_n|^{r(x)} \overset{*}{\rightharpoonup} \nu$ *in* $\mathcal{M}_B(\mathbb{R}^N)$.

We define the quantities:

$$\mu_\infty = \lim_{\{x\in\mathbb{R}^N; |x|>R\}} \limsup_{n\to+\infty} \int_{\{x\in\mathbb{R}^N; |x|>R\}} (|\nabla u_n|^{h(x)} + b(x)|u_n|^{h(x)})dx$$

and

$$\nu_\infty = \lim_{R\to\infty} \limsup_{n\to+\infty} \int_{\{x\in\mathbb{R}^N; |x|>R\}} |u_n|^{r(x)}dx.$$

The quantities μ_∞ *and* ν_∞ *are well defined and satisfy*

$$\limsup_{n\to+\infty} \int_{C_h} (|\nabla u_n|^{h(x)} + b(x)|u_n|^{h(x)})dx = \int_{C_h} d\mu + \mu_\infty \text{ and } \limsup_{n\to+\infty} \int_{C_h} |u_n|^{r(x)}dx = \int_{C_h} d\nu + \nu_\infty.$$

Additionally, the following inequality holds:

$$\nu_\infty \le C^* \max\left\{\mu_\infty^{r^+/h^-}, \mu_\infty^{r^-/h^+}\right\}.$$

2.2 | Limit index theory

In this subsection, we shall introduce the limit index theory due to Li [33]. In order to do that, we recall the following definitions (the interested readers can refer to Szulkin [47] and Willem [49]).

Definition 2.9 (see [33]). The action of a topological group G on a normed space Z is a continuous map $G \times Z \to Z : [g, z] \mapsto gz$ such that

$$1.z = z, \quad (gh)z = g(hz), \qquad z \mapsto gz \text{ is linear for all } g, h \in G.$$

The action is isometric if $\|gz\| = \|z\|$, for all $g \in G, z \in Z$, in which case, Z is called the G-space. The set of invariant points is defined by

$$\text{Fix}G := \{z \in Z : gz = z \text{ for all g} \in G\}.$$

A set $A \subset Z$ is invariant if $gA = A$ for every $g \in G$. A function $\varphi : Z \to R$ is invariant if $\varphi \circ g = \varphi$ for every $g \in G, z \in Z$. A map $f : Z \to Z$ is equivariant if $g \circ f = f \circ g$ for every $g \in G$.

Suppose that Z is a G-Banach space, that is, there is a G isometric action on Z. Let

$$\Sigma := \{A \subset Z : A \text{ is closed and } gA = A, \text{ for all } g \in G\}$$

be a family of all G-invariant closed subset of Z, and let

$$h := \left\{h \in C^0(Z,Z) : h(gu) = g(hu), \text{ for all } g \in G\right\}$$

be the class of all G-equivariant mappings of Z. Finally, the set $O(u) := \{gu : g \in G\}$ is called the G-orbit of u.

Definition 2.10 (see [33]). An index for (G, Σ, h) is a mapping $i : \Sigma \to \mathbb{Z}_+ \cup \{+\infty\}$, where $\mathbb{Z}_+$ is the set of all nonnegative integers, such that for all $A, B \in \Sigma$, $h \in h$, the following conditions are satisfied:

(1) $i(A) = 0 \Leftrightarrow A = \emptyset$.
(2) (Monotonicity) $A \subset B \Rightarrow i(A) \le i(B)$.
(3) (Subadditivity) $i(A \cup B) \le i(A) + i(B)$.
(4) (Supervariance) $i(A) \le i(\overline{h(A)})$, for all $h \in h$.
(5) (Continuity) If A is compact and $A \cap \mathrm{Fix}G = \emptyset$, then $i(A) < +\infty$ and there is a G-invariant neighborhood N of A such that $i(\overline{N}) = i(A)$.
(6) (Normalization) If $x \notin \mathrm{Fix}G$, then $i(O(x)) = 1$.

Definition 2.11 (see [4]). An index theory is said to satisfy the d-dimension property if there is a positive integer d such that $i(V^{dk} \cap S_1(0)) = k$, for all dk-dimensional subspaces $V^{dk} \in \Sigma$ such that $V^{dk} \cap \mathrm{Fix}G = \{0\}$, where $S_1(0))$ is the unit sphere in Z.

Suppose U and V are G-invariant closed subspaces of Z such that $Z = U \oplus V$, where V is infinite dimensional and

$$V = \overline{\bigcup_{j=1}^{\infty} V_j},$$

where V_j is dn_j-dimensional G-invariant subspaces of V, $j = 1, 2, \ldots$, and $V_1 \subset V_2 \subset \cdots \subset V_n \subset \ldots$. Let $Z_j = U \oplus V_j$ and for all $A \in \Sigma$, let $A_j = A \cap Z_j$.

Definition 2.12 (see [33]). Let i be an index theory satisfying the d-dimension property. A limit index with respect to (Z_j) induced by i is a mapping $i^\infty : \Sigma \to \mathbb{Z} \cup \{-\infty; +\infty\}$, given by $i^\infty(A) = \limsup_{j\to\infty}(i(A_j) - n_j)$.

Proposition 2.13 (see [33]). *Let $A, B \in \Sigma$. Then, i^∞ satisfies the following:*

(1) $A = \emptyset \Rightarrow i^\infty = -\infty$.
(2) (Monotonicity) $A \subset B \Rightarrow i^\infty(A) \le i^\infty(B)$.
(3) (Subadditivity) $i^\infty(A \cup B) \le i^\infty(A) + i^\infty(B)$.
(4) If $V \cap \mathrm{Fix}G = \{0\}$, then $i^\infty(S_\rho(0) \cap V) = 0$, where $S_\rho(0) = \{z \in Z : \|z\| = \rho\}$.
(5) If Y_0 and $\tilde{Y}_0$ are G-invariant closed subspaces of V such that $V = Y_0 \oplus \tilde{Y}_0$, $\tilde{Y}_0 \subset V_{j_0}$ for some j_0 and $\dim \tilde{Y}_0 = dm$, then $i^\infty(S_\rho(0) \cap Y_0) \ge -m$.

Definition 2.14 (see [49]). A functional $E \in C^1(Z, \mathbb{R})$ is said to satisfy condition $(PS)_c$ if any sequence $\{u_{nk}\}_k$, $u_{nk} \in Z_{nk}$, such that $E_{n_k}(u_{nk}) \to c$, $E'_{nk}(u_{nk}) \to 0$, as $n_k \to \infty$, possesses a convergent subsequence, where Z_{nk} is the n_k-dimensional subspace of Z as in Definition 2.11 and $E_{n_k} = E|Z_{nk}$.

Theorem 2.15 (see [33]). *Assume the following:*

(B_1) *$E \in C^1(Z, \mathbb{R})$ is G-invariant.*
(B_2) *There exist G-invariant closed subspaces U and V such that V is infinite dimensional and $Z = U \oplus V$.*

(B_3) *There is a sequence of G-invariant finite-dimensional subspaces*

$$V_1 \subset V_2 \subset \cdots \subset V_j \subset \ldots, \quad \dim V_j = dn_j,$$

such that $V = \overline{\bigcup_{j=1}^{\infty} V_j}$.

(B_4) *There is an index theory i on Z satisfying the d-dimension property.*

(B_5) *There are G-invariant subspaces* $Y_0, \tilde{Y}_0, Y_1$ *of V such that* $V = Y_0 \oplus \tilde{Y}_0$, $Y_1, \tilde{Y}_0 \subset V_{j_0}$ *for some* j_0 *and* $\dim \tilde{Y}_0 = dm < dk = \dim Y_1$.

(B_6) *There are* $\mathfrak{M}$ *and* $\mathfrak{N}$, $\mathfrak{M} < \mathfrak{N}$ *such that E satisfies* $(PS)_c$, *for all* $c \in [\mathfrak{M}, \mathfrak{N}]$.

(B_7) *The following holds:*

$$\begin{cases} (1)\ \textit{either } FixG \subset U \oplus Y_1 \textit{ or } FixG \cap V = \{0\}, \\ (2)\ \textit{there is } \rho > 0 \textit{ Such that for all } u \in Y_0 \cap S_\rho(0), \textit{ we have } E(z) \geq \mathfrak{M}, \\ (3)\ \textit{for all } z \in U \oplus Y_1, \textit{ we have } E(z) \leq \mathfrak{N}. \end{cases}$$

If i^∞ *is the limit index corresponding to i, then the numbers* $c_j := \inf_{i^\infty(A) \geq j} \sup_{z \in A} E(z)$, $-k+1 \leq j \leq -m$,

are critical values of E, and $\mathfrak{M} \leq c_{-k+1} \leq \cdots \leq c_{-m} \leq \mathfrak{N}$. *Moreover, if* $c = c_l = \cdots = c_{l+r}$, $r \geq 0$, *then* $i(\mathbb{K}_c) \geq r+1$, *where* $\mathbb{K}_c = \{z \in Z : E'(z) = 0, E(z) = c\}$.

Notations. $X = W_b^{1,p(x)}(\mathbb{R}^N) \cap W_b^{1,h(x)}(\mathbb{R}^N)$, $Y = X \times X$, $G_1 = O(\mathbb{N})$ is the group of orthogonal linear transformations in $\mathbb{R}^N$,

$$X_{G_1} := W_{b,G_1}^{1,p(x)}(\mathbb{R}^N) \cap W_{b,G_1}^{1,h(x)}(\mathbb{R}^N) = \{u \in W_b^{1,p(x)}(\mathbb{R}^N) \cap W_b^{1,h(x)}(\mathbb{R}^N) : gu(x) = u(g^{-1}x) = u(x), g \in G_1\}$$

$$= \{u \in W_b^{1,p(x)}(\mathbb{R}^N) \cap W_b^{1,h(x)}(\mathbb{R}^N) : u \text{ and } v \text{ are radially symmetric}\},$$

and $Z = Y_{G_1} = X_{G_1} \times X_{G_1}$.

3 | PROOF OF THE MAIN RESULT

To prove the main result of this paper, Theorem 1.1, we shall perform a careful analysis of the behavior of minimizing sequences in Lemma 3.1, by using the CCP for the weighted-variable exponent Sobolev space stated above, which will allow us to recover compactness below some critical threshold.

Since $\tilde{E}_\lambda \in C^1(Y, \mathbb{R})$, the weak solutions for problem (1.1) coincide with the critical points of $\tilde{E}_\lambda$. On the other hand, by condition (F_5), it is immediate that $\tilde{E}_\lambda$ is G_1-invariant. Therefore, by the principle of symmetric criticality of Krawcewicz and Marzantowicz [32], we know that (u, v) is a critical point of $\tilde{E}_\lambda$ if and only if (u, v) is a critical point of $E_\lambda = \tilde{E}_\lambda|_{Z = X_{G_1} \times X_{G_1}}$. So it suffices to prove the existence of a sequence of critical points of E_λ on Z.

Let X be a Banach space and a functional $f \in C^1(X, \mathbb{R})$. Given sequence $\{u_n\}_n$ in X, if there exist $c \in \mathbb{R}$ such that

$$f(u_n) \to c \text{ and } f'(u_n) \to 0 \text{ in } X',$$

we say that $\{u_n\}_n$ is a PS sequence with energy level c (or $\{u_n\}$ is $(PS)_c$ for short). When any $(PS)_c$ sequence for f possesses some strongly convergent subsequence in X, we say that f satisfies the PS condition at level c (or f is $(PS)_c$ short).

In order to prove that E_λ satisfies $(PS)_c$, we recall some properties of the Banach space X. According to Triebel [48, section 4.9.4], there exists a Schauder basis $\{e_n'\}_{n=1}^{\infty}$ for X. Let $e_n = \int_{G_1} e_n'(g(x)) d_{\mu_g}$. We are going to, if necessary, select one in identical elements. We know that $\{e_n\}_{n=1}^{\infty}$ is a Schauder basis for X_{G_1}. Furthermore, since X_{G_1} is reflexive, $\{e_n^*\}_{n=1}^{\infty}$ the biorthogonal functionals associated to the basis $\{e_n\}_{n=1}^{\infty}$ (which is characterized by relations

$$\langle e_m^*, e_n \rangle = \delta_{m,n} = \begin{cases} 1 \text{ if } n = m, \\ 0 \text{ if } n \neq m. \end{cases} \quad),$$

form a basis for $X_{G_1}^{\star}$ with the following properties—see Lindenstrauss and Tzafriri [39, cf. Proposition l.b.1 and Theorem l.b.51]. Denote

$$X_{G_1}^{(n)} = \operatorname{span}\{e_1, \dots, e_n\},\ X_{G_1}^{(n)^\perp} = \overline{\operatorname{span}\{e_{n+1}, \dots\}},\ X_{G_1}^{*(n)} = \operatorname{span}\{e_1^\star, \dots, e_n^\star\}.$$

Let $P_n : X_{G_1} \to X_{G_1}^{(n)}$ be the projector corresponding to decomposition $X_{G_1} = X_{G_1}^{(n)} \oplus X_{G_1}^{(n)^\perp}$ and $P_n^* : X_{G_1}^* \to X_{G_1}^{*(n)}$ the projector corresponding to the decomposition $X_{G_1}^* = X_{G_1}^{*(n)} \oplus X_{G_1}^{*(n)^\perp}$. Then, $P_n u \to u$, $P_n^* v^* \to v^*$ for any $u \in X_{G_1}$, $v^* \in X_{G_1}^*$ as $n \to \infty$ and $\langle P_n^* v^*, u\rangle = \langle v^*, P_n u\rangle$. Set $Z = Y_{G_1} = X_{G_1} \times X_{G_1}$, $\quad Z_n = X_{G_1} \times X_{G_1}^{(n)}$. We shall prove the following local PS condition.

Lemma 3.1. *Assume that the conditions $(H_{a_1}) - (H_{a_4})$, $(K_1) - (K_2)$, and $(F_1) - (F_5)$ are satisfied. Then, the functional E_λ satisfies the local $(PS)_c$ with*

$$c \in \left(-\infty, \left(\frac{1}{\vartheta^-} - \frac{1}{r^-}\right) \max\left\{\left(\frac{\mathfrak{K}_0 D}{S^{\frac{h^-}{r^+}}}\right)^{\frac{r^+}{r^+-h^-}}, \left(\frac{\mathfrak{K}_0 D}{S^{\frac{h^-}{r^-}}}\right)^{\frac{r^-}{r^--h^-}}\right\}\right),$$

where $D = (1 - \mathcal{H}(\kappa_\star^3)) \min\{\kappa_1^0, \kappa_2^0\} + \mathcal{H}(\kappa_\star^3) \min\{\kappa_1^2, \kappa_2^2\}$, in the following sense: If $\{y_{n_k}\} \subset Y$ is a sequence such that $y_{n_k} = (u_{n_k}, v_{n_k})$ and

$$E_{\lambda_{n_k}}(u_{n_k}, v_{n_k}) \to c \text{ and } E'_{\lambda_{n_k}}(u_{n_k}, v_{n_k}) \to 0, \text{ as } k \to \infty,$$

where $E_{\lambda_{n_k}} = E_\lambda|_{Z_{n_k}}$ with $Z_{n_k} = X_{G_1} \times X_{n_k}$, then $\{(u_{n_k}, v_{n_k})\}_k$ possesses a subsequence converging strongly in Z to a critical point of the functional E_λ.

Proof. First, we show that $\{(u_{n_k}, v_{n_k})\}$ is bounded in Z. If not, we may assume that $\|u_{n_k}\|_{b,h} > 1$ and $\|v_{n_k}\|_{b,h} > 1$ for any integer n. We have by condition (F_3),

$$\begin{aligned}
o(1)\|u_{n_k}\|_{b,h} &\geq \left\langle -E'_{\lambda_{n_k}}(u_{n_k}, v_{n_k}), (u_{n_k}, 0)\right\rangle \\
&= K\big(\mathcal{B}(u_{n_k})\big) \int_{\mathbb{R}^N} \big(\mathcal{A}_1(\nabla u_{n_k}).\nabla u_{n_k} + b(x)\mathcal{A}_2(u)u\big)dx \\
&\quad + \int_{\mathbb{R}^N} |u_{n_k}|^{r(x)}dx + \int_{\mathbb{R}^N} \lambda(x)\frac{\partial \mathcal{F}}{\partial u}(x, u_{n_k}, v_{n_k})u_{n_k}dx, \\
&\geq K\big(\mathcal{B}(u_{n_k})\big) \int_{\mathbb{R}^N} \big(a_1(|\nabla u_{n_k}|^{p(x)})|\nabla u_{n_k}|^{p(x)} + b(x)a_2(|u_{n_k}|^{p(x)})|u_{n_k}|^{p(x)}\big)dx.
\end{aligned}$$

Therefore, by using (K_1) and (H_{a_2}), we have

$$\begin{aligned}
\left\langle -E'_{\lambda_{n_k}}(u_{n_k}, v_{n_k}), (u_{n_k}, 0)\right\rangle \geq \mathfrak{K}_0 \Bigg[&\min\{\kappa_1^0, \kappa_2^0\} \int_{\mathbb{R}^N} \big(|\nabla u_{n_k}|^{p(x)} + b(x)|u_{n_k}|^{p(x)}\big)dx \\
&+ \min\{\kappa_1^2, \kappa_2^2\}\mathcal{H}(\kappa_\star^3) \int_{\mathbb{R}^N} \big(|\nabla u_{n_k}|^{q(x)} + b(x)|u_{n_k}|^{q(x)}\big)dx\Bigg]. \qquad (3.1)
\end{aligned}$$

Let us assume, for the sake of contradiction, that there exists a subsequence, still denoted by $\{u_{n_k}\}$, such that $\left\|u_{n_k}\right\|_{b,h} \to +\infty$. If $\kappa_\star^3 = 0$, from Proposition 2.2, we have

$$\left\langle -E'_{\lambda_{n_k}}(u_{n_k}, v_{n_k}), (u_{n_k}, 0)\right\rangle \geq C_1\|u_{n_k}\|_{b,p}^{p^-},$$

thus

$$o(1)\|u_{n_k}\|_{b,h} \geq C_1\|u_{n_k}\|_{b,p}^{p^-}.$$

However, this is a contradiction since $p^- > 1$. Therefore, we can conclude that $\{u_{n_k}\}$ is bounded in X_{G_1}.

Next, if $\kappa_\star^3 > 0$, we need to analyze the following cases:

(1) $\left\|u_{n_k}\right\|_{b,p} \to +\infty$ and $\left\|u_{n_k}\right\|_{b,q} \to +\infty$ as $k \to +\infty$;

(2) $\left\|u_{n_k}\right\|_{b,p} \to +\infty$ and $\left\|u_{n_k}\right\|_{b,q}$ is bounded;

(3) $\left\|u_{n_k}\right\|_{b,p}$ is bounded and $\left\|u_{n_k}\right\|_{b,q} \to +\infty$.

We shall investigate each of these cases separately. In case (1), for m large enough, $\|u_{n_k}\|_{b,q}^{q^-} \geq \|u_{n_k}\|_{b,q}^{p^-}$. Hence, using relation (3.1), we get

$$\begin{aligned} c + o_{n_k}(1) &\geq C_1\|u_{n_k}\|_{b,p}^{p^-} + C_2\mathcal{H}(\kappa_\star^3)\|u_{n_k}\|_{b,q}^{q^-} \\ &\geq C_1\|u_{n_k}\|_{b,p}^{p^-} + C_2\mathcal{H}(\kappa_\star^3)\|u_{n_k}\|_{b,q}^{p^-}, \geq C_3\|u_{nk}\|_{b,p}^{p^-}, \end{aligned}$$

which leads to an absurd result.

In case (2), by relation (3.1), we obtain

$$c + o_{n_k}(1) \geq C_1\|u_{n_k}\|_{b,p}^{p^-}.$$

Taking the limit as $k \to +\infty$, we get a contradiction.

Case (3) can be handled in a manner similar to case (2). Hence, we conclude that $\{u_{n_k}\}$ is bounded in X_{G_1}.

On the one hand, we get

$$\begin{aligned} c + o(1)\|v_{n_k}\|_{b,h} &\geq E_{\lambda_{n_k}}(0, v_{n_k}) - \left\langle E'_{\lambda_{n_k}}(u_{n_k}, v_{n_k}), \left(0, \frac{v_{n_k}}{\vartheta(x)}\right)\right\rangle \\ &= \widehat{K}\left(\mathcal{B}(v_{n_k})\right) - \int_{\mathbb{R}^N} \frac{1}{r(x)}|v_{n_k}|^{r(x)}dx - \int_{\mathbb{R}^N} \lambda(x)\mathcal{F}(x, 0, v_{n_k})dx \\ &\quad - K\left(\mathcal{B}(v_{n_k})\right)\int_{\mathbb{R}^N}\left(\mathcal{A}_1(\nabla v_{n_k})\nabla\left(\frac{v_{n_k}}{\vartheta(x)}\right)dx + b(x)\mathcal{A}_2(v_{n_k})\frac{v_{n_k}}{\vartheta(x)}\right)dx \\ &\quad + \int_{\mathbb{R}^N}\frac{1}{\vartheta(x)}|v_{n_k}|^{r(x)}dx + \int_{\mathbb{R}^N}\lambda(x)\frac{\partial\mathcal{F}}{\partial v}(x, 0, v_{n_k})\frac{v_{n_k}}{\vartheta(x)}dx, \end{aligned}$$

that is,

$$\begin{aligned} c + o(1)\|v_{n_k}\|_{b,h} &= \widehat{K}\left(\mathcal{B}(v_{n_k})\right) - K\left(\mathcal{B}(v_{n_k})\right)\int_{\mathbb{R}^N}\left(\mathcal{A}_1(\nabla v_{n_k})\nabla(\frac{v_{n_k}}{\vartheta(x)}) + b(x)\mathcal{A}_2(v_{n_k})\frac{v_{n_k}}{\vartheta(x)}\right)dx \\ &\quad + \int_{\mathbb{R}^N}\left(\frac{1}{\vartheta(x)} - \frac{1}{r(x)}\right)|v_{n_k}|^{r(x)}dx + \int_{\mathbb{R}^N}\lambda(x)\left(\frac{\partial\mathcal{F}}{\partial v}(x, 0, v_{nk})\frac{v_{n_k}}{\vartheta(x)} - \mathcal{F}(x, 0, v_{n_k})\right)dx. \end{aligned}$$

Next, by using $(H_{a_4}), (K_1) - (K_2)$, and (F_2), we obtain

$$\begin{aligned} c + o(1)\|v_{n_k}\|_{b,h} \geq \mathfrak{K}_0\int_{\mathbb{R}^N}&\left(\left(\frac{\gamma}{\alpha_1 p(x)} - \frac{1}{\vartheta(x)}\right)a_1(|\nabla v_{n_k}|^{p(x)})|\nabla v_{n_k}|^{p(x)}\right. \\ &\left.+\left(\frac{\gamma}{\alpha_2 p(x)} - \frac{1}{\vartheta(x)}\right)b(x)a_2(|v_{n_k}|^{p(x)})|v_{n_k}|^{p(x)}\right)dx \\ &+ \mathfrak{K}_0\int_{\mathbb{R}^N}\frac{v_{n_k}}{\vartheta(x)^2}a_1(|\nabla v_{n_k}|^{p(x)})|\nabla v_{n_k}|^{p(x)-2}\nabla v_{n_k}\nabla\vartheta dx + \int_{\mathbb{R}^N}\left(\frac{1}{\vartheta(x)} - \frac{1}{r(x)}\right)|v_{n_k}|^{r(x)}dx, \end{aligned}$$

$$\geq \mathfrak{K}_0 \int_{\mathbb{R}^N} \left(\frac{\gamma}{\max\{\alpha_1, \alpha_2\} p(x)} - \frac{1}{\vartheta(x)} \right) \left[a_1(|\nabla v_{n_k}|^{p(x)}) |\nabla v_{n_k}|^{p(x)} + b(x) a_2(|v_{n_k}|^{p(x)}) |v_{n_k}|^{p(x)} \right] dx$$

$$+ \mathfrak{K}_0 \int_{\mathbb{R}^N} \frac{v_{n_k}}{\vartheta(x)^2} a_1(|\nabla v_{n_k}|^{p(x)}) |\nabla v_{n_k}|^{p(x)-2} \nabla v_{n_k} \nabla \vartheta dx + \int_{\mathbb{R}^N} \left(\frac{1}{\vartheta(x)} - \frac{1}{r(x)} \right) |v_{n_k}|^{r(x)} dx.$$

Denote

$$\eth_1 := \inf_{x \in \mathbb{R}^N} \left(\frac{\gamma}{\max\{\alpha_1, \alpha_2\} p(x)} - \frac{1}{\vartheta(x)} \right) > 0, \text{ and } \eth_2 := \inf_{x \in \mathbb{R}^N} \left(\frac{1}{\vartheta(x)} - \frac{1}{r(x)} \right) > 0.$$

Then, using (F_2), we get

$$E_{\lambda_{n_k}}(0, v_{k_n}) - \left\langle E'_{\lambda_{n_k}}(u_{n_k}, v_{n_k}), \left(0, \frac{v_{n_k}}{\vartheta(x)}\right) \right\rangle \geq \mathfrak{K}_0 \Big(\eth_1 \min\left\{\kappa_1^0, \kappa_2^0\right\} \int_{\mathbb{R}^N} \left(|\nabla v_{n_k}|^{p(x)} + b(x) |v_{n_k}|^{p(x)} \right) dx$$

$$+ \eth_1 \min\left\{\kappa_1^2, \kappa_2^2\right\} \mathcal{H}(\kappa_\star^3) \int_{\mathbb{R}^N} \left(|\nabla v_{n_k}|^{q(x)} + b(x) |v_{n_k}|^{q(x)} \right) dx \Big)$$

$$+ \mathfrak{K}_0 \int_{\mathbb{R}^N} \frac{v_{n_k}}{\vartheta(x)^2} a_1(|\nabla v_{n_k}|^{p(x)}) |\nabla v_{n_k}|^{p(x)-2} \nabla v_{n_k} \nabla \vartheta dx + \int_{\mathbb{R}^N} \eth_2 |v_{n_k}|^{r(x)} dx.$$

On the other hand, we obtain

$$\left| \frac{v_{n_k}}{\vartheta(x)^2} a_1(|\nabla v_{n_k}|^{p(x)}) |\nabla v_{n_k}|^{p(x)-2} \nabla v_{n_k} \nabla \vartheta \right| \leq \left| \kappa_1^1 \frac{v_{n_k}}{\vartheta(x)^2} |\nabla v_{n_k}|^{p(x)-2} \nabla v_{n_k} \nabla \vartheta \right| + \left| \kappa_\star^3 \frac{v_{n_k}}{\vartheta(x)^2} |\nabla v_{n_k}|^{q(x)-2} \nabla v_{n_k} \nabla \vartheta \right|.$$

By use the Young inequality, for any $\varepsilon \in (0, 1)$, there exist $c_1(\varepsilon)$ and $c_2(\varepsilon) > 0$ such that

$$\left| \frac{v_{n_k}}{\vartheta(x)^2} |\nabla v_{n_k}|^{p(x)-2} \nabla v_{n_k} \nabla \vartheta \right| \leq \varepsilon |\nabla v_{n_k}|^{p(x)} + c_1(\varepsilon) |v_{n_k}|^{p(x)}, \tag{3.2}$$

$$\left| \frac{v_{n_k}}{\vartheta(x)^2} |\nabla v_{n_k}|^{q(x)-2} \nabla v_{n_k} \nabla \vartheta \right| \leq \epsilon |\nabla v_{n_k}|^{q(x)} + c_2(\epsilon) |v_{n_k}|^{q(x)}. \tag{3.3}$$

Hence, by relations (3.2) and (3.3), we get

$$c + o(1) \|v_{n_k}\|_{b,h} \geq \mathfrak{K}_0 \Big(\int_{\mathbb{R}^N} \left((\eth_\star - \epsilon) |\nabla v_{n_k}|^{p(x)} + (\eth_\star b(x) - c_1(\epsilon)) |v_{n_k}|^{p(x)} \right) dx$$

$$\mathcal{H}(\kappa_\star^3) \int_{\mathbb{R}^N} \left((\eth_\star - \epsilon) |\nabla v_{n_k}|^{q(x)} + (\eth_\star b(x) - c_2(\epsilon)) |v_{n_k}|^{q(x)} \right) dx \Big),$$

where $\eth_\star = \min\left\{ \eth_1 \min\left\{\kappa_1^0, \kappa_2^0\right\}, \eth_1 \min\left\{\kappa_1^2, \kappa_2^2\right\} \right\}$.

Let $\epsilon < \eth_\star/2$ and $w_0 = 2\max(c_1(\epsilon), c_2(\epsilon))/\eth_\star$, we get

$$c + o(1) \|v_{n_k}\|_{b,h} \geq \mathfrak{K}_0 \frac{\eth_\star}{2} \Big(\int_{\mathbb{R}^N} \left(|\nabla v_{n_k}|^{p(x)} + b(x) |v_{n_k}|^{p(x)} \right) dx$$

$$\mathcal{H}(\kappa_\star^3) \int_{\mathbb{R}^N} \left(|\nabla v_{n_k}|^{q(x)} + b(x) |v_{n_k}|^{q(x)} \right) dx \Big)$$

$$\geq C_1 \|v_{n_k}\|_{b,p}^{p^-} + C_2 \mathcal{H}(\kappa_\star^3) \|v_{n_k}\|_{b,q}^{q^-}.$$

This implies that $\{v_{n_k}\}$ is bounded in X_{G_1}, This implies that $\|u_{nk}\|_{b,h} + \|v_{n_k}\|_{b,h}$ is bounded in Z.

In the sequel, we shall prove that $\{(u_{n_k}, v_{n_k})\}$ contains a subsequence converging strongly in Z. We note that the sequence $\{(u_{n_k}\}$ is bounded in X_{G_1}. Therefore, up to a subsequence, $u_{n_k} \rightharpoonup u$ in X_{G_1} and $u_{n_k} \to u$ a.e. in $\mathbb{R}^N$.

$$
\begin{aligned}
o(1)\|u_{n_k} - u\|_{b,h} \geq & \left\langle -E'_{\lambda_{n_k}}(u_{n_k} - u, v_{n_k}), (u_{n_k} - u, 0)\right\rangle \\
&= K\big(\mathcal{B}(u_{n_k} - u)\big) \int_{\mathbb{R}^N} \big(\mathcal{A}_1(\nabla(u_{n_k} - u)).\nabla(u_{n_k} - u) + b(x)\mathcal{A}_2(u_{n_k} - u)(u_{n_k} - u)\big)dx \\
&\quad + \int_{\mathbb{R}^N} |u_{n_k} - u|^{r(x)}dx + \int_{\mathbb{R}^N} \lambda(x)\frac{\partial \mathcal{F}}{\partial u}(x, u_{n_k} - u, v_{n_k})dx \\
&\geq \mathfrak{K}_0 \int_{\mathbb{R}^N} \big(a_1(|\nabla u_{n_k} - u|^{p(x)})|\nabla u_{n_k} - u|^{p(x)} + b(x)a_2(|u_{n_k} - u|^{p(x)})|u_{n_k} - u|^{p(x)}\big)dx \\
&\quad + \int_{\mathbb{R}^N} |u_{n_k} - u|^{r(x)}dx + \inf_{x\in\mathbb{R}^N} \lambda(x) \int_{\mathbb{R}^N} \frac{\partial \mathcal{F}}{\partial u}(x, u_{n_k} - u, v_{n_k})dx \\
&\geq \mathfrak{K}_0 \int_{\mathbb{R}^N} \big(a_1(|\nabla(u_{n_k} - u)|^{p(x)})|\nabla(u_{n_k} - u)|^{p(x)} + b(x)a_2(|u_{n_k} - u|^{p(x)})|u_{n_k} - u|^{p(x)}\big)dx \\
&\geq C_1\|u_{n_k} - u\|_{b,p}^{p^-} + C_2\mathcal{H}(\kappa_\star^3)\|u_{n_k} - u\|_{b,q}^{q^-}.
\end{aligned}
$$

This implies that u_{n_k} converges strongly to u in X_{G_1}.

Next, we shall prove that there exists $v \in X_{G_1}$ such that $v_{n_k} \to v$ strongly in X_{G_1}. As X_{G_1} is reflexive, passing to a subsequence, still denoted by v_{n_k}, we may assume that there exists $v \in X_{G_1}$ such that $v_{n_k} \rightharpoonup v$ in X_{G_1} and $v_{n_k}(x) \to v(x)$ a.e. in $\mathbb{R}^N$. We can also obtain that $v_{n_k} \rightharpoonup v$ in X_{G_1}, as $k \to \infty$. So there exist two positive and bounded measures μ and ν on $\mathbb{R}^N$ and some at least countable family of points $(x_i)_{i\in I} \subset C_h = \{x \in \mathbb{R}^N : r(x) = h^*(x)\}$ and of positive numbers $(\nu_i)_{i\in I}$ and $(\mu_i)_{i\in I}$ such that

$$
|\nabla u_{n_k}|^{h(x)} + b(x)|u_{n_k}|^{h(x)} \overset{*}{\rightharpoonup} \mu \text{ in } \mathcal{M}_B(\mathbb{R}^N),
$$

$$
|u_{n_k}|^{r(x)} \overset{*}{\rightharpoonup} \nu \text{ in } \mathcal{M}_B(\mathbb{R}^N).
$$

According to Theorem 2.7, we have

$$
\mu = |\nabla u|^{h(x)} + b(x)|u|^{h(x)} + \sum_{i\in I} \mu_i\delta_{x_i} + \tilde{\mu} \quad \mu(C_h) \leq 1,
$$

$$
\nu = |u|^{r(x)} + \sum_{i\in I} \nu_i\delta_{x_i} \quad \nu(C_h) \leq C^*,
$$

where δ_{x_i} is the Dirac mass at x_i, I is a countable index set, and $\tilde{\mu}$ is a nonatomic measure

$$
\nu(C_h) \leq 2^{\frac{h^+r^+}{h^-}} C^* \max\left\{\mu\Big(C_h\Big)^{\frac{r^+}{h^-}}, \mu(C_h\Big)^{\frac{r^-}{h^+}}\right\}, \tag{3.4}
$$

$$
\nu_i \leq C^* \max\{\mu_i^{\frac{r^+}{h^-}}, \mu_i^{\frac{r^-}{h^+}}\}, \text{ for all } i \in I. \tag{3.5}
$$

Concentration at infinity of the sequence $\{v_{n_k}\}$ is described by the following quantities:

$$
\mu_\infty := \lim_{R\to\infty} \limsup_{n_k\to\infty} \int_{\{x\in\mathbb{R}^N : |x|>R\}} (|\nabla v_{n_k}|^{h(x)} + b(x)|v_{n_k}|^{h(x)})dx,
$$

$$
\nu_\infty := \lim_{R\to\infty} \limsup_{n_k\to\infty} \int_{\{x\in\mathbb{R}^N : |x|>R\}} |v_{n_k}|^{r(x)}dx.
$$

We claim that I is finite and for $i \in I$, either $\upsilon_i = 0$ or

$$\nu_i \geq \max\left\{\left(\frac{(1-\mathcal{H}(\kappa_\star^3))\min\{\kappa_1^0,\kappa_2^0\}+\mathcal{H}(\kappa_\star^3)\min\{\kappa_1^2,\kappa_2^2\}}{S^{\frac{h^-}{r^+}}}\right)^{\frac{r^+}{r^+-h^-}}, \left(\frac{(1-\mathcal{H}(\kappa_\star^3))\min\{\kappa_1^0,\kappa_2^0\}+\mathcal{H}(\kappa_\star^3)\min\{\kappa_1^2,\kappa_2^2\}}{S^{\frac{h^-}{r^-}}}\right)^{\frac{r^-}{r^--h^-}}\right\}.$$

Let $x_i \in \mathcal{C}_h$ be a singular point of the measures μ and ν. We choose $\phi \in C_0^\infty(\mathbb{R}^N, [0,1])$ such that $|\nabla\phi|_\infty \leq 2$ and

$$\phi(x) = \begin{cases} 1, & \text{if } |x| < 1, \\ 0, & \text{if } |x| \geq 2. \end{cases}$$

We define, for any $\varepsilon > 0$ and $i \in I$, the function

$$\phi_{i,\varepsilon} := \phi\left(\frac{x - x_i}{\varepsilon}\right), \qquad \text{for all } x \in \mathbb{R}^N.$$

Note that $\phi_{i,\varepsilon} \in C_0^\infty(\mathbb{R}^N, [0,1])$, $|\nabla\phi_{i,\varepsilon}|_\infty \leq \frac{2}{\varepsilon}$, and

$$\phi_{i,\varepsilon}(x) = \begin{cases} 1, & x \in B_\varepsilon(x_i), \\ 0, & x \in \mathbb{R}^N \setminus B_{2\varepsilon}(x_i). \end{cases}$$

It is clear that $\{\upsilon_{n_k}\phi_{i,\varepsilon}\}$ is bounded in X_{G_1}. From this, we can conclude that $\langle E'_{\lambda_{n_k}}(u_{n_k}, \upsilon_{n_k}), (0, \upsilon_{n_k}\phi_{i,\varepsilon})\rangle \to 0$ as $n_k \to +\infty$, that is, we obtain

$$\begin{aligned}\left\langle E'_{\lambda_{n_k}}(u_{n_k}, \upsilon_{n_k}), (0, \upsilon_{n_k}\phi_{i,\varepsilon})\right\rangle &= K\big(\mathcal{B}(\upsilon_{n_k})\big)\int_{\mathbb{R}^N}\big(a_1(|\nabla\upsilon_{n_k}|^{p(x)})|\nabla\upsilon_{n_k}|^{p(x)-2}\nabla\upsilon_{n_k}\nabla(\upsilon_{n_k}\phi_{i,\varepsilon}) \\ &\quad + b(x)a_2(|\upsilon_{n_k}|^{p(x)})|\upsilon_{n_k}|^{p(x)-2}\upsilon_{n_k}(\upsilon_{n_k}\phi_{i,\varepsilon})\big)\,dx - \int_{\mathbb{R}^N}|\upsilon_{n_k}|^{r(x)-2}\upsilon_{n_k}(\upsilon_{n_k}\phi_{i,\varepsilon})\,dx \\ &\quad - \int_{\mathbb{R}^N}\lambda(x)\frac{\partial\mathcal{F}}{\partial\upsilon}(x, u_{n_k}, \upsilon_{n_k})\upsilon_{n_k}\phi_{i,\varepsilon}\,dx \to 0 \text{ as } n_k \to +\infty.\end{aligned}$$

That is,

$$\begin{aligned}K\big(\mathcal{B}(\upsilon_{n_k})\big)\int_{\mathbb{R}^N} a_1(|\nabla\upsilon_{n_k}|^{p(x)})|\nabla\upsilon_{n_k}|^{p(x)-2}\nabla\upsilon_{n_k}\nabla\phi_{i,\varepsilon}\upsilon_{n_k}\,dx &= -K\big(\mathcal{B}(\upsilon_{n_k})\big)\int_{\mathbb{R}^N}\Big(a_1(|\nabla\upsilon_{n_k}|^{p(x)})|\nabla\upsilon_{n_k}|^{p(x)} \\ + b(x)a_2(|\upsilon_{n_k}|^{p(x)})|\upsilon_{n_k}|^{p(x)}\Big)\phi_{i,\varepsilon}\,dx + \int_{\mathbb{R}^N}|\upsilon_{n_k}|^{r(x)}\phi_{i,\varepsilon}\,dx &+ \int_{\mathbb{R}^N}\lambda(x)\frac{\partial\mathcal{F}}{\partial\upsilon}(x, u_{n_k}, \upsilon_{n_k})\upsilon_{n_k}\phi_{i,\varepsilon}\,dx + o_{n_k}(1).\end{aligned} \tag{3.6}$$

Now, we shall prove that

$$\lim_{\varepsilon\to 0}\left\{\limsup_{n_k\to+\infty} K\big(\mathcal{B}(\upsilon_{n_k})\big)\int_{\mathbb{R}^N} a_1(|\nabla\upsilon_{n_k}|^{p(x)})|\nabla\upsilon_{n_k}|^{p(x)-2}\nabla\upsilon_{n_k}\nabla\phi_{i,\varepsilon}\upsilon_{n_k}\,dx\right\} = 0. \tag{3.7}$$

Note that, due to the hypotheses (H_{a_2}) enough to show that

$$\lim_{\varepsilon\to 0}\left\{\limsup_{n_k\to+\infty} K\big(\mathcal{B}(\upsilon_{n_k})\big)\int_{\mathbb{R}^N}|\nabla\upsilon_{n_k}|^{p(x)-2}\nabla\upsilon_{n_k}\nabla\phi_{i,\varepsilon}\upsilon_{n_k}\,dx\right\} = 0 \tag{3.8}$$

and

$$\lim_{\varepsilon\to 0}\left\{\limsup_{n_k\to+\infty} K\big(\mathcal{B}(\upsilon_{n_k})\big)\int_{\mathbb{R}^N}|\nabla\upsilon_{n_k}|^{q(x)-2}\nabla\upsilon_{n_k}\nabla\phi_{i,\varepsilon}\upsilon_{n_k}\,dx\right\} = 0. \tag{3.9}$$

First, by using the Hölder's inequality, we have

$$\left|\int_{\mathbb{R}^N} |\nabla v_{n_k}|^{p(x)-2}\nabla v_{n_k}\nabla\phi_{i,\varepsilon}v_{n_k}\,dx\right| \leq 2\left|\left|\nabla v_{n_k}\right|^{p(x)-1}\right|_{\frac{p(x)}{p(x)-1}}\left|\nabla\phi_{i,\varepsilon}v_{n_k}\right|_{p(x)},$$

given that $\{v_{n_k}\}$ is bounded, the sequence of real values $\left|\left|\nabla v_{n_k}\right|^{p(x)-1}\right|_{\frac{p(x)}{p(x)-1}}$ is also bounded. Therefore, there exists a positive constant C such that

$$\left|\int_{\mathbb{R}^N} |\nabla v_{n_k}|^{p(x)-2}\nabla v_{n_k}\nabla\phi_{i,\varepsilon}v_{n_k}\,dx\right| \leq C|\nabla\phi_{i,\varepsilon}v_{n_k}|_{p(x)}.$$

Moreover, $\{v_{n_k}\}$ is bounded in $W_b^{1,p(x)}(B_{2\varepsilon}(x_i))$, then there exists a subsequence denoted again $\{v_{n_k}\}$ weakly convergent to v in $L^{p(x)}(B_{2\varepsilon}(x_i))$. Hence,

$$\begin{aligned}\limsup_{n_k\to+\infty}\left|\int_{\mathbb{R}^N} |\nabla v_{n_k}|^{p(x)-2}\nabla v_{n_k}\nabla\phi_{i,\varepsilon}v_{nk}\,dx\right| &\leq C|\nabla\phi_{i,\varepsilon}v_{n_k}|_{p(x)}\\ &\leq 2C\limsup_{\varepsilon\to 0}||\nabla\phi_{i,\varepsilon}|^{p(x)}|_{\left(\frac{p^\star(x)}{p(x)}\right)',B_{2\varepsilon}(x_i)}||v|^{p(x)}|_{\frac{p^\star(x)}{p(x)},B_{2\varepsilon}(x_i)}\\ &\leq 2C\limsup_{\varepsilon\to 0}||\nabla\phi_{i,\varepsilon}|^{p(x)}|_{\frac{N}{p(x)},B_{2\varepsilon}(x_i)}||v|^{p(x)}|_{\frac{N}{N-p(x)},B_{2\varepsilon}(x_i)}.\end{aligned}$$

Note that

$$\int_{B_{2\varepsilon}(x_i)}(|\nabla\phi_{i,\varepsilon}|^{p(x)})^{\left(\frac{p^\star(x)}{p(x)}\right)'}dx = \int_{B_{2\varepsilon}(x_i)}|\nabla\phi_{i,\varepsilon}|^N dx \leq \left(\frac{2}{\varepsilon}\right)^N meas(B_{2\varepsilon}(x_i)) = \frac{4^N}{N}\omega_N,$$

where ω_N is the surface area of an N-dimensional unit sphere. Since $\int_{B_{2\varepsilon}(x_i)}(|v_{nk}|^{p(x)})^{\frac{p^\star(x)}{p(x)}}dx \to 0$ as $\varepsilon\to 0$, we can conclude that $|\nabla\phi_{i,\varepsilon}v_{n_k}|_{p(x)}\to 0$, which implies

$$\lim_{\varepsilon\to 0}\left\{\limsup_{n_k\to+\infty}\left|\int_{\mathbb{R}^N}|\nabla v_{nk}|^{p(x)-2}\nabla v_{nk}\nabla\phi_{i,\varepsilon}v_{nk}\,dx\right|\right\} = 0. \tag{3.10}$$

Since $\{v_{nk}\}$ is bounded in $W_b^{1,p(x)}(\mathbb{R}^N)$, we may assume that $\mathcal{B}(v_{nk})\to t\geq 0$ as $n_k\to+\infty$. We note that $K(t)$ is continuous, we then have

$$K(\mathcal{B}(v_{nk}))\to K(t)\geq \mathfrak{K}_0 > 0, \quad \text{as } n_k\to+\infty.$$

Hence, by relation (3.10), we obtain

$$\lim_{\varepsilon\to 0}\left\{\limsup_{n_k\to+\infty}K(\mathcal{B}(v_{nk}))\int_{\mathbb{R}^N}|\nabla v_{nk}|^{p(x)-2}\nabla v_{nk}\nabla\phi_{i,\varepsilon}v_{nk}\,dx\right\} = 0. \tag{3.11}$$

Analogously, we verify relation (3.9). Therefore, we conclude the proof of relation (3.7).

Similarly, we can also get

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^N}\lambda(x)\frac{\partial\mathcal{F}}{\partial v}(x,u_{n_k},v_{n_k})\phi_{i,\varepsilon}v_{n_k}\,dx = 0, \text{ as } k\to+\infty. \tag{3.12}$$

Indeed, by use Hölder's inequality with assumption ($\mathbf{F}_2$) and since $0 \leq \phi_{i,\varepsilon} \leq 1$, we obtain

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{\mathbb{R}^N}\lambda(x)\frac{\partial\mathcal{F}}{\partial v}(x,u_{n_k},v_{n_k})\phi_{i,\varepsilon}v_{n_k}\,dx &\leq \lim_{\varepsilon\to 0}\sup_{x\in\mathbb{R}^N}\lambda(x)\int_{\mathbb{R}^N}\left(f_1(x)|u_{n_k}|^{\ell(x)}+f_1(x)|v_{n_k}|^{\ell(x)}\right)\phi_{i,\varepsilon}v_{n_k}\,dx,\\ &\leq \lim_{\varepsilon\to 0}\lambda^+\int_{\mathbb{R}^N}\left(f_1(x)|u_{n_k}|^{\ell(x)}+f_1(x)|v_{n_k}|^{\ell(x)}\right)|\phi_{i,\varepsilon}v_{n_k}|\,dx\\ &\leq \lim_{\varepsilon\to 0}c_1\left(|f_1|_{l(x)}||u_{n_k}|^{\ell}|_{h^\star(x)}+|f_2|_{m(x)}||v_{n_k}|^{\ell}|_{h^\star(x)}\right)|\phi_{i,\varepsilon}v_{n_k}|_{h^\star(x)}.\end{aligned}$$

The above propositions yield

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^N}\lambda(x)\frac{\partial\mathcal{F}}{\partial v}(x,u_{n_k},v_{n_k})\phi_{i,\varepsilon}v_{n_k}\,dx \leq \lim_{\varepsilon\to 0}c_1\left(|f_1|_{l(x)}\|u_{n_k}\|^{\ell}_{h(x)}+|f_2|_{l(x)}\|v_{n_k}\|^{\ell}_{h(x)}\right)\|v_{k_n}\|_{h(x),B_{2\varepsilon}(x_i)},$$

and this last goes to zero because

$$|f_1|_{\ell(x)}\|u_{n_k}\|^{\ell}_{h(x)}+|f_2|_{\ell(x)}\|v_{n_k}\|^{\ell}_{h(x)}<\infty.$$

Since $\phi_{i,\varepsilon}$ has compact support, going to the limit $n_k \to +\infty$ and letting $\varepsilon \to 0$ in relation (3.6), from relations (3.7) and (3.8), we get

$$\begin{aligned}0 &= \nu_i - \lim_{\varepsilon\to 0}\left(\limsup_{n_k\to+\infty}K(\mathcal{B}(v_{nk}))\int_{\mathbb{R}^N}\left(a_1(|\nabla v_{n_k}|^{p(x)})|\nabla v_{n_k}|^{p(x)}\phi_{i,\varepsilon}+b(x)a_2(|v_{n_k}|^{p(x)})|v_{n_k}|^{p(x)}\phi_{i,\varepsilon}\right)dx\right)\\ &\leq \nu_i - \lim_{\varepsilon\to 0}\left(\limsup_{n_k\to+\infty}\mathfrak{K}_0\left(\int_{\mathbb{R}^N}\left[a_1(|\nabla v_{n_k}|^{p(x)})|\nabla v_{n_k}|^{p(x)}+b(x)a_2(|v_{n_k}|^{p(x)})|v_{n_k}|^{p(x)}\right]\phi_{i,\varepsilon}\,dx\right)\right),\end{aligned}$$

and by applying assumption (H_{a_2}), we obtain

$$\begin{aligned}0 \leq \nu_i - \lim_{\varepsilon\to 0}\Bigg(\limsup_{n_k\to+\infty}\mathfrak{K}_0\Bigg(&\min\left\{\kappa_1^0,\kappa_2^0\right\}\int_{\mathbb{R}^N}\left(|\nabla v_{n_k}|^{p(x)}+b(x)|v_{n_k}|^{p(x)}\right)dx\\ &+\min\left\{\kappa_1^2,\kappa_2^2\right\}\mathcal{H}(\kappa_\star^3)\left(|\nabla u_{n_k}|^{q(x)}+b(x)|u_{n_k}|^{q(x)}\right)dx,\Bigg)\Bigg).\end{aligned}\tag{3.13}$$

Note that, when $\kappa_\star^3 = 0$, we have $h(x) = p(x)$. Hence, from Theorem 2.7 and the aforementioned arguments, we obtain

$$\begin{aligned}0 &\leq \nu_i - \mathfrak{K}_0\min\left\{\kappa_1^0,\kappa_2^0\right\}\lim_{\varepsilon\to 0}\int_{\mathbb{R}^N}\phi_{i,\varepsilon}\,d\mu\\ &\leq \nu_i - \mathfrak{K}_0\min\left\{\kappa_1^0,\kappa_2^0\right\}\left(\mu_i - \lim_{\varepsilon\to 0}\int_{\mathbb{R}^N}(|\nabla v|^{p(x)}+b(x)|v|^{p(x)})\phi_{i,\varepsilon}\,dx\right).\end{aligned}$$

By using Lebesgue dominated convergence theorem, we get

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^N}(|\nabla v|^{p(x)}+b(x)|v|^{p(x)})\phi_{i,\varepsilon}\,dx = 0.$$

Then, we get

$$\mathfrak{K}_0\min\left\{\kappa_1^0,\kappa_2^0\right\}\mu_i \leq \nu_i.\tag{3.14}$$

On the other hand, if $\kappa_\star^3 > 0$, we have $h(x) = q(x)$ Therefore, it follows from Theorem 2.7 and relation (3.13) that

$$\begin{aligned}0 &\leq \nu_i - \mathfrak{K}_0\min\left\{\kappa_1^0,\kappa_2^0\right\}\mathcal{H}(\kappa_\star^3)\lim_{\varepsilon\to 0}\left[\limsup_{n_k\to\infty}\left(\int_{\mathbb{R}^N}\left(|\nabla v_{n_k}|^{q(x)}+b(x)|v_{n_k}|^{q(x)}\right)\phi_{i,\varepsilon}\,dx\right)\right]\\ &\leq \nu_i - \mathfrak{K}_0\min\left\{\kappa_1^2,\kappa_2^2\right\}\mathcal{H}(\kappa_\star^3)\lim_{\varepsilon\to 0}\int_{\mathbb{R}^N}\phi_{i,\varepsilon}\,d\mu\\ &\leq \nu_i - \mathfrak{K}_0\min\left\{\kappa_1^2,\kappa_2^2\right\}\mathcal{H}(\kappa_\star^3)\left(\mu_i - \lim_{\varepsilon\to 0}\int_{\mathbb{R}^N}(|\nabla v|^{q(x)}+b(x)|v|^{q(x)})\phi_{i,\varepsilon}\,dx\right),\end{aligned}$$

and by applying the Lebesgue dominated convergence theorem, we obtain

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^N}(|\nabla v|^{q(x)}+b(x)|v|^{q(x)})\phi_{i,\varepsilon}\,dx=0.$$

Then, we get

$$\mathfrak{K}_0\min\left\{\kappa_1^2,\kappa_2^2\right\}\mathcal{H}(\kappa_\star^3)\mu_i\le\nu_i. \tag{3.15}$$

Now, by combining relations (3.14) and (3.15), we have

$$\mathfrak{K}_0\big((1-\mathcal{H}(\kappa_i^3))\min\left\{\kappa_1^0,\kappa_2^0\right\}+\mathcal{H}(\kappa_\star^3)\min\left\{\kappa_1^2,\kappa_2^2\right\}\big)\mu_i\le\nu_i. \tag{3.16}$$

Using relation (3.5), we obtain

$$\nu_i\le C^*\max\left\{\left(\frac{\nu_i}{\mathfrak{K}_0 D}\right)^{\frac{r^+}{h^-}},\left(\frac{\nu_i}{\mathfrak{K}_0 D}\right)^{\frac{r^-}{h^+}}\right\},$$

where $D=(1-\mathcal{H}(\kappa_\star^3))\min\left\{\kappa_1^0,\kappa_2^0\right\}+\mathcal{H}(\kappa_\star^3)\min\left\{\kappa_1^2,\kappa_2^2\right\}$. which implies that $\nu_i=0$ or

$$\nu_i\ge\max\left\{\left(\frac{\mathfrak{K}_0 D}{S^{\frac{h^-}{r^+}}}\right)^{\frac{r^+}{r^+-h^-}},\left(\frac{\mathfrak{K}_0 D}{S^{\frac{h^-}{r^-}}}\right)^{\frac{r^-}{r^--h^-}}\right\} \tag{3.17}$$

for all $i\in I$, which implies that I is finite. The claim is therefore proved.

To analyze the concentration at ∞, we choose a suitable cut-off function $\psi\in C_0^\infty(\mathbb{R}^N,[0,1])$ such that $\psi(x)\equiv 0$ on $B_R(0)$ and $\psi(x)\equiv 1$ on $\mathbb{R}^N\backslash B_{2R}(0)$. We set $\psi_R(x)=\psi\left(\frac{x}{R}\right)$, we can easily observe that $\{v_{n_k}\psi_R\}$ is bounded in X_{G_1} and $\lim_{n_k\to\infty}\left\langle E'_{\lambda_{n_k}}(u_{n_k},v_{n_k}),(0,v_{n_k}\psi_R)\right\rangle=0$,

$$\begin{aligned}\left\langle E'_{\lambda_{n_k}}(u_{n_k},v_{n_k}),(0,v_{n_k}\psi_R)\right\rangle&=K\big(\mathcal{B}(v_{n_k})\big)\int_{\mathbb{R}^N}\big(a_1(|\nabla v_{n_k}|^{p(x)})|\nabla v_{n_k}|^{p(x)-2}\nabla v_{n_k}\nabla(v_{n_k}\psi_R)\\&\quad+b(x)a_2(|v_{n_k}|^{p(x)})|v_{n_k}|^{p(x)-2}v_{n_k}(v_{n_k}\psi_R)\big)\,dx-\int_{\mathbb{R}^N}|v_{n_k}|^{r(x)-2}v_{n_k}(v_{n_k}\psi_R)\,dx\\&\quad-\int_{\mathbb{R}^N}\lambda(x)\frac{\partial\mathcal{F}}{\partial v}(x,u_{n_k},v_{n_k})v_{n_k}\psi_R\,dx\to 0\text{ as }n_k\to+\infty.\end{aligned}$$

In other words,

$$\begin{aligned}K\big(\mathcal{B}_1(v_{n_k})\big)\int_{\mathbb{R}^N}a_1(|\nabla v_{n_k}|^{p(x)})|\nabla v_{n_k}|^{p(x)-2}\nabla v_{n_k}\nabla\psi_R v_{n_k}\,dx&=\int_{\mathbb{R}^N}\lambda(x)\frac{\partial\mathcal{F}}{\partial v}(x,u_{n_k},v_{n_k})v_{n_k}\psi_R\,dx\\+\int_{\mathbb{R}^N}|v_{n_k}|^{r(x)}\psi_R\,dx-K\big(\mathcal{B}(v_{n_k})\big)\int_{\mathbb{R}^N}\Big(a_1(|\nabla v_{n_k}|^{p(x)})|\nabla v_{n_k}|^{p(x)}&+b(x)a_2(|v_{n_k}|^{p(x)})|v_{n_k}|^{p(x)}\Big)\phi_{i,\varepsilon}\,dx+o_{n_k}(1).\end{aligned} \tag{3.18}$$

As in the previous proof, we can find that $\lim_{n_k\to\infty}\left|\nabla\psi_R v_{n_k}\right|_{p(x)}=0$ when $R\to\infty$, and

$$\left|\int_{\mathbb{R}^N}|\nabla v_{n_k}|^{p(x)-2}\nabla v_{nk}\psi_R v_{n_k}\,dx\right|\le 2\left|\left|\nabla v_{n_k}\right|^{p(x)-1}\right|_{\frac{p(x)}{p(x)-1}}\left|\nabla\psi_R v_{n_k}\right|_{p(x)},$$

since $\{v_{n_k}\}$ is bounded, the real-valued sequence $\left|\left|\nabla v_{n_k}\right|^{p(x)-1}\right|_{\frac{p(x)}{p(x)-1}}$ is also bounded, then

$$\lim_{R\to\infty}\limsup_{n_k\to+\infty} K\big(\mathcal{B}(v_{n_k})\big)\int_{\mathbb{R}^N} ||\nabla v_{n_k}|^{p(x)-2}\nabla v_{n_k}\psi_R v_{n_k}|\,dx = 0. \tag{3.19}$$

Similarly, we can also get

$$\lim_{R\to\infty}\limsup_{n_k\to+\infty} K\big(\mathcal{B}(v_{n_k})\big)\int_{\mathbb{R}^N} ||\nabla v_{n_k}|^{q(x)-2}\nabla v_{n_k}\psi_R v_{n_k}|\,dx = 0. \tag{3.20}$$

Therefore, we have

$$\lim_{R\to\infty}\limsup_{n_k\to+\infty}\int_{\mathbb{R}^N} a_1(|\nabla v_{n_k}|^{p(x)})|\nabla v_{n_k}|^{p(x)-2}\nabla v_{n_k}\nabla\psi_R v_{n_k}\,dx = 0.$$

Note that $v_{n_k}\to v$ weakly in X_{G_1}, so $\int_{\mathbb{R}^N}\lambda(x)\frac{\partial\mathcal{F}}{\partial v}(x,u,v)(v_{n_k}-v)\psi_R dx\to 0$. As

$$\left|\int_{\mathbb{R}^N}\lambda(x)\left(\frac{\partial\mathcal{F}}{\partial v}(x,u_{n_k},v_{n_k}\right)-\frac{\partial\mathcal{F}}{\partial v}(x,u,v))v_{n_k}\psi_R dx\right| \le c\left|\left(\frac{\partial\mathcal{F}}{\partial v}(x,u_{n_k},v_{n_k})-\frac{\partial\mathcal{F}}{\partial v}(x,u,v)\right)\psi_R\right|_{(h^\star(x))'}|v_{n_k}|_{h^\star(x)},$$
$$\le c\left|\frac{\partial\mathcal{F}}{\partial v}(x,u_{n_k},v_{n_k})-\frac{\partial\mathcal{F}}{\partial v}(x,u,v)\right|_{L^{(h^\star(x))'}(\mathbb{R}^N\setminus B_R(0))}.$$

According to assumption (F_1), analogous to Fu and Zhang to [24, Theorem 4.3], for any $\epsilon>0$, there exists $R_1>0$ such that when $R>R_1$, $|\frac{\partial\mathcal{F}}{\partial v}(x,u_{n_k},v_{n_k})-\frac{\partial\mathcal{F}}{\partial v}(x,u,v)|_{L^{(h^\star(x))'}(\mathbb{R}^N\setminus B_R(0))}<\epsilon$, for any $n\in\mathbb{N}$.

Note that $\int_{\mathbb{R}^N}\frac{\partial\mathcal{F}}{\partial v}(x,u,v)v\psi_R dx\to 0$ as $R\to\infty$. Thus, we obtain that

$$\begin{aligned}
&\lim_{R\to\infty}\limsup_{k\to+\infty}\int_{\mathbb{R}^N}\lambda(x)\frac{\partial\mathcal{F}}{\partial v}(x,u_{nk},v_{nk})v_{nk}\psi_R dx\\
&\le \sup_{x\in\mathbb{R}^N}\lambda(x)\lim_{R\to\infty}\limsup_{n_k\to+\infty}\int_{\mathbb{R}^N}\frac{\partial\mathcal{F}}{\partial v}(x,u_{nk},v_{nk})v_{nk}\psi_R dx\\
&=\lambda^+\lim_{R\to\infty}\limsup_{n_k\to+\infty}\int_{\mathbb{R}^N}\left(\frac{\partial\mathcal{F}}{\partial v}(x,u_{n_k},v_{n_k})-\frac{\partial\mathcal{F}}{\partial v}(x,u,v)\right)v_{n_k}\psi_R+\frac{\partial\mathcal{F}}{\partial v}(x,u,v))(v_{n_k}-v)+\frac{\partial\mathcal{F}}{\partial v}(x,u,v)v\psi_R dx,\\
&=\lambda^+\lim_{R\to\infty}\left(\limsup_{n_k\to+\infty}\int_{\mathbb{R}^N}\left(\frac{\partial\mathcal{F}}{\partial v}(x,u_{n_k},v_{n_k})-\frac{\partial\mathcal{F}}{\partial v}(x,u,v)\right)v_{n_k}\psi_R dx+\int_{\mathbb{R}^N}\frac{\partial\mathcal{F}}{\partial v}(x,u,v)v\psi_R dx\right),\\
&=\lambda^+\left(\lim_{R\to\infty}\limsup_{n_k\to+\infty}\int_{\mathbb{R}^N}\left(\frac{\partial\mathcal{F}}{\partial v}(x,u_{n_k},v_{n_k}\right)-\frac{\partial\mathcal{F}}{\partial v}(x,u,v))v_{n_k}\psi_R dx+\lim_{R\to\infty}\int_{\mathbb{R}^N}\frac{\partial\mathcal{F}}{\partial v}(x,u,v)v\psi_R dx\right),\\
&=0.
\end{aligned}$$

Since ψ_R has compact support, going to the limit $n_k\to+\infty$ and letting $R\to\infty$ in relation (3.18), we get

$$\mathfrak{K}_0\big((1-\mathcal{H}(\kappa_\star^3))\min\{\kappa_1^0,\kappa_2^0\}+\mathcal{H}(\kappa_\star^3)\min\{\kappa_1^2,\kappa_2^2\}\big)\mu_\infty\le\nu_\infty.$$

According to Theorem 2.8, we have either $\nu_\infty=0$ or

$$\nu_\infty\ge\max\left\{\left(\frac{\mathfrak{K}_0 D}{S^{\frac{h^-}{r^+}}}\right)^{\frac{r^+}{r^+-h^-}},\left(\frac{\mathfrak{K}_0 D}{S^{\frac{h^-}{r^-}}}\right)^{\frac{r^-}{r^--h^-}}\right\} \tag{3.21}$$

for all $i\in I$, where $D=(1-\mathcal{H}(\kappa_\star^3))\min\{\kappa_1^0,\kappa_2^0\}+\mathcal{H}(\kappa_\star^3)\min\{\kappa_1^2,\kappa_2^2\}$.

Next, we claim that relations (3.17) and (3.21) cannot occur. If the case (3.21) holds, for some $i \in I$, then by using $(H_{a_4}), (K_1)-(K_2)$, and (F_2), we get that

$$
\begin{aligned}
c &= \lim_{n_k\to\infty}\left(E_{\lambda_{n_k}}(0,v_{n_k}) - \left\langle E'_{\lambda_{n_k}}(u_{n_k},v_{n_k}), \left(0, \frac{v_{n_k}}{\vartheta(x)}\right)\right\rangle\right)\\
&= \widehat{K}\big(\mathcal{B}(v_{n_k})\big) - \int_{\mathbb{R}^N}\frac{1}{r(x)}|v_{n_k}|^{r(x)}dx - \int_{\mathbb{R}^N}\lambda(x)\mathcal{F}(x,0,v_{n_k})dx - K\big(\mathcal{B}_1(v_{n_k})\big)\int_{\mathbb{R}^N}\left(\mathcal{A}_1(\nabla v_{n_k})\nabla\left(\frac{v_{n_k}}{\vartheta(x)}\right)\right.\\
&\quad\left.+b(x)\mathcal{A}_2(v_{n_k})\frac{v_{n_k}}{\vartheta(x)}\right)dx + \int_{\mathbb{R}^N}\frac{1}{\vartheta(x)}|v_{n_k}|^{r(x)}dx + \int_{\mathbb{R}^N}\lambda(x)\frac{\partial\mathcal{F}}{\partial v}(x,0,v_{n_k})\frac{v_{n_k}}{\vartheta(x)}dx,\\
&\geq \mathfrak{K}_0\int_{\mathbb{R}^N}\left(\frac{\gamma}{\max\{\alpha_1,\alpha_2\}p(x)} - \frac{1}{\vartheta(x)}\right)\left[a_1(|\nabla v_{nk}|^{p(x)})|\nabla v_{nk}|^{p(x)} + b(x)a_2(|v_{nk}|^{p(x)})|v_{nk}|^{p(x)}\right]dx\\
&\quad+\mathfrak{K}_0\int_{\mathbb{R}^N}\frac{v_{nk}}{\vartheta(x)^2}a_1(|\nabla v_{nk}|^{p(x)})|\nabla v_{nk}|^{p(x)-2}\nabla v_{nk}\nabla\vartheta dx\\
&\quad+\int_{\mathbb{R}^N}\left(\frac{1}{\vartheta(x)} - \frac{1}{r(x)}\right)|v_{nk}|^{r(x)}dx + \lambda^-\int_{\mathbb{R}^N}\left(\frac{\partial\mathcal{F}}{\partial v}(x,0,v_{nk})\frac{v_{nk}}{\vartheta(x)} - \mathcal{F}(x,0,v_{nk})\right)dx,\\
&\geq \left(\frac{1}{\vartheta^-} - \frac{1}{r^-}\right)\nu_\infty.
\end{aligned}
$$

So, by relation (3.21), we have

$$
c \geq \left(\frac{1}{\vartheta^-} - \frac{1}{r^-}\right)\max\left\{\left(\frac{\mathfrak{K}_0 D}{S^{\frac{h^-}{r^+}}}\right)^{\frac{r^+}{r^+-h^-}}, \left(\frac{\mathfrak{K}_0 D}{S^{\frac{h^-}{r^-}}}\right)^{\frac{r^-}{r^--h^-}}\right\}.
$$

This is impossible. Therefore, $\nu_\infty = 0$ for all $i \in I$. Similarly, we can prove that (3.17) cannot occur for any i. Then,

$$
\limsup_{n_k\to+\infty}\int_{\mathbb{R}^N}|v_{nk}|^{r(x)}dx \to \int_{\mathbb{R}^N}|v|^{r(x)}dx.
$$

Note that if $|v_{n_k} - v|^{r(x)} \leq 2^{r^+}(|v_{n_k}|^{r(x)} + |v|^{r(x)})$, then by the Fatou Lemma, we have

$$
\begin{aligned}
\int_{\mathbb{R}^N}2^{r^+}|v|^{r(x)}dx &= \int_{\mathbb{R}^N}\liminf_{n_k\to+\infty}\left(2^{r^+}(|v_{n_k}|^{r(x)} + |v|^{r(x)}) - |v_{n_k} - v|^{r(x)}\right)dx,\\
&\leq \liminf_{n_k\to+\infty}\int_{\mathbb{R}^N}\left(2^{r^+}|v_{n_k}|^{r(x)} + 2^{r^+}|v|^{r(x)} - |v_{n_k} - v|^{r(x)}\right)dx,\\
&\leq \int_{\mathbb{R}^N}2^{r^++1}|v|^{r(x)}dx - \limsup_{n_k\to+\infty}\int_{\mathbb{R}^N}|v_{n_k} - v|^{r(x)}dx.
\end{aligned}
$$

Thus, $\int_{\mathbb{R}^N}|v_{n_k} - v|^{r(x)}dx \to 0$, we have $v_{n_k} \to v$ strongly in $L^{r(x)}(\mathbb{R}^N)$.

Now, let us define the operator Φ as follows:

$$
[\Phi(v), \tilde{v}] := \int_{\mathbb{R}^N}(\mathcal{A}_1(\nabla v)\nabla\tilde{v} + b(x)\mathcal{A}_2(v)\tilde{v})dx
$$

for any $(v,\tilde{v}) \in X_{G_1} \times X_{G_1}$. Using Hölder's inequality and the condition $(\mathbf{H}_{a_2})$, we can establish that

$$
|\langle\Phi(v), \tilde{v}\rangle| \leq c\|v\|_{b,h}^{q^--1}\|\tilde{v}\|_{b,h}.
$$

Thus, the linear functional $\Phi(v)$ is continuous on X_{G_1} for each $v \in X_{G_1}$. Therefore, due to the weak convergence of v_{n_k} in X_{G_1}, we obtain

$$\lim_{n_k\to\infty} \left\langle \Phi(v_{n_k}), v_0 \right\rangle = \langle \Phi(v_0), v_0\rangle \quad \text{and} \lim_{n_k\to\infty} \left\langle \Phi(v_0), v_{n_k} - v_0 \right\rangle = 0. \tag{3.22}$$

Clearly, $\left\langle \Phi(v_{n_k}), v_{n_k} - v_0 \right\rangle \to 0$ as $n_k \to \infty$. Hence, based on relation (3.22), we can deduce that

$$\lim_{n_k\to\infty} \left\langle \Phi(v_{n_k}) - \Phi(v_0), v_{n_k} - v_0 \right\rangle = \lim_{n_k\to\infty} \int_{\mathbb{R}^N} (\mathcal{R}_n(x) + b(x)\mathcal{Q}_n(x))dx = 0,$$

with

$$\mathcal{R}_n(x) = \left\langle a_1(|\nabla v_{n_k}|^{p(x)})|\nabla v_{n_k}|^{p(x)-2}\nabla v_{n_k} - a_1(|\nabla v_0|^{p(x)})|\nabla v_0|^{p(x)-2}\nabla v, \nabla v_{n_k} - \nabla v_0 \right\rangle$$

for all $x \in \mathbb{R}^N$ and all $n \in \mathbb{N}$, and

$$\mathcal{Q}_n(x) = \left\langle a_2(|v_{n_k}|^{p(x)})|v_{n_k}|^{p(x)-2}v_{n_k} - a_2(|v_0|^{p(x)})|v_0|^{p(x)-2}v, v_{n_k} - v_0 \right\rangle$$

for all $x \in \mathbb{R}^N$ and all $n \in \mathbb{N}$. Hence, by applying some elementary inequalities (see, e.g., Hurtado et al. [29, Auxiliary Results]), for any $\eta, \xi \in \mathbb{R}^N$,

$$\begin{cases} |\eta - \xi|^{p(x)} \le c_p \left\langle a_i(|\eta|^{p(x)})|\eta|^{p(x)-2}\eta - a_i(|\xi|^{p(x)})|\xi|^{p(x)-2}\xi, \eta - \xi \right\rangle & \text{if } p(x) \ge 2 \\ |\eta - \xi|^2 \le c(|\eta| + |\xi|)^{2-p(x)} \left\langle a_i(|\eta|^{p(x)})|\eta|^{p(x)-2}\eta - a_i(|\xi|^{p(x)})|\xi|^{p(x)-2}\xi, \eta - \xi \right\rangle & \text{if } 1 < p(x) < 2 \end{cases}. \tag{3.23}$$

By replacing η and ξ with ∇v_{n_k} and ∇v_0, respectively, and integrating over $\mathbb{R}^N$, we obtain

$$\int_{\mathbb{R}^N} \mathcal{R}_n(x)dx \ge C \int_{\{x\in\mathbb{R}^N;\ p(x)\ge 2\}} |\nabla v_{n_k} - \nabla v_0|^{p(x)}dx.$$

Thus,

$$\lim_{n_k\to\infty} \int_{\{x\in\mathbb{R}^N;\ p(x)\ge 2\}} |\nabla v_{n_k} - \nabla v_0|^{p(x)}dx = 0. \tag{3.24}$$

On the other hand, by using relation (3.23), we get

$$\int_{\mathbb{R}^N} \mathcal{R}_n(x)dx \ge C \int_{\{x\in\mathbb{R}^N;\ 1<p(x)<2\}} \sigma_1(x)^{p(x)-2}|\nabla v_{n_k} - \nabla v_0|^2 dx,$$

where $\sigma_1(x) = C(|v_{n_k}| + |\nabla v_0|)$. Therefore, by Hölder's inequality, we have

$$\begin{aligned}
\int_{\{x\in\mathbb{R}^N;\ 1<p(x)<2\}} |\nabla v_{n_k} - \nabla v_0|^{p(x)}dx &= \int_{\{x\in\mathbb{R}^N;\ 1<p(x)<2\}} \sigma_1^{\frac{p(x)(p(x)-2)}{2}} \left(\sigma_1^{\frac{p(x)(p(x)-2)}{2}} |\nabla v_{n_k} - \nabla v_0|^{p(x)} \right) dx \\
&\le C \|\sigma_1^{\frac{p(x)(2-p(x))}{2}}\|_{L^{\frac{2}{2-p(x)}}(\{x\in\mathbb{R}^N;\ 1<p(x)<2\})} \|\sigma_1^{\frac{p(x)(p(x)-2)}{2}} |\nabla v_{n_k} - \nabla v|^{p(x)}\|_{L^{\frac{2}{p(x)}}(\{x\in\mathbb{R}^N;\ 1<p(x)<2\})} \\
&\le C \max\left\{ \|\sigma_1\|_{L^{p(x)}(\{x\in\mathbb{R}^N;\ 1<p(x)<2\})}^{\left(\frac{p(x)(p(x)-2)}{2}\right)^-}, \|\sigma_1\|_{L^{p(x)}(\{x\in\mathbb{R}^N;\ 1<p(x)<2\})}^{\left(\frac{p(x)(p(x)-2)}{2}\right)^+} \right\} \times \\
&\max\left\{ \left(\int_{\{x\in\mathbb{R}^N;\ 1<p(x)<2\}} \sigma_1^{p(x)-2}|\nabla v_{n_k} - \nabla v_0|^2 dx \right)^{\frac{p^-}{2}}, \left(\int_{\{x\in\mathbb{R}^N;\ 1<p(x)<2\}} \sigma_1^{p(x)-2}|\nabla v_{n_k} - \nabla v_0|^2 dx \right)^{\frac{p^+}{2}} \right\}.
\end{aligned}$$

As the last term on the right-hand side of the above inequality tends to zero, we can conclude

$$\lim_{n_k\to\infty}\int_{\{x\in\mathbb{R}^N;\ 1<p(x)<2\}}|\nabla v_{n_k}-\nabla v_0|^{p(x)}dx=0. \tag{3.25}$$

Now, combining relation (3.24) with relation (3.25), we obtain

$$\lim_{n_k\to\infty}\int_{\mathbb{R}^N}|\nabla v_{n_k}-\nabla v_0|^{p(x)}dx=0.$$

The same arguments can be used to prove that

$$\lim_{n_k\to\infty}\int_{\mathbb{R}^N}b(x)|v_{n_k}-v_0|^{p(x)}dx=0.$$

In conclusion, we have shown that the sequence $\{v_{n_k}\}$ converges strongly to v_0 in X_{G_1}. Therefore, we can conclude that $\{(u_{n_k},v_{n_k})\}$ contains a subsequence converging strongly in Z. □

Now, we are ready to prove Theorem 1.1.

Proof. The proof immediately follows from Theorem 2.15. More precisely, it suffices to check the conditions of Theorem 2.15. Set

$$Z=U\oplus V,\quad U=X_{G_1}\times\{0\},\quad V=\{0\}\times X_{G_1},$$

and

$$Y_0=\{0\}\times X_{G_1}^{(m)^\perp},\quad Y_1=\{0\}\times X_{G_1}^{(k)},$$

where m and k are yet to be determined.

Define a group action $G=\{1,\tau\}\cong\mathbb{Z}_2$ by setting $\tau(u,v)=(-u,-v)$, then $\mathrm{Fix}G=\{0\}\times\{0\}$ (also denote$\{0\}$). It is clear that U and V are G-invariant closed subspaces of Z, and Y_0 and Y_1 are G-invariant closed subspaces of V and $\mathrm{codim}_V Y_0=m$, $\dim Y_1=k$.

Let

$$\Sigma:=\{A\subset Z\backslash\{0\}:\ \text{A is closed in } X \text{ and } (u,v)\in A\Rightarrow(-u,-v)\in A\}.$$

Define an index χ on Σ by

$$\chi(A)=\begin{cases}\min\{N\in\mathbb{Z}:\exists h\in C(A,\mathbb{R}^N\backslash\{0\})\text{ such that } h(-u,-v)=h(u,v)\},\\ 0 \text{ if } \ A=\emptyset,\\ +\infty \text{ if such } h \text{ does not exist.}\end{cases}$$

Then, from Huang and Li [28], we deduce that χ is an index theory satisfying the properties given in Definition 2.10. Moreover, χ satisfies the one-dimensional property. According to Definition 2.12, we can obtain a limit index χ^∞ with respect to (Z_n) from χ.

Now we shall verify the conditions of Theorem 2.15. It is easy to verify that the conditions (B_1), (B_2), (B_4) in Theorem 2.15 are satisfied. Set

$$V_j=X_{G_1}^{(j)}=\mathrm{span}\{e_1,e_2,\dots,e_j\}.$$

Hence, (B_3) in Theorem 2.15 is also satisfied. In the sequel, we shall verify the condition (B_7) in Theorem 2.15. Note that $\mathrm{Fix}G=\{0\}$, which implies that $\mathrm{Fix}G\cap V=\{(0,0)\}$, satisfying condition (1) of (B_7). Now, we need to verify the conditions (2) and (3) of (B_7).

Hereafter, we shall focus our attention on the case when $z = (u, v) \in Z$ satisfies $\|u\|_{b,h} \leq 1$ and $\|v\|_{b,h} \leq 1$.

(i) Let $(0, v) \in Y_0 \cap S_{\rho_m}(0)$ (where ρ_m is yet to be determined). Thus, by using assumptions $(\mathbf{F}_1)$ and $(\mathbf{H}_{a_3})$, we have

$$
\begin{aligned}
E_\lambda(0, v) &= \widehat{K}(\mathcal{B}(v)) - \int_{\mathbb{R}^N} \frac{1}{r(x)}|v|^{r(x)}dx - \int_{\mathbb{R}^N} \lambda(x)\mathcal{F}(x, 0, v)dx, \\
&\geq \gamma \mathfrak{K}_0 \int_{\mathbb{R}^N} \left(\frac{A_1(|v|^{p(x)})}{p(x)} + b(x)\frac{A_2(|v|^{p(x)})}{p(x)} \right) dx - \frac{1}{r^-}\int_{\mathbb{R}^N} |v|^{r(x)}dx - \sup_{x\in\mathbb{R}^N} \lambda(x) \int_{\mathbb{R}^N} \frac{f_2(x)}{\ell(x)}|v|^{\ell(x)}dx, \\
&\geq C\left[\|v\|_{h,p}^{p^+} + \mathcal{H}(\kappa_\star)\|v\|_{h,q}^{q^+}\right] - \frac{1}{r^-}\int_{\mathbb{R}^N} |v|^{r(x)}dx - \lambda^+ \int_{\mathbb{R}^N} \frac{f_2(x)}{\ell(x)}|v|^{\ell(x)}dx.
\end{aligned}
$$

Denote

$$
\delta_m = \sup_{v\in X_{G_1}^{m\perp}, \|v\|_{b,h}\leq 1} \int_{\mathbb{R}^N} \frac{f_2(x)}{\ell(x)}|v|^{\ell(x)}dx \quad \text{and } \tau_m = \sup_{v\in X_{G_1}^{m\perp}, \|v\|_{b,h}\leq 1} \int_{\mathbb{R}^N} |v|^{r(x)}dx.
$$

We invoke here Fan and Han [20, Lemma 3.3] to obtain that $\delta_m \to 0$, as $m \to \infty$.

Next, we need to verify that $\tau_m \to 0$ as $m \to \infty$. We know that $0 \leq \tau_m + 1 \leq \tau_m$, which implies that $\tau_m \to \tau \geq 0$ as $m \to \infty$. Therefore, there exist $v_m \in X_{G_1}^{m\perp}$ such that

$$
0 \leq \tau_m - \int_{\mathbb{R}^N} |v_m|^{r(x)}dx < \frac{1}{m},
$$

for every $m = 1, 2, \ldots$. As X_{G_1} is reflexive, we can pass to a subsequence, still denoted by $\{v_m\}$, such that there exists $v \in X_{G_1}$ satisfying $v_m \rightharpoonup v$ weakly in X_{G_1} as $m \to \infty$.

We claim $v = 0$. in fact, for any $e_k^* \in \{e_1^*, e_2^*, \ldots, e_m^*, \ldots\}$, we have $e_k^*(v_m) = 0$ when $m > k$, which implies that $e_k^*(v_m) \to 0$ as $m \to \infty$. It is immediate that $e_k^*(v) = 0$ for any $k \in \mathbb{N}$. Since $(X_{G_1})^* = \overline{\operatorname{span}\{e_1^*, e_2^*, \ldots, e_k^*, \ldots\}}$, we can conclude that $v = 0$.

By Theorem 2.7, there exist a finite measure ν and sequences $\{x_i\} \subset \mathcal{C}_h$ such that $|v_m|^{r(x)} \overset{*}{\rightharpoonup} \nu = \sum_{i\in I} \nu_i \delta_{x_i}$ in $\mathcal{M}_B(\mathbb{R}^N)$. where I is a countable set. Following a similar discussion as in Lemma 3.1, we can conclude that $v_i = v(\{x_i\}) = 0$ for any $i \in I$ where $x_i \neq 0$.

On the other hand, for any $0 < t < R$, take $\theta \in C_0^\infty(B_{2R}(0))$ such that $0 \leq \theta \leq 1$; $\theta \equiv 1$ in $B_{2R}(0)\backslash B_{2t}(0)$, $\theta \equiv 0$ in $B_t(0)$. Then,

$$
\int_{\mathbb{R}^N} |v_m|^{r(x)}\theta dx \longrightarrow \int_{\mathbb{R}^N} \theta d\nu = \int_{\{x\in\mathbb{R}^N;\ t\leq|x|\leq R\}} \theta d\nu = 0, \text{ as } m \to \infty.
$$

Since

$$
\int_{\{x\in\mathbb{R}^N;\ 2t\leq|x|\leq 2R\}} |v_m|^{r(x)}dx \leq \int_{\mathbb{R}^N} |v_m|^{r(x)}\theta dx,
$$

we obtain $\lim_{m\to\infty} \int_{\mathbb{R}^N} |v_m|^{r(x)}\, dx = 0$. Therefore, $\tau_m \to 0$, as $m \to \infty$.

Then, we have

$$
\begin{aligned}
E_\lambda(0, v) &\geq C\left[\|v\|_{b,p}^{p^+} + \mathcal{H}(\kappa_\star^3)\|v\|_{b,q}^{q^+}\right] - \frac{\tau_m^{r^-}}{r^-}\|v\|_{b,h}^{r^-} - \lambda^+\delta_m^{\ell^-}\|v\|_{b,h}^{\ell^-} \\
&\geq c\|v\|_{b,h}^{q^+} - \tau_m^{r^-}\|v\|_{b,h}^{r^-} - \lambda^+\delta_m^{\ell^-}\|v\|_{b,h}^{\ell^-}, \\
&\geq c\|v\|_{b,h}^{q^+} - (\tau_m^{r^-} + \lambda^+\delta_m^{\ell^-})\|v\|_{b,h}^{r^-}.
\end{aligned}
$$

Let

$$\rho_m = \left(\frac{cq^+}{r^-(\tau_m^{r^-} + \lambda \delta_m^{\ell^-})} \right)^{\frac{1}{r^- - q^+}}.$$

When $(0, v) \in Y_0 \cap S_{\rho_m}(0)$ and $\|v\|_{b,h} = \rho_m$, for sufficiently large m, we have

$$\sup E_\lambda(0, v)|_{Y_0 \cap S_{\rho_m}(0)} \geq \left(\frac{cq^+}{r^-} \right)^{\frac{r^-}{r^- - q^+}} \left(\frac{r^- - q^+}{q^+} \right) \left(\frac{1}{\tau_m^{r^-} + \lambda^+ \delta_m^{\ell^-}} \right)^{\frac{q^+}{r^- - q^+}},$$

where τ_m and $\delta_m \to 0$ as $m \to \infty$, thus we have

$$\sup E_\lambda(0, v)|_{Y_0 \cap S_{\rho_m}(0)} \geq \left(\frac{cq^+}{r^-} \right)^{\frac{r^-}{r^- - q^+}} \left(\frac{r^- - q^+}{q^+} \right) \left(\frac{1}{\tau_m^{r^-} + \lambda^+ \delta_m^{\ell^-}} \right)^{\frac{q^+}{r^- - q^+}} = \mathfrak{M}_m \to \infty \text{ as } m \to \infty,$$

that is, the condition (2) of ($\mathbf{B}_7$) holds.

(ii) By ($\mathbf{K}_1$) and ($\mathbf{K}_2$), for any $u \in X_{G_1}$, we have

$$\begin{aligned} E_\lambda(u, 0) &= -\widehat{K}(\mathcal{B}(u)) - \int_{\mathbb{R}^N} \frac{1}{r(x)} |u|^{r(x)} dx - \int_{\mathbb{R}^N} \lambda(x) \mathcal{F}(x, 0, v) dx, \\ &\leq 0. \end{aligned}$$

Hence, we can choose $\mathfrak{M}$ such that

$$\mathfrak{M} > \sup_{u \in X_{G_1}} E_\lambda(u, 0). \tag{3.26}$$

On the other hand, from ($\mathbf{K}_2$), we can obtain for $\xi > \xi_0$

$$\widehat{K}(\xi) \leq \frac{\widehat{K}(\xi_0)}{\xi_0^{\frac{1}{\gamma}}} \xi^{\frac{1}{\gamma}} \leq c \xi^{\frac{1}{\gamma}}. \tag{3.27}$$

About the latter condition and relation (3.26), for all $(u, v) \in U \oplus Y_1$, we have

$$\begin{aligned} E_\lambda(u, v) &= -\widehat{K}(\mathcal{B}(u)) + \widehat{K}(\mathcal{B}(v)) - \int_{\mathbb{R}^N} \frac{1}{r(x)} |u|^{r(x)} dx - \int_{\mathbb{R}^N} \frac{1}{r(x)} |v|^{r(x)} dx - \int_{\mathbb{R}^N} \lambda(x) \mathcal{F}(x, u, v) dx, \\ &\leq c \|v\|_{b,h}^{\frac{q^+}{\gamma}} - c |v|_{r(x)}^{r^-} + \mathfrak{M}. \end{aligned}$$

Since $|.|_{r(x)}$ is also a norm on Y_1, and Y_1 is a finite-dimensional space, thus $\|.\|_{b,h}$ and $|.|_{r(x)}$ are equivalent. Then, we get

$$E_\lambda(u, v) \leq c \|v\|_{b,h}^{\frac{q^+}{\gamma}} - c_{p^*} \|v\|_{b,h}^{r^-} + \mathfrak{M}.$$

Given that $\gamma > \frac{q^+}{r^-}$, we have

$$\sup E_\lambda|_{U \oplus Y_1} < +\infty.$$

Therefore, we can choose $k > m$ and $\mathfrak{N}_k > \mathfrak{M}_m$ such that

$$E_\lambda|_{U \oplus Y_1} \leq \mathfrak{N}_k,$$

which satisfies the condition (3) in ($\mathbf{B}_7$). According to Lemma 3.1, $E_\lambda(u, v)$ satisfies the condition of $(PS)_c$ for any $c \in [\mathfrak{M}_m, \mathfrak{N}_k]$, thus (B_6) in Theorem 2.15 holds. Consequently, based on Theorem 2.15, we can conclude that

$$c_i = \sup_{\chi^\infty(A)\leq i} \sup_{z=(u,v)\in A} E_\lambda(u, v), \quad -k+1 \leq i \leq -m,$$

represent critical values of E_λ, where $\mathfrak{M}_m \leq c_{-k+1} \leq \ldots \leq c_{-m} \leq \mathfrak{N}_k$. As we let $m \to \infty$, we can obtain an unbounded sequence of critical values c_i. Due to the even nature of the functional E, this results in two critical points $\mp z_i$ of E_λ corresponding to each c_i. □

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