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Idempotent Convexity and Algebras for the Capacity Monad and its Submonads

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Abstract Idempotent analogues of convexity are introduced. It is proved that the category of algebras for the capacity monad in the category of compacta is isomorphic to the category of (max, min)-idempotent biconvex compacta and their biaffine maps. It is also shown that the category of algebras for the monad of sup-measures ((max, min)-idempotent measures) is isomorphic to the category of (max, min)-idempotent convex compacta and their affine maps.

Keywords Capacity functor · Algebra for a monad · Idempotent semimodule · Idempotent convexity

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1 Introduction

Monads (also called *triples* [2, 7]) in topological categories and algebras for these monads are closely related to important objects in analysis and topological algebra. Świrszcz [17] proved that algebras and their morphisms for the probability measure monad are precisely the convex compact maps of locally convex vector topological spaces and continuous affine maps.

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By a result of Day (Theorem 3.3 of [5]), the category of algebras for the filter monad in the category of sets is the category of continuous lattices and their mappings that preserve directed joins and arbitrary meets. Due to Wyler [19] algebras for the hyperspace monad are the compact Lawson semilattices. Zarichnyi [20] proved that the category of algebras for the superextension monad is isomorphic to the category of compacta with (fixed) almost normal T_2-subbase and their convex maps. We will use a result of Radul [15] who introduced the inclusion hyperspace triple and proved that its algebras and their morphisms are in fact the compact Lawson lattices and their complete homomorphisms.

Unlike probability (normed additive) measures which are a traditional object of investigation by means of categorical topology, their non-additive analogues were paid less attention from this point of view. Meanwhile, the capacities (normed non-additive measures) that were introduced by Choquet [3] and rediscovered by Sugeno under the name *fuzzy measures*, have found numerous applications, e.g. in decision making under uncertainty [6, 16]. One of the most promising classes of non-additive measures is one of idempotent measures [1]. For other important classes of capacities and their topological properties see [14]. Upper semicontinuous capacities on compact spaces were systematically studied in [13].

Therefore it seems natural to use methods of categorical topology to study non-additive measures. Nykyforchyn and Zarichnyi [21] defined the capacity functor and the capacity monad in the category of compacta, and proved the basic topological properties of capacities on metrizable and non-metrizable compacta. Two important dual subfunctors of the capacity functor, namely of $\cup$-capacities (possibility measures) and of $\cap$-capacities (necessity measures) were introduced in [8], and it was shown that they lead to submonads of the capacity monad. The aim of this paper is to describe categories of algebras for the capacity monad, for the monads of $\cup$-capacities and of $\cap$-capacities, and to present internal relations of the capacity monad and its submonads with idempotent mathematics and generalizations of convexity (in the form of join geometry).

2 Preliminaries

A *compactum* is a compact Hausdorff topological space. We regard the unit segment $I = [0; 1]$ as a subspace of the real line with the natural topology. We write $A \underset{\text{cl}}{\subset} B$ (resp. $A \underset{\text{op}}{\subset} B$) if A is a closed (resp. open) subset of the space B. For a set X, the identity mapping $X \to X$ is denoted by $\mathbf{1}_X$. For a compactum X, we denote by $\exp X$ the set of all nonempty closed subsets of X with the *Vietoris topology*. A base of this topology consists of all sets of the form:

$$\langle U_1, U_2, \ldots, U_n \rangle = \big\{F \in \exp X \mid F \subset U_1 \cup U_2 \cup \cdots \cup U_n,\ F \cap U_i \neq \varnothing \text{ for all } 1 \leqslant i \leqslant n\big\},$$

where $n \in \mathbb{N}$ and all $U_i \subset X$ are open. The space $\exp X$ for a compactum X is a compactum as well. A nonempty closed subset $\mathcal{F} \subset \exp X$ is called an *inclusion hyperspace* if for all $A, B \in \exp X$, the inclusion $A \subset B$ and $A \in \mathcal{F}$ imply $B \in \mathcal{F}$. The set GX of all inclusion hyperspaces is closed in $\exp(\exp X)$. For more on $\exp X$ and GX see [18].

We regard any set S with an idempotent, commutative and associative binary operation $\oplus : S \times S \to S$ (with an additive notation) as an upper semilattice with the partial order $x \leqslant y \iff y \geqslant x \iff x \oplus y = y$ and the pairwise supremum $x \oplus y$ for $x, y \in S$. Similarly, given an idempotent, commutative and associative operation $\otimes : S \times S \to S$ (with a multiplicative notation), we regard S as a lower semilattice with the partial order $x \leqslant y \iff y \geqslant x \iff x \otimes y = x$ and $x \otimes y$ being the infimum of $x, y \in S$.

If two operations $\oplus, \otimes : L \times L \to L$ are idempotent, commutative and associative, and the distributive laws and the laws of absorption are valid, then L is a distributive lattice w.r.t. the partial order $x \leqslant y \iff y \geqslant x \iff x \oplus y = y \iff x \otimes y = x$, and $x \oplus y$ and $x \otimes y$ are the pairwise supremum and the pairwise infimum of $x, y \in L$.

If f, g are functions with the same domain and values in a poset, then by $f \vee g$ and $f \wedge g$ we also define their pointwise supremum and infimum. If f is a function with values in a set L with an operation "$\oplus$" (or "$\otimes$"), and $\alpha \in L$, then $(\alpha \oplus f)(x) = \alpha \oplus f(x)$ (resp. $(\alpha \otimes f)(x) = \alpha \otimes f(x)$) for any valid variable x.

An *idempotent semiring* is a set R with binary operations $\oplus, \otimes : R \times R \to R$ such that $(R, \oplus)$ is an abelian monoid with a neutral element 0, "$\oplus$" is idempotent, i.e. $a \oplus a = a$ for all $a \in R$, $(R, \otimes)$ is a monoid with a neutral element 1, the operation "$\otimes$" is distributive over "$\oplus$": $a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c)$ for all $a, b, c \in R$, and $0 \otimes a = a \otimes 0 = 0$ for all $a \in R$. The most popular idempotent semiring is the *tropical semiring* $(\mathbb{R} \cup \{-\infty\}, \oplus, \otimes)$, where $x \oplus y = \max\{x, y\}$, $x \otimes y = x + y$, which is the basis of *tropical mathematics* [9].

Somewhat less extensively studied is the idempotent semiring $\left(\mathbb{R} \cup \{\pm\infty\}, \oplus, \otimes\right)$, where $x \oplus y = \max\{x, y\}$, $x \otimes y = \min\{x, y\}$. We will use a semiring which is algebraically and topologically isomorphic to it, but more convenient for our purpose, namely $(I, \oplus, \otimes)$ with $x \oplus y = \max\left\{x, y\right\}$, $x \otimes y = \min\{x, y\}$. In general, any distributive lattice $(L, \oplus, \otimes)$ with top and bottom elements is an idempotent semiring.

We refer to [2, 11] for definitions of category, morphism, functor, natural transformation, monad, algebra for a monad, morphism of algebras, tripleability and related facts. By $\mathbf{1}_{\mathcal{C}}$ we denote the identity functor in the category $\mathcal{C}$. Recall that for a fixed monad $\mathbb{F}$, all $\mathbb{F}$-algebras and all their morphisms form a *category of* $\mathbb{F}$*-algebras*.

It was proved in [18] that constructions exp and G can be extended to functors in $\mathcal{C}$omp which are functorial parts of monads. For a continuous map of compacta $f : X \to Y$ the maps $\exp f : \exp X \to \exp Y$ and $Gf : GX \to GY$ are defined by the formulae $\exp f(F) = \left\{f(x) \mid x \in F\right\}$, $F \in \exp X$ and $Gf(\mathcal{F}) = \left\{B \underset{\mathrm{cl}}{\subset} Y \mid B \supset f(A) \text{ for some } A \in \mathcal{F}\right\}$, $\mathcal{F} \in GX$. For the *inclusion hyperspace monad* $\mathbb{G} = (G, \eta_G, \mu_G)$ the components $\eta_G X : X \to GX$ and $\mu_G X : G^2 X \to GX$ of the unit and the multiplication are defined as follows: $\eta_G(x) = \left\{F \in \exp X \mid x \in F\right\}$, $x \in X$ and $\mu_G X(\mathrm{F}) = \bigcup\left\{\bigcap \mathcal{A} \mid \mathcal{A} \in \mathrm{F}\right\}, \mathrm{F} \in G^2 X$.

We denote the *category of compacta* that consists of all compacta and their continuous mappings by $\mathcal{C}$omp. If there is a natural transformation of one functor in $\mathcal{C}$omp to another with all components being topological embeddings, then the first functor is called a *subfunctor* of the latter [18]. Similarly, an *embedding of monads* in $\mathcal{C}$omp is a morphism of monads with all components being topological embeddings. If there exists an embedding of one monad in $\mathcal{C}$omp into another, then the first monad is called a *submonad* of the latter.

Now we present the main notions and results of [8, 21] which concern capacities on compacta, the capacity functor and the capacity monad. We call a function $c : \exp X \cup \{\varnothing\} \to I$ a *capacity* on a compactum X if the following three properties hold for all closed subsets F, G of X:

(1) $c(\varnothing) = 0, c(X) = 1$;
(2) if $F \subset G$, then $c(F) \leqslant c(G)$ (monotonicity); and
(3) if $c(F) < a$, then there exists an open set $U \supset F$ such that $G \subset U$ implies $c(G) < a$ (upper semicontinuity).

We extend a capacity c to all open subsets in X by the formula:

$$c(U) = \sup \left\{ c(F) \middle| F \underset{\text{cl}}{\subset} X, F \subset U \right\}, \quad U \underset{\text{op}}{\subset} X.$$

It was proved in [21] that the set MX of all capacities on a compactum X is a compactum as well, if a topology on MX is determined by a subbase that consists of all sets of the form

$$O_-(F, a) = \big\{ c \in MX \mid c(F) < a \big\},$$

where $F \underset{\text{cl}}{\subset} X, a \in \mathbb{R}$, and

$$\begin{aligned} O_+(U, a) &= \big\{ c \in MX \mid c(U) > a \big\} \\ &= \big\{ c \in MX \mid \text{there exists a compactum } F \subset U, c(F) > a \big\}, \end{aligned}$$

where $U \underset{\text{op}}{\subset} X, a \in \mathbb{R}$.

The assignment M extends to the *capacity functor* M in the category of compacta, if for a continuous map of compacta $f : X \to Y$, the map $Mf : MX \to MY$ is defined by the formula:

$$Mf(c)(F) = c\big(f^{-1}(F)\big),$$

where $c \in MX$, $F \underset{\text{cl}}{\subset} Y$. This functor is the functorial part of the *capacity monad* $\mathbb{M} = (M, \eta, \mu)$ which was described in [21]. Its unit and multiplication are defined by the formulae:

$$\eta X(x) = \delta_x, \text{ where } \delta_x(F) = \begin{cases} 1, & \text{if } x \in F, \\ 0, & \text{if } x \notin F, \end{cases} \qquad \text{(a Dirac measure concentrated in } x\text{)}$$

and

$$\mu X(\mathcal{C})(F) = \sup \big\{ \alpha \in I \mid \mathcal{C}\big(\{c \in MX \mid c(F) \geqslant \alpha\}\big) \geqslant \alpha \big\},$$

where $x \in X, \mathcal{C} \in M^2 X, F \underset{\text{cl}}{\subset} X$.

We call a capacity $c \in MX$ a $\cup$-*capacity* (also called *sup-measure* or *possibility measure*) if $c(A \cup B) = \max\{c(A), c(B)\}$ for all $A, B \underset{\text{cl}}{\subset} X$. A capacity $c \in MX$ is a $\cap$-*capacity* (or a *necessity measure*) [8] if $c(A \cap B) = \min\{c(A), c(B)\}$ for all $A, B \underset{\text{cl}}{\subset} X$. The sets of all $\cup$-capacities and of all $\cap$-capacities on a compactum X are denoted by $M_\cup X$ and $M_\cap X$.

It was proved in [8] that $M_\cup X$ and $M_\cap X$ are closed in MX, $Mf(M_\cup X) \subset M_\cup Y$ and $Mf(M_\cap X) \subset M_\cap Y$ for any continuous map of compacta $f : X \to Y$. Therefore one obtains subfunctors $M_\cup$, $M_\cap$ of the capacity functor M. Moreover, we get submonads $\mathbb{M}_\cup$ and $\mathbb{M}_\cap$ of the capacity monad $\mathbb{M}$.

Observe that for a $\cup$-capacity c and a closed set $F \subset X$ we have $c(F) = \max\{c(x) \mid x \in F\}$, and c is completely determined by its values on singletons. Therefore we often identify c with the upper semicontinuous function $X \to I$ which sends each $x \in X$ to $c(\{x\})$, and we write $c(x)$ instead of $c(\{x\})$. Conversely, each upper semicontinuous function $c : X \to I$ with $\max c = 1$ determines a $\cup$-capacity by the formula $c(F) = \max\{c(x) \mid x \in F\}$, $F \underset{\text{cl}}{\subset} X$. A similar, but a little more complicated observation is valid for $\cap$-capacities.

3 Algebras for the Monads of ∪-Capacities and ∩-Capacities

Let an operation $ic : X \times I \times X \to X$ be given for a set X. In the sequel, for the sake of shortness, we shall denote $ic(x, \alpha, y)$ by $x \oplus (\alpha \otimes y)$ or simply by $x \oplus \alpha y$. We call ic an *idempotent convex combination* of two points in X if the following equalities are valid for all $x, y, z \in X$, and all $\alpha, \beta \in I$:

(1) $x \oplus \alpha x = x$;
(2) $(x \oplus \alpha y) \oplus \beta z = (x \oplus \beta z) \oplus \alpha y$;
(3) $x \oplus \alpha(y \oplus \beta z) = (x \oplus \alpha y) \oplus (\alpha \otimes \beta)z$;
(4) $x \oplus 1y = y \oplus 1x$; and
(5) $x \oplus 0y = x$.

We call the set

$$\Delta^n_\oplus = \left\{(\alpha_0, \alpha_1, \ldots, \alpha_n) \in I^{n+1} \middle| \alpha_0 \oplus \alpha_1 \oplus \ldots \oplus \alpha_n = 1\right\}$$

the *(idempotent) n-dimensional* $\oplus$*-simplex*. Now, assuming (1)–(5), for any coefficients $(\alpha_0, \alpha_1, \ldots, \alpha_n) \in \Delta^n_\oplus$ and elements $x_0, x_1, \ldots, x_n \in X$, we define the *idempotent convex combination* of $n+1$ points as follows (assume that $\alpha_k = 1$ for some $0 \leqslant k \leqslant n$):

$$\begin{aligned}&\alpha_0 x_0 \oplus \alpha_1 x_n \oplus \ldots \oplus \alpha_n x_n \\ &\quad = \Big(\ldots\big((x_k \oplus \alpha_0 x_0) \oplus \ldots\big) \oplus \alpha_{k-1}x_{k-1}\big) \oplus \alpha_{k+1}x_{k+1}\Big) \oplus \ldots\Big) \oplus \alpha_n x_n.\end{aligned}$$

Conditions (2) and (4) guarantee that the combination is well-defined and does not depend on the order of summands. Obviously, $1x \oplus \alpha y = x \oplus \alpha y$. By (5), the summands with zero coefficients can be dropped, and by (1) and (3), if two summands contain the same point, then a summand with the greater coefficient absorbs a summand with the smaller coefficient. Conditions (2) and (3) also imply the "big associative law":

$$\alpha_0\left(\beta_0^0 x_0^0 \oplus \ldots \oplus \beta_{k_0}^0 x_{k_0}^0\right) \oplus \alpha_1\left(\beta_0^1 x_0^1 \oplus \ldots \oplus \beta_{k_1}^1 x_{k_1}^1\right) \oplus \ldots \oplus \alpha_n\left(\beta_0^n x_0^n \oplus \ldots \oplus \beta_{k_n}^n x_{k_n}^n\right)$$
$$= \left(\alpha_0 \otimes \beta_0^0\right) x_0^0 \oplus \ldots \oplus \left(\alpha_0 \otimes \beta_{k_0}^0\right) x_{k_0}^0 \oplus \left(\alpha_1 \otimes \beta_0^1\right) x_0^1 \oplus \ldots \oplus \left(\alpha_1 \otimes \beta_{k_1}^1\right) x_{k_1}^1$$
$$\oplus \ldots \oplus \left(\alpha_n \otimes \beta_0^n\right) x_0^n \oplus \ldots \oplus \left(\alpha_n \otimes \beta_{k_n}^n\right) x_{k_n}^n,$$

where $x_j^i \in X$, $\left(\alpha_0, \alpha_1, \ldots, \alpha_n\right) \in \Delta_\oplus^n$, $\left(\beta_0^i, \beta_0^i, \ldots, \beta_{k_i}^i\right) \in \Delta_\oplus^{k_i}$ for $i = 0, 1, \ldots, n$.

Properties (1)–(4) imply that the operation $\vee : X \times X \to X$, $x \vee y = x \oplus 1y$ for all $x, y \in X$, is commutative, associative and idempotent, thus $(X, \vee)$ is an upper semilattice with a partial order $x \leqslant y \iff x \vee y = y$ for which $x \vee y$ is a pairwise supremum of x and y. If X is a compactum such that

(6) for any neighborhood U of any element $x \in X$ there is a neighborhood V of x, $V \subset U$, such that $y \oplus 1z \in V$ for all $y, z \in V$;

then each point of X has a local base consisting of subsemilattices, and $(X, \vee)$ is a compact Lawson upper semilattice [10]. We shall call a pair (X, ic) of a compactum X with an idempotent convex combination ic which satisfies the property (6), a (max, min)-*idempotent convex compactum*.

Theorem 3.1 *Let X be any compactum. Then there exist a one-to-one correspondence between continuous maps $\xi : M_\cup X \to X$ such that the pair (X, ξ) is an $\mathbb{M}_\cup$-algebra, and continuous idempotent convex combinations $ic : X \times I \times X \to X$ such that (X, ic) is a* (max, min)*-idempotent convex compactum.*

If for a continuous $ic : X \times I \times X \to X$ conditions (1)–(5) *are satisfied, then condition* (6) *implies the following stronger property:*

(6+) *for any neighborhood U of any element $x \in X$ there is a neighborhood V of x, $V \subset U$, such that $y \oplus \alpha z \in V$ for all $y, z \in V$, $\alpha \in I$.*

Proof Let (X, ξ) be an $\mathbb{M}_\cup$-algebra. Define the operation $ic : X \times I \to X$ by the formula $ic(x, \alpha, y) = \xi(\delta_x \oplus \alpha\delta_y)$. It is obvious that ξ is well-defined, continuous and satisfies conditions (1), (4), and (5). In order to verify condition (2), observe that by the definition of an algebra for a monad we obtain:

$$(x \oplus \alpha y) \oplus \beta z = \xi\left(\delta_{\xi(x \oplus \alpha y)} \oplus \beta\delta_z\right)$$
$$= \xi \circ M_\cup \xi\left(\delta_{\delta_x \oplus \alpha\delta_y} \oplus \beta\delta_{\delta_z}\right) = \xi \circ \mu_\cup X\left(\delta_{\delta_x \oplus \alpha\delta_y} \oplus \beta\delta_{\delta_z}\right)$$
$$= \xi\left(\delta_x \oplus \alpha\delta_y \oplus \beta\delta_z\right) = \xi\left(\delta_x \oplus \beta\delta_z \oplus \alpha\delta_y\right) = \left(x \oplus \beta z\right) \oplus \alpha y.$$

Proof of (3) is essentially analogous. Thus the map ic is an idempotent convex combination of two points, and we consider idempotent convex combinations of arbitrary finite number of points to be defined as described above.

Let U be a neighborhood of $x \in X$. By continuity of ξ and the equality $\xi(\delta_x) = x$, there is a neighborhood $\tilde{U} \subset M_\cup X$ of δ_x such that for all $c \in \tilde{U}$ we have $\xi(c) \in U$.

There also exists a neighborhood $\tilde{V} \ni x$ such that for all $y_0, y_1, \ldots, y_n \in \tilde{V}$, $(\alpha_0, \alpha_1, \ldots, \alpha_n) \in \Delta^n_\oplus$ we have $\alpha_0\delta_{y_0} \oplus \alpha_1\delta_{y_1} \oplus \ldots \oplus \alpha_n\delta y_n \in \tilde{U}$. It is straightforward to verify that the set

$$V = \big\{\alpha_0 y_0 \oplus \alpha_1 y_1 \oplus \ldots \oplus \alpha_n y_n \mid n \in \{0, 1, \ldots\}, \\ (\alpha_0, \alpha_1, \ldots, \alpha_n) \in \Delta^n_\oplus, y_0, y_1, \ldots, y_n \in \tilde{V}\big\}$$

is a neighborhood of x requested by (6+), which implies (6). Thus it is proved that an $\mathbb{M}_\cup$-algebra (X, ξ) determines a continuous operation *ic* that satisfies conditions (1)–(6).

Now assume that we are given a compactum X and a continuous operation $ic : X \times I \times X \to X$ that satisfies conditions (1)–(6). Recall that X with the operation $\vee : X \times X \to X$, defined by the formula $x \vee y = x \oplus 1y$, is a compact Lawson upper semilattice, therefore for all nonempty closed subsets $F \subset X$, there exists $\sup F$ which continuously depends on F w.r.t. the Vietoris topology [12].

Let $c \in M_\cup X$ and $c(x_0) = 1$ for some $x_0 \in X$. We put $\xi(c) = \sup\{x_0 \oplus \alpha x \mid x \in X, \alpha \leqslant c(x)\}$. We shall prove that $\xi : M_\cup X \to X$ is well-defined (i.e. it does not depend on the choice of x_0) and it is continuous.

For each $x \in X$ let $gr(x)$ be the collection $(x \vee y)_{y\in X} \in X^X$. Then the map of compacta $gr : X \to X^X$ is continuous and injective, therefore is it is an embedding.

The equality:

$$\begin{aligned} \xi(c) \vee y &= \sup\big\{x_0 \oplus \alpha x \mid x \in X, \alpha \leqslant c(x)\big\} \vee y \\ &= \sup\big\{y \oplus 1x_0 \oplus \alpha x \mid x \in X, \alpha \leqslant c(x)\big\} \\ &= \sup\big\{\big(y \oplus 1x_0\big) \vee \big(y \oplus \alpha x\big) \mid x \in X, \alpha \leqslant c(x)\big\} \\ &= \big(y \oplus 1x_0\big) \vee \sup\big\{\big(y \oplus \alpha x\big) \mid x \in X, \alpha \leqslant c(x)\big\} \\ &= \sup\big\{\big(y \oplus \alpha x\big) \mid x \in X, \alpha \leqslant c(x)\big\}. \end{aligned}$$

holds for each $y \in X$, and the latter expression does not depend on x_0. This implies that $gr(\xi(c))$ and thus also $\xi(c)$ are uniquely determined. Moreover, $pr_y \circ gr(\xi(c))$ is the supremum of the image of the closed set $\{(x, \alpha) \mid x \in X, \alpha \in I, \alpha \leqslant c(x)\} \subset X \times I$ under the continuous map which sends (x, α) to $y \oplus \alpha x \in X$. Taking into account that this set (the *hypograph* of the function $c : X \to I$) depends on $c \in M_\cup X$ continuously, we obtain that the correspondence $c \mapsto gr(\xi(c))$ is continuous, which implies the continuity of $\xi : M_\cup X \to X$.

To show that (X, ξ) is an $\mathbb{M}_\cup$-algebra, we again assume $c(x_0) = 1$ for a capacity $c \in M_\cup X$. Then

$$\begin{aligned} y \oplus \alpha\xi(c) &= y \oplus \alpha \sup\big\{x_0 \oplus \beta x \mid x \in X, \beta \leqslant c(x)\big\} \\ &= \sup\big\{y \oplus \alpha x_0 \oplus (\alpha \otimes \beta)x \mid x \in X, \beta \leqslant c(x)\big\} \\ &= \sup\big\{(y \oplus \alpha x_0) \vee \big(y \oplus (\alpha \otimes \beta)x\big) \mid x \in X, \beta \leqslant c(x)\big\} \\ &= \big(y \oplus \alpha x_0\big) \vee \sup\big\{y \oplus (\alpha \otimes \beta)x \mid x \in X, \beta \leqslant c(x)\big\} \\ &= \sup\big\{y \oplus (\alpha \otimes \beta)x \mid x \in X, \beta \leqslant c(x)\big\} \end{aligned}$$

holds for each $y \in X$, $\alpha \in I$. The second equality sign follows from an "infinite distributive law" $y \oplus \alpha \sup F = \sup\{y \oplus \alpha x \mid x \in F\}$, with F a nonempty subset of X.

This law is first proved for finite F and then extended to the infinite case by continuity of lowest upper bounds.

It is obvious that $\xi(\delta_x) = x$ for a point $x \in X$, i.e. $\xi \circ \eta_{\cup} X = \mathbf{1}_X$. We choose a capacity $\mathcal{C} \in M^2_{\cup} X$ and compare $\xi \circ M_{\cup}\xi(\mathcal{C})$ and $\xi \circ \mu_{\cup} X(\mathcal{C})$. For any point $y \in X$ we have

$$\begin{aligned} y \vee \big(\xi \circ M_{\cup}\xi(\mathcal{C})\big) &= \sup\big\{(y \oplus \alpha x \mid x \in X, \alpha \leqslant M_{\cup}\xi(\mathcal{C})(x)\big\} \\ &= \sup\big\{(y \oplus \alpha \xi(c) \mid c \in M_{\cup} X, \alpha \leqslant \mathcal{C}(c)\big\} \\ &= \sup\big\{ \sup\{y \oplus (\alpha \otimes \beta)x \mid x \in X, \beta \leqslant c(x)\big\}, c \in M_{\cup} X, \alpha \leqslant \mathcal{C}(c)\big\} \\ &= \sup\big\{(y \oplus \alpha x \mid x \in X, c \in M_{\cup} X, \alpha \leqslant \min\{\mathcal{C}(c), c(x)\}\big\} \\ &= y \vee \big(\xi \circ \mu_{\cup} X(\mathcal{C})\big). \end{aligned}$$

This implies $\xi \circ M_{\cup}\xi = \xi \circ \mu_{\cup} X$, i.e. (X, ξ) is a $\mathbb{M}_{\cup}$-algebra.

To prove that the correspondence "$\mathbb{M}_{\cup}$-algebra $\leftrightarrow$ idempotent convex combination that satisfies (1)–(6)" is one-to-one, assume that for some continuous $ic : X \times I \times X$ satisfying (1)–(6), there is a continuous map $\xi' : M_{\cup} X \to X$ such that (X, ξ') is a $\mathbb{M}_{\cup}$-algebra and $ic(x, \alpha, y) = \xi'(\delta_x \oplus \alpha\delta_y)$ for all $x, y \in X, \alpha \in I$. Therefore $\xi'(\delta_x \oplus \alpha\delta_y) = \xi(\delta_x \oplus \alpha\delta_y)$ for the map ξ constructed above. Let $1 \geqslant \alpha_1 \geqslant \alpha_2 \geqslant 0$, $x_0, x_1, x_2 \in X$. Then

$$\begin{aligned} \xi\big(\delta_{x_0} \oplus \alpha_1\delta_{x_1} \oplus \alpha_2\delta_{x_2}\big) &= \xi \circ \mu_{\cup} X\left(\delta_{\delta_{x_0}} \oplus \alpha_1\delta_{\delta_{x_1} \oplus \alpha_2\delta_{x_2}}\right) \\ &= \xi \circ M_{\cup}\xi\left(\delta_{\delta_{x_0}} \oplus \alpha_1\delta_{\delta_{x_1} \oplus \alpha_2\delta_{x_2}}\right) \\ &= \xi\left(\delta_{x_0} \oplus \alpha_1\delta_{\xi(\delta_{x_1} \oplus \alpha_2\delta_{x_2})}\right) \\ &= \xi'\left(\delta_{x_0} \oplus \alpha_1\delta_{\xi'(\delta_{x_1} \oplus \alpha_2\delta_{x_2})}\right) \\ &= \ldots \\ &= \xi'\left(\delta_{x_0} \oplus \alpha_1\delta_{x_1} \oplus \alpha_2\delta_{x_2}\right). \end{aligned}$$

In a similar manner we prove by induction that:

$$\xi\left(\delta_{x_0} \oplus \alpha_1\delta_{x_1} \oplus \alpha_2\delta_{x_2} \oplus \ldots \oplus \alpha_n\delta_{x_n}\right) = \xi'\left(\delta_{x_0} \oplus \alpha_1\delta_{x_1} \oplus \alpha_2\delta_{x_2} \oplus \ldots \oplus \alpha_n\delta_{x_n}\right)$$

for arbitrary integer $n \geqslant 0$. By continuity we deduce that $\xi(c) = \xi'(c)$ for all $c \in M_{\cup} X$. □

Let $ic : X \times I \times X \to X$ and $ic' : X' \times I \times X' \to X'$ be idempotent convex combinations. We say that a map $f : (X, ic) \to (X', ic')$ is *affine* if it preserves idempotent convex combination, i.e. $f(ic(x, \alpha, y)) = ic'(f(x), \alpha, f(y))$ for all $x, y \in X, \alpha \in I$.

Theorem 3.2 *Let (X, ξ), (X', ξ') be $\mathbb{M}_{\cup}$-algebras and let $ic : X \times I \times X \to X$ and $ic' : X' \times I \times X' \to X'$ be the respective idempotent convex combinations. Then a continuous map $f : X \to Y$ is a morphism of $\mathbb{M}_{\cup}$-algebras $(X, \xi) \to (X', \xi')$ if and only if $f : (X, ic) \to (X', ic')$ is affine.*

Proof

Necessity Let $f : (X, \xi) \to (X', \xi')$ be a morphism of $\mathbb{M}_{\cup}$-algebras and let $x, y \in X$, $\alpha \in I$. Then

$$f\big(ic(x, \alpha, y)\big) = f \circ \xi\big(\delta_x \vee \alpha\delta_y\big) = \xi' \circ Mf\big(\delta_x \vee \alpha\delta_y\big)$$
$$= \xi'\left(\delta_{f(x)} \vee \alpha\delta_{f(y)}\right) = ic'\big(f(x), \alpha, f(y)\big).$$

Sufficiency Let $f : (X, ic) \to (X', ic')$ be affine. Then $f(x \vee y) = f(x) \vee f(y)$ for all $x, y \in X$. Continuity of f implies that f preserves suprema of closed sets. For $c \in M_{\cup}X$ we choose a point $x_0 \in X$ such that $c(x_0) = 1$, then $M_{\cup} f(x)(f(x_0)) = 1$. Therefore:

$$\begin{aligned}
\xi' \circ M_{\cup} f(c) &= \sup\left\{ f(x_0) \oplus \alpha x' \mid x' \in X', \alpha \leqslant M_{\cup} f(c)(x')\right\}\\
&= \sup\left\{ f(x_0) \oplus \alpha f(x) \mid x \in X, \alpha \leqslant c(x)\right\}\\
&= \sup\left\{ f(x_0 \oplus \alpha x) \mid x \in X, \alpha \leqslant c(x)\right\}\\
&= f\big(\sup\left\{x_0 \oplus \alpha x \mid x \in X, \alpha \leqslant c(x)\right\}\big)\\
&= f \circ \xi(c),
\end{aligned}$$

and f is a morphism of $\mathbb{M}_{\cup}$-algebras. □

Remark 3.3 It is easy to see that (max, min)-idempotent convex compacta and their affine continuous maps constitute a category $\mathcal{C}\mathrm{onv}_{\max,\min}$ of (max, min)-idempotent convex compacta that by the latter theorem is monadic (= tripleable) [17] over the category of compacta.

Convex compacta are usually defined as compact closed subsets of locally convex topological vector spaces. To obtain a similar description for (max, min)-idempotent convex compacta, we need some extra definitions and facts. For an idempotent semiring (cf. [4]) $\mathcal{S} = (S, \oplus, \otimes, 0, 1)$, a (left idempotent) $\mathcal{S}$-semimodule is a set L with operations $\oplus : L \times L \to L$ and $\otimes : S \times L \to L$ such that for all $x, y, z \in L, \alpha, \beta \in S$:

(1) $x \oplus y = y \oplus x$;
(2) $(x \oplus y) \oplus z = x \oplus (y \oplus z)$;
(3) there is an (obviously unique) element $\bar{0} \in L$ such that $x \oplus \bar{0} = x$ for all x;
(4) $\alpha \otimes (x \oplus y) = (\alpha \otimes x) \oplus (\alpha \otimes y)$, $(\alpha \oplus \beta) \otimes x = (\alpha \otimes x) \oplus (\beta \otimes x)$;
(5) $(\alpha \otimes \beta) \otimes x = \alpha \otimes (\beta \otimes x)$;
(6) $1 \otimes x = x$; and
(7) $0 \otimes x = \bar{0}$.

We adopt the usual convention and write αx instead of $\alpha \otimes x$. Observe that these axioms imply $\alpha\bar{0} = \bar{0}$, $x \oplus x = x$. Informally speaking, an idempotent semimodule is a vector space over an idempotent semiring.

If $\mathcal{S} = (I, \max, \min, 0, 1)$, we talk about a (max, min)-*idempotent semimodule*. In this case we define an operation $ic : L \times I \times L \to L$ by the formula $ic(x, \alpha, y) = x \oplus (\alpha \otimes y)$ ($\oplus$ and $\otimes$ are from L). It is easy to see that ic satisfies (1)–(5). The combination $\alpha_0 x_0 \oplus \alpha_1 x_0 \oplus \ldots \oplus \alpha_n x_n$ of points $x_0, x_1, \ldots, x_n$ is defined in an obvious way and it coincides with the operation described above if $(\alpha_0, \alpha_1, \ldots, \alpha_n) \in \Delta^n_{\oplus}$. A subset A of a (max, min)-idempotent semimodule L is called *convex* if $x \oplus \alpha y \in A$ whenever $x, y \in A$, $\alpha \in I$. A convex subset $A \subset L$ contains all idempotent convex combinations of its elements.

Let a (max, min)-idempotent semimodule L be a compactum and suppose that the operations $\oplus$ and $\otimes$ are continuous and the topology on L satisfies the following additional condition:

(8) for any neighborhood U of any element $x \in L$ there is a neighborhood V of x, $V \subset U$, such that $y \oplus z \in V$ for all $y, z \in V$.

Then we call $(L, \oplus, \otimes)$ a *compact Lawson* (max, min)-*idempotent semimodule*. By the above theorem L is a $\mathbb{M}_{\cup}$-algebra, which implies the following:

(8+) for any neighborhood U of any element $x \in L$ there is a neighborhood V of x, $V \subset U$, such that $y \oplus \alpha z \in V$ for all $y, z \in V$, $\alpha \in I$.

Thus for every point of L there is a local base that consists of convex neighborhoods, and we say that L is *locally convex*.

The nature of a compactum X with an idempotent convex combination that satisfies (1)–(6) is clarified by the following:

Theorem 3.4 *A pair of a compactum X and a continuous map $ic : X \times I \times X \to X$ is a* (max, min)*-idempotent convex compactum if and only if X is a closed convex subset of a compact Lawson* (max, min)*-idempotent semimodule $(L, \oplus, \otimes)$ such that $ic(x, \alpha, y) \equiv \underbrace{x \oplus \alpha y}_{in\ L}$.*

Proof Sufficiency is obvious. To prove necessity, assume that X is a compactum and a continuous map $ic : X \times I \times X \to X$ satisfies conditions (1)–(6). We define an equivalence relation "$\sim$" on $X \times I$ as follows: $(x_1, a_1) \sim (x_2, a_2)$ if $y \oplus a_1 x_1 = y \oplus a_2 x_2$ for all $y \in X$. This relation is closed in $(X \times I) \times (X \times I)$, therefore the quotient space $X \times I/\sim$, which we denote by $\bar{X}$, is a compact Hausdorff space. We denote by $[(x, a)]$ the equivalence class of the pair (x, a). The map $i : X \to \bar{X}$ that sends a point $x \in X$ to $[(x, 1)]$ is an embedding because $(x_1, 1) \sim (x_2, 1)$ is possible only if $x_1 = x_2$.

We define operations $\otimes : I \times \bar{X} \to \bar{X}$ and $\oplus : \bar{X} \times \bar{X} \to \bar{X}$ by the formulae $\alpha \otimes [(x, a)] = [(x, \alpha \otimes a)]$ and

$$[(x, a)] \oplus [(y, b)] = \begin{cases} \big[(x \oplus b y, a)\big], a \geqslant b, \\ \big[(y \oplus a y, b)\big], a \leqslant b. \end{cases}$$

The element $\bar{0} = [(x, 0)]$ does not depend on x and it satisfies (3). Properties (5), (6), (7) are obvious. Verification that $\oplus$, $\otimes$ are well-defined, continuous and satisfy (1), (2), (4), (8), is more convenient with a generalization of the mapping $gr : X \to X^X$ that was defined in the proof of the latter theorem. To avoid introducing extra denotations, we denote by $gr(x, \alpha)$, where $x \in X$, $\alpha \in I$, the collection $(t \oplus \alpha x)_{t \in X}$. Then the map $gr : X \times I \to X^X$ is continuous (but, as can be shown, not injective). It is obvious that $(x_1, \alpha_1) \sim (x_2, \alpha_2)$ if and only if $gr(x_1, \alpha_1) = gr(x_2, \alpha_2)$, thus we shall identify the image of the map gr with the quotient space $\bar{X} = X \times I/\sim$, and gr with the quotient map.

Let $\bar{x}, \bar{y}, \bar{z}$ be points in $\bar{X}$, and $\bar{x} = gr(x, a) = (x_t)_{t\in X}$, $\bar{y} = gr(y, b) = (y_t)_{t\in X}$, $\bar{z} = gr(z, c) = (z_t)_{t\in X}$. Observe that $x \oplus y = (x_t \vee y_t)_{t\in}$, $\alpha \otimes \bar{x} = (t \oplus \alpha x_t)_{t\in X}$, therefore $\bar{x} \oplus \bar{y}$ and $\alpha \otimes \bar{x}$ are uniquely determined and continuous w.r.t. $\bar{x}$, $\bar{y}$ and α, $\bar{x}$ resp. Similar expressions can be written for $x \oplus z$ and $y \oplus z$, and 1),2) are easily seen. Next, $\alpha \otimes \bar{x} = (t \oplus \alpha x_t)_{t\in X}$, $\alpha \otimes \bar{y} = (t \oplus \alpha y_t)_{t\in X}$, thus

$$\begin{aligned}(\alpha \otimes \bar{x}) \oplus (\alpha \otimes \bar{y}) &= \big((t \oplus \alpha x_t) \vee (t \oplus \alpha x_t)\big)_{t\in X} \\ &= \big((t \oplus \alpha(x_t \vee \alpha x_t)\big)_{t\in X} = \alpha \otimes (\bar{x} \oplus \bar{y}).\end{aligned}$$

Similarly,

$$\begin{aligned}(\alpha \otimes \bar{x}) \oplus (\beta \otimes \bar{x}) &= \big((t \oplus \alpha x_t) \vee (t \oplus \beta x_t)\big)_{t\in X} \\ &= \big((t \oplus \alpha x_t \oplus \beta x_t)\big)_{t\in X} = (\alpha \oplus \beta) \otimes \bar{x},\end{aligned}$$

and condition (4) holds.

Let $G \subset \bar{X}$ be a closed nonempty set, then $G = gr(F)$ for some closed $F \subset X \times I$. There is $(x_0, a_0) \in F$ such that $a_0 = \max\{a \mid (x, a) \in F\}$. It is easy to show that $\sup G$ in $\bar{X}$ is equal to $[(x', a_0)]$ where $x' = \sup\{x_0 \oplus ax \mid (x, a) \in F\}$, thus the upper semilattice $\bar{X}$ is complete. It is also clear that:

$$gr(x', a_0) = \big(\sup\{t \oplus ax \mid (x, a) \in F\}\big)_{t\in X} = \big(\sup\{x_t \mid (x_t)_{t\in X} \in G\}\big)_{t\in X},$$

therefore $\sup G$ depends on G continuously w.r.t. Vietoris topology. This is a statement equivalent to (8) [12]. □

As triples $\mathbb{M}_\cup$ and $\mathbb{M}_\cap$ are isomorphic through a natural transformation $\varkappa$ defined in [8], and the map $I \to I$ that sends each t to $1 - t$ is an isomorphism of the idempotent semirings $(I, \oplus, \otimes, 0, 1)$ and $(I, \otimes, \oplus, 1, 0)$, we can immediately state by duality, an analogue of Theorem 3.1. Its proof can be obtained by replacing $M_\cup$ by $M_\cap$, $\otimes$ by $\oplus$, 1 by 0, upper semilattices by lower ones, $\vee$ by $\wedge$, sup by inf, $\Delta_\oplus$ by the *(idempotent) n-dimensional* $\otimes$*-simplex*

$$\Delta^n_\otimes = \big\{(\alpha_0, \alpha_1, \ldots, \alpha_n) \in I^{n+1} \mid \alpha_0 \otimes \alpha_1 \otimes \ldots \otimes \alpha_n = 0\big\}$$

and vice versa, wherever necessary.

Thus we define *dual idempotent convex combinations* and (min, max)-*idempotent convex compacta* that are precisely $\mathbb{M}_\cap$-algebras. We omit the obvious details. Observe that for a given $\mathbb{M}_\cap$-algebra (X, ξ) the respective dual idempotent convex combination $ci : X \times I \times X \to X$ is determined by the equality $ci(x, \alpha, y) = \xi(\delta_x \wedge (\alpha \vee \delta_y))$. Conversely, the value $\xi(c)$ for a capacity $c \in M_\cup X$ (assuming that $c(X \setminus \{x_0\}) = 0$) is equal to $\xi(c) = \inf\{ci(x_0, \alpha, x) \mid x \in X, \alpha \geqslant c(X \setminus \{x\}\}$.

It is also easy to formulate analogues of Theorems 3.2 and 3.4.

4 Algebras for the Capacity Monad

In the sequel, a (min, max)-*idempotent biconvex compactum* is a compactum X with four operations $\bar{\oplus} : X \times X \to X$, $\otimes : I \times X \to X$, $\bar{\otimes} : X \times X \to X$, and $\oplus : I \times X \to X$ such that $(X, \bar{\oplus}, \bar{\otimes})$ is a Lawson lattice, $(X, \bar{\oplus}, \otimes)$ is an $(I, \oplus, \otimes)$-semimodule, $(X, \bar{\otimes}, \oplus)$ is an $(I, \otimes, \oplus)$-semimodule, the associative laws $(\alpha \oplus x) \bar{\oplus}$

$y = \alpha \oplus (x \bar{\oplus} y)$, $(\alpha \otimes x) \bar{\otimes} y = \alpha \otimes (x \bar{\otimes} y)$ and the distributive laws $\alpha \otimes (\beta \oplus x) = (\alpha \otimes \beta) \oplus (\alpha \otimes x)$, $\alpha \oplus (\beta \otimes x) = (\alpha \oplus \beta) \otimes (\alpha \oplus x)$ are valid for all $x, y \in X$, $\alpha, \beta \in I$.

Theorem 4.1 *Let X be any compactum. Then there is a one-to-one correspondence between:*

(1) *continuous maps $\xi : MX \to X$ such that the pair (X, ξ) is an $\mathbb{M}$-algebra;*
(2) *quadruples $(\bar{\oplus}, \otimes, \bar{\otimes}, \oplus)$ of continuous operations $\bar{\oplus} : X \times X \to X$, $\otimes : I \times X \to X$, $\bar{\otimes} : X \times X \to X$, $\oplus : I \times X \to X$ such that $(X, \bar{\oplus}, \otimes, \bar{\otimes}, \oplus)$ is a* (max, min)*-idempotent biconvex compactum;*
(3) *quadruples $(\bar{\oplus}, \otimes, p, m)$ of continuous maps $\bar{\oplus} : X \times X \to X$, $\bar{\otimes} : X \times X \to X$, $p, m : I \to X$ such that*
    (a) *$(X, \bar{\oplus}, \bar{\otimes})$ is a Lawson lattice;*
    (b) *$p : (I, \oplus) \to (X, \bar{\oplus})$ is a morphism of upper semilattices that preserves the top element;*
    (c) *$m : (I, \otimes) \to (X, \bar{\otimes})$ is a morphism of lower semilattices that preserves the bottom element;*
    (d) *for all $\alpha, \beta \in I$ we have $m(\alpha) \otimes p(\beta) = p(\alpha \otimes \beta)$, $m(\alpha) \oplus p(\beta) = m(\alpha \oplus \beta)$.*

In the case (2) *the following property of* local biconvexity *holds: for any neighborhood U of any element $x \in X$ there is a neighborhood V of x, $V \subset U$, such that $y \bar{\oplus} (\alpha \bar{\otimes} z) \in V$, $y \bar{\otimes} (\alpha \bar{\oplus} z) \in V$ for all $y, z \in V$, $\alpha \in I$.*

Proof

(1)→(3) Let (X, ξ) be an $\mathbb{M}$-algebra. We use the fact that $\mathbb{G}$ is a submonad of the capacity monad $\mathbb{M}$. The components of an embedding $i_G : \mathbb{G} \hookrightarrow \mathbb{M}$ are of the form

$$i_G X(\mathcal{A})(F) = \begin{cases} 1, & \text{if } F \in \mathcal{A}, \\ 0 & \text{— otherwise.} \end{cases}$$

Therefore $(X, \xi \circ i_G X)$ is a $\mathbb{G}$-algebra. Theorem 2 of [15] states that for a $\mathbb{G}$-algebra (X, θ) the operations $\bar{\oplus} : X \times X \to X$, $\otimes : I \times X \to X$ defined by the formulae $x \bar{\oplus} y = \theta(\eta_G X(x) \cap \eta_G X(y))$ and $x \bar{\otimes} y = \theta(\eta_G X(x) \cup \eta_G X(y))$ are such that $(X, \bar{\oplus}, \bar{\otimes})$ is a Lawson lattice. We apply this theorem to $\theta = \xi \circ i_G X$ and obtain that X with the operations $x \bar{\oplus} y = \xi(\delta_x \vee \delta_y)$ and $x \bar{\otimes} y = \xi(\delta_x \wedge \delta_y)$ is a Lawson lattice.
We denote by $\bar{0}$ and $\bar{1}$ its least and greatest elements. Now we put $p(\alpha) = \xi(\delta_{\bar{0}} \vee \alpha \otimes \delta_{\bar{1}})$, $m(\alpha) = \xi(\delta_{\bar{1}} \wedge \alpha \oplus \delta_{\bar{0}})$. It is obvious that p, m are continuous and $p(1) = \bar{0} \bar{\oplus} \bar{1} = \bar{1}$, $m(0) = \bar{1} \bar{\otimes} \bar{0} = \bar{0}$. Next, for all $\alpha, \beta \in I$:

$$\begin{aligned} p(\alpha \oplus \beta) &= \xi\left(\delta_{\bar{0}} \vee (\alpha \oplus \beta) \otimes \delta_{\bar{1}}\right) \\ &= \xi \circ \mu X \left(\delta_{\delta \bar{0} \vee \alpha \otimes \delta_{\bar{1}}} \vee \delta_{\delta \bar{0} \vee \beta \otimes \delta_{\bar{1}}}\right) \\ &= \xi \circ M\xi \left(\delta_{\delta \bar{0} \vee \alpha \otimes \delta_{\bar{1}}} \vee \delta_{\delta \bar{0} \vee \beta \otimes \delta_{\bar{1}}}\right) \\ &= \xi\left(\delta_{p(\alpha)} \vee \delta_{p(\beta)}\right) \\ &= p(\alpha) \bar{\oplus} p(\beta). \end{aligned}$$

Similarly, $m(\alpha \otimes \beta) = m(\alpha) \bar{\otimes} m(\beta)$ for all $\alpha, \beta \in I$. We also have

$$\begin{aligned} m(\alpha) \otimes p(\beta) &= \xi\left(\delta_{\xi(\delta_{\bar{1}} \wedge \alpha \oplus \delta_{\bar{0}})} \wedge \delta_{\xi(\delta_{\bar{0}} \vee \beta \otimes \delta_{\bar{1}})}\right) \\ &= \xi \circ M\xi\left(\delta_{\delta_{\bar{1}} \wedge \alpha \oplus \delta_{\bar{0}}} \wedge \delta_{\delta_{\bar{0}} \vee \beta \otimes \delta_{\bar{1}}}\right) \\ &= \xi \circ \mu X\left(\delta_{\delta_{\bar{1}} \wedge \alpha \oplus \delta_{\bar{0}}} \wedge \delta_{\delta_{\bar{0}} \vee \beta \otimes \delta_{\bar{1}}}\right) \\ &= \xi\left(\delta_{\bar{0}} \vee (\alpha \otimes \beta) \otimes \delta_{\bar{1}}\right) \\ &= p\big(\alpha \otimes \beta\big), \end{aligned}$$

as well as $m(\alpha) \oplus p(\beta) = m(\alpha \oplus \beta)$.

(3)→(2) It is sufficient to put $\alpha \otimes x = m(\alpha) \bar{\otimes} x$, $\alpha \oplus x = p(\alpha) \bar{\oplus} x$, and it is clear that all conditions of (2) are satisfied due to the commutative, associative and distributive laws in $(X, \bar{\oplus}, \bar{\otimes})$.

Observe also that, if m, p are determined by an $\mathbb{M}$-algebra (X, ξ) as described above, then

$$\begin{aligned} \alpha \otimes x &= \xi\left(\delta_{\xi(\delta_{\bar{1}} \wedge \alpha \oplus \delta_{\bar{0}})} \wedge \delta_x\right) \\ &= \xi\left(\delta_{\xi(\delta_{\bar{1}} \wedge \alpha \oplus \delta_{\bar{0}})} \wedge \delta_x\right) \\ &= \xi \circ M\xi\left(\delta_{\delta_{\bar{1}} \wedge \alpha \oplus \delta_{\bar{0}}} \wedge \delta_{\delta_x}\right) \\ &= \xi \circ \mu X\left(\delta_{\delta_{\bar{1}} \wedge \alpha \oplus \delta_{\bar{0}}} \wedge \delta_{\delta_x}\right) \\ &= \xi\left(\delta_{\bar{1}} \wedge \alpha \oplus \delta_{\bar{0}} \wedge \delta_x\right) \\ &= \xi \circ \mu X\left(\delta_{\delta_x \wedge \alpha \oplus \delta_{\bar{0}}} \wedge \delta_{\delta_{\bar{1}}}\right) \\ &= \xi \circ M\xi\left(\delta_{\delta_x \wedge \alpha \oplus \delta_{\bar{0}}} \wedge \delta_{\delta_{\bar{1}}}\right) \\ &= \xi\left(\delta_x \wedge \alpha \oplus \delta_{\bar{0}}\right) \bar{\otimes}\, \bar{1} \\ &= \xi\left(\delta_x \wedge \alpha \oplus \delta_{\bar{0}}\right), \end{aligned}$$

and similarly $\alpha \oplus x = \xi(\delta_x \vee \alpha \otimes \delta_{\bar{1}})$ for all $x \in X, \alpha \in I$. In the same manner we can show that $x \bar{\oplus}(\alpha \otimes y) = \xi(\delta_x \vee \alpha \otimes \delta_y)$, $x \bar{\otimes}(\alpha \oplus y) = \xi(\delta_x \wedge \alpha \oplus \delta_y)$ for all $x, y \in X$, $\alpha \in I$. These formulae are the same that were used to define idempotent semiconvex combinations and dual idempotent semiconvex combinations in the proofs of Theorem 3.1 and the dual theorem.

(2)→(1) Now let $(X, \bar{\oplus}, \otimes, \bar{\otimes}, \oplus)$ be a $(\min, \max)$-idempotent biconvex compactum. If $ic(x, \alpha, y) = x \bar{\oplus}(\alpha \otimes y)$, $ci(x, \alpha, y) = x \bar{\otimes}(\alpha \oplus y)$, then it is obvious that (X, ic) is a $(\min, \max)$-idempotent convex compactum and (X, ci) is a $(\max, \min)$-idempotent convex compactum. Thus by Theorem 3.1 and the dual theorem, if mappings $\xi_{\cup} : M_{\cup}X \to X$ and $\xi_{\cap} : M_{\cap}X \to X$ are defined by the formulae

$$\xi_{\cup}(c) = \sup\left\{x_0 \bar{\oplus}(\alpha \otimes x) \mid x \in X, \alpha \leqslant c(x)\right\},$$
$$c \in M_{\cup}X, x_0 \in X, c(x_0) = 1,$$

and

$$\xi_{\cap}(c) = \inf\left\{x_0 \,\bar{\otimes}(\alpha \oplus x) \mid x \in X, \alpha \geqslant c(X \setminus \{x\})\right\},$$
$$c \in M_{\cap}X, x_0 \in X, c(X \setminus \{x_0\}) = 0,$$

then the pairs $(X, \xi_{\cup})$ and $(X, \xi_{\cap})$ are an $\mathbb{M}_{\cup}$-algebra and an $\mathbb{M}_{\cap}$-algebra, respectively. In our case we can define $\xi_{\cup}, \xi_{\cap}$ by simpler but equivalent formulae (the second "=" sign in each equality is due to complete distributivity of a compact Lawson lattice):

$$\begin{aligned}\xi_{\cup}(c) &= \sup\left\{c(x) \otimes x \mid x \in X\right\}\\ &= \inf\left\{c(X \setminus A) \oplus \sup A \mid A \underset{\text{cl}}{\subset} X\right\}, \quad c \in M_{\cup}X,\end{aligned}$$

and

$$\begin{aligned}\xi_{\cap}(c) &= \inf\left\{c(X \setminus \{x\}) \oplus x \mid x \in X\right\}\\ &= \sup\left\{c(X \setminus A) \otimes \inf A \mid A \underset{\text{cl}}{\subset} X\right\}, \quad c \in M_{\cap}X.\end{aligned}$$

If $\xi, \xi' : MX \to X$ are continuous maps such that the pairs (X, ξ), (X, ξ') are $\mathbb{M}$-algebras and $\xi|_{M_{\cup}X} = \xi'|_{M_{\cup}X} = \xi_{\cup}$, $\xi|_{M_{\cap}X} = \xi'|_{M_{\cap}X} = \xi_{\cap}$, then the following two diagrams have to be commutative (we omit explicit notations for restrictions):

$$\begin{array}{ccc} M_{\cup}M_{\cap}X & \xrightarrow{\mu X} & MX \\ {\scriptstyle M_{\cup}\xi_{\cap}}\downarrow & & \downarrow{\scriptstyle \xi} \\ M_{\cup} & \xrightarrow[\xi_{\cup}]{} & X \end{array} \quad (*) \qquad \begin{array}{ccc} M_{\cap}M_{\cup}X & \xrightarrow{\mu X} & MX \\ {\scriptstyle M_{\cap}\xi_{\cup}}\downarrow & & \downarrow{\scriptstyle \xi'} \\ M_{\cap} & \xrightarrow[\xi_{\cap}]{} & X \end{array} \quad (**)$$

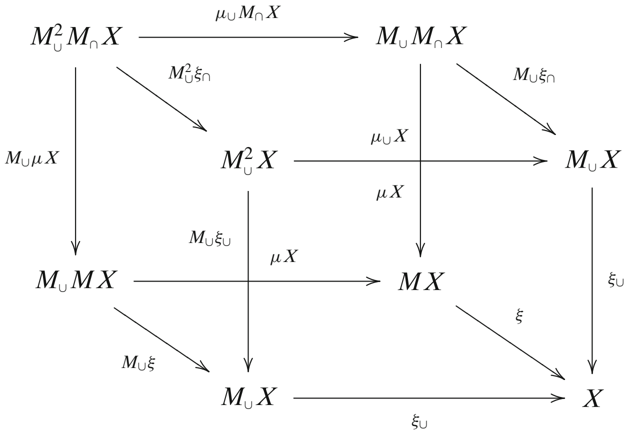
We show that if $\mathcal{C}, \mathcal{C}' \in M_{\cap}M_{\cup}X$ are such that $\mu X(\mathcal{C}) = \mu X(\mathcal{C})$, then $\xi_{\cup} \circ M_{\cup}\xi_{\cap}(\mathcal{C}) = \xi_{\cup} \circ M_{\cup}\xi_{\cap}(\mathcal{C}')$. Observe that $\mu X(\mathcal{C}) = \mu X(\mathcal{C})$ implies that for all $A \underset{\text{cl}}{\subset} X$ and $\alpha \in I$ the existence of $c \in M_{\cap}X$ such that $\mathcal{C}(c) \geqslant \alpha$ and $c(A) \geqslant \alpha$ is equivalent to the existence of $c' \in M_{\cap}X$ such that $\mathcal{C}'(c') \geqslant \alpha$ and $c'(A) \geqslant \alpha$.

It is also obvious that the same statement is valid for any open $A \subset X$. Thus:

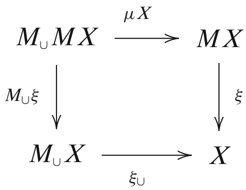
$$\begin{aligned}\xi_{\cup} \circ M_{\cup}\xi_{\cap}(\mathcal{C}) &= \sup\left\{M_{\cup}\xi_{\cap}(\mathcal{C})(x) \otimes x \mid x \in X\right\}\\ &= \sup\left\{\mathcal{C}(c) \otimes \xi_{\cap}(c) \mid c \in M_{\cap}X\right\}\\ &= \sup\left\{\mathcal{C}(c) \otimes \sup\{c(X \setminus A) \otimes \inf A \mid A \underset{\text{cl}}{\subset} X\} \mid c \in M_{\cap}X\right\}\\ &= \sup\left\{\mathcal{C}(c) \otimes c(X \setminus A) \otimes \inf A \mid A \underset{\text{cl}}{\subset} X, c \in M_{\cap}X\right\}\\ &= \sup\left\{\alpha \otimes \inf A \mid A \underset{\text{cl}}{\subset} X, c \in M_{\cap}X, \alpha \in I, \alpha \leqslant \mathcal{C}(c), \alpha \leqslant c(X \setminus A)\right\}\\ &= \sup\left\{\alpha \otimes \inf A \mid A \underset{\text{cl}}{\subset} X, c' \in M_{\cap}X, \alpha \in I, \alpha \leqslant \mathcal{C}'(c'), \alpha \leqslant c'(X \setminus A)\right\}\\ &= \dots\\ &= \xi_{\cup} \circ M_{\cup}\xi_{\cap}(\mathcal{C}').\end{aligned}$$

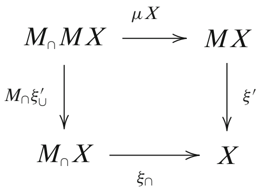
An obvious dual statement is also valid. Taking into account that by Theorem 8 of [8] for a compactum X the equality $\mu(M_\cap M_\cup X) = \mu(M_\cup M_\cap X) = MX$ is valid, and $\mu_X|_{M_\cap M_\cup X} : M_\cap M_\cup X \to MX$ and $\mu_X|_{M_\cup M_\cap X} : M_\cup M_\cap X \to MX$ are quotient maps as continuous surjective maps of compacta, we obtain that the diagrams (*) and (**) uniquely determine continuous maps $\xi, \xi' : MX \to X$.

In the diagram

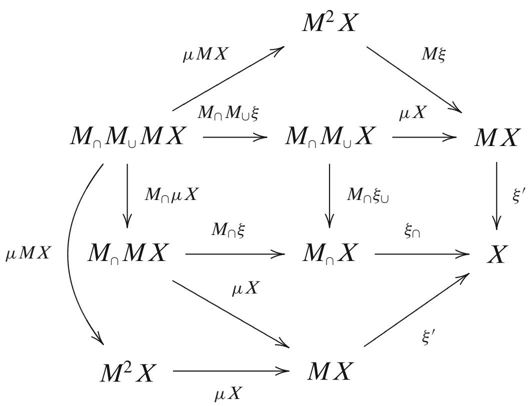


the top square and the side squares are commutative, and the leftmost vertical arrow is an epimorphsm, therefore the bottom square commutes as well. Using also dual arguments, we show that the following two diagrams are commutative:





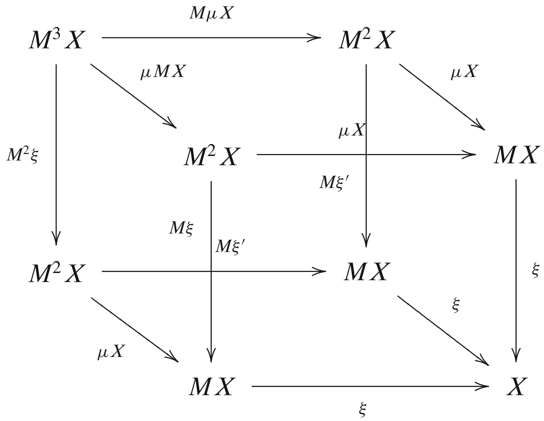
We apply the functor $M_\cap$ to the left diagram and combine it with (**):



The restriction of μMX to $M_{\cap}M_{\cup}MX$ is an epimorphism, therefore the commutativity of the outer contour implies that the left of the two following diagrams commutes. The right diagram is commutative by dual arguments.

$$\begin{array}{ccc} M^2X & \xrightarrow{\mu X} & MX \\ {\scriptstyle M\xi}\downarrow & & \downarrow{\scriptstyle \xi'} \\ MX & \xrightarrow[\xi']{} & X \end{array} \qquad \begin{array}{ccc} M^2X & \xrightarrow{\mu X} & MX \\ {\scriptstyle M\xi'}\downarrow & & \downarrow{\scriptstyle \xi} \\ MX & \xrightarrow[\xi]{} & X \end{array}$$

Therefore in the diagram



the front square is commutative, which is the "harder part" of the definition of $\mathbb{M}$-algebra. A proof of the "easier part" $\xi \circ \eta X = \mathbf{1}_X$ is straightforward. Thus (X, ξ) is a *unique* $\mathbb{M}$-algebra such that $x \,\bar{\oplus}(\alpha \otimes y) = \xi(\delta_x \vee (\alpha \otimes \delta_y))$ and $x \,\bar{\otimes}(\alpha \oplus y) = \xi(\delta_x \wedge (\alpha \oplus \delta_y))$ for all $x, y \in X$, $\alpha \in I$. As a by-product we obtain that $\xi = \xi'$, i.e. defininions of ξ by the diagrams (*) and (**) are equivalent.

To prove local biconvexity, for a given neighborhood U of a point x by continuity of ξ and the equality $\xi(\delta_x) = x$ we choose a neighborhood $\tilde{U} \subset M_{\cup}X$ of δ_x such that for all $c \in \tilde{U}$ we have $\xi(c) \in U$. There exists a neighborhood $\hat{U} \ni x$ such that for all $x_0, x_1, \ldots, x_n \in \hat{U}$, $(\alpha_0, \alpha_1, \ldots, \alpha_n) \in \Delta^n_{\oplus}$ we have $\alpha_0\delta_{x_0} \vee \alpha_1\delta_{x_1} \vee \cdots \vee \alpha_n\delta x_n \in \tilde{U}$.

Next we choose a neighborhood $\tilde{\tilde{U}} \subset M_{\cap}X$ of δ_x such that for all $c \in \tilde{U}$ we have $\xi(c) \in \hat{U}$. There is also a neighborhood $\hat{\hat{U}} \ni x$ such that $y_0, y_1, \ldots, y_n \in \hat{\hat{U}}$, $(\alpha_0, \alpha_1, \ldots, \alpha_n) \in \Delta^n_{\otimes}$ imply $\alpha_0\delta_{y_0} \wedge \alpha_1\delta_{y_1} \wedge \cdots \wedge \alpha_n\delta y_n \in \tilde{\tilde{U}}$. Now we put

$$\tilde{V} = \big\{(\alpha_{\oplus}0y_0)\,\bar{\otimes}(\alpha_1 \oplus y_1)\,\bar{\otimes}\ldots\bar{\otimes}(\alpha_n \oplus y_n) \mid$$
$$n \in \{0, 1, \ldots\}, (\alpha_0, \alpha_1, \ldots, \alpha_n) \in \Delta^n_{\otimes}, y_0, y_1, \ldots, y_n \in \hat{\hat{U}}\big\},$$

and the set

$$V = \big\{(\alpha_0 \otimes y_0)\,\bar{\oplus}(\alpha_1 \otimes y_1)\,\bar{\oplus}\ldots\bar{\oplus}(\alpha_n \otimes y_n) \mid$$
$$n \in \{0, 1, \ldots\}, (\alpha_0, \alpha_1, \ldots, \alpha_n) \in \Delta^n_{\oplus}, x_0, x_1, \ldots, x_n \in \tilde{V}\big\}$$

is a neighborhood of x requested by local bicommutativity. □

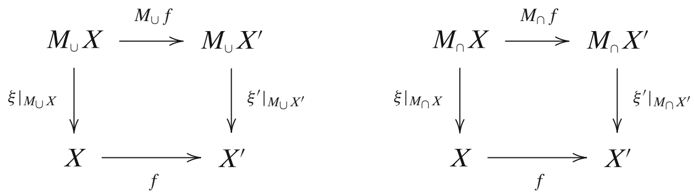
For (max, min)-idempotent biconvex compacta $(X, \bar{\oplus}, \otimes, \bar{\otimes}, \oplus)$ and $(X', \bar{\oplus}, \otimes, \bar{\otimes}, \oplus)$ we say that a map $f : X \to X'$ is *biaffine* if it preserves idempotent convex combination and the dual idempotent convex combination, i.e. $f(x \bar{\oplus}(\alpha \otimes y)) = f(x) \bar{\oplus}(\alpha \otimes f(y))$, $f(x \bar{\otimes}(\alpha \oplus y)) = f(x) \bar{\otimes}(\alpha \oplus f(y))$ whenever $x, y \in X, \alpha \in I$.

Theorem 4.2 *Let (X, ξ), (X', ξ') be $\mathbb{M}$-algebras and let quadruples $(\bar{\oplus}, \otimes, \bar{\otimes}, \oplus)$ of continuous operations be determined on X and X' by ξ and ξ' resp. (in the sense of Theorem* 4.1*).Then a continuous map $f : X \to Y$ is a morphism of $\mathbb{M}_\cup$-algebras $(X, \xi) \to (X', \xi')$ if and only if $(X, \bar{\oplus}, \otimes, \bar{\otimes}, \oplus) \to (X', \bar{\oplus}, \otimes, \bar{\otimes}, \oplus)$ is biaffine.*

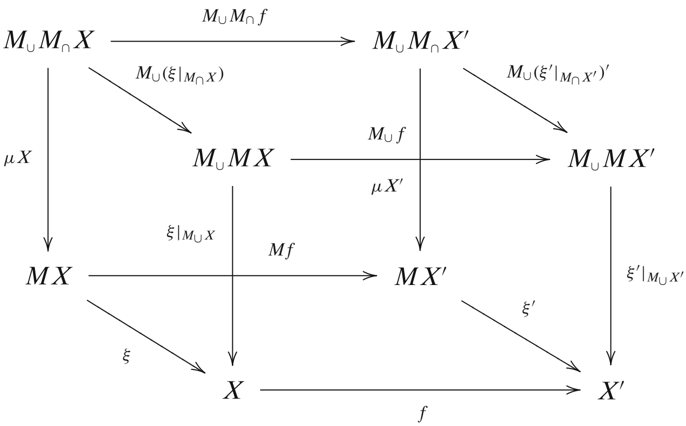
Proof

Necessity Let f be a morphism of algebras. It was shown in the proof of the previous theorem that the idempotent convex combination and the dual idempotent convex combination of points $x, y \in X$ are determined by the formulae $x \bar{\oplus}(\alpha \otimes y) = \xi(\delta_x \vee \alpha \otimes \delta_y)$, $x \bar{\otimes}(\alpha \oplus y) = \xi(\delta_x \wedge \alpha \oplus \delta_y)$ (in X' the same but ξ replaced with ξ'). Now we follow the line of the proof of Theorem 3.2.

Sufficiency Let f be biaffine. Then by Theorem 3.2 and a dual theorem, f is a morphism of $M_\cup$-algebras $(X, \xi|_{M_\cup X}) \to (X', \xi'|_{M_\cup X'})$ and a morphism of $M_\cap$-algebras $(X, \xi|_{M_\cap X}) \to (X', \xi'|_{M_\cap X'})$, i.e. the diagrams



are commutative. Therefore the top face and the side faces of the diagram



commute. The leftmost arrow $\mu X : M_\cup M_\cap X \to MX$ is an epimorphism, thus the bottom face commutes as well, i.e. f is a morphism of $\mathbb{M}$-algebras. □

Remark 4.3 The latter theorem implies that the category $\mathcal{B}i\mathcal{C}\mathrm{onv}_{\max,\min}$ of (max, min)-idempotent biconvex compacta and their continuous biaffine maps is monadic over the category of compacta.

Remark 4.4 Note that a biaffine map $f : (X, \bar{\oplus}, \otimes, \bar{\otimes}, \oplus) \to (X', \bar{\oplus}, \otimes, \bar{\otimes}, \oplus)$ not necessarily preserves operations $\oplus$ and $\otimes$ (although it preserves $\bar{\oplus}$ and $\bar{\otimes}$). For example, let $X = X' = I$, $\oplus = \bar{\oplus} = \max$, $\otimes = \bar{\otimes} = \min$, $f(x) = \max\{x, \frac{1}{2}\}$. Then f is biaffine, but $f(0 \otimes 1) = \frac{1}{2} \neq 0 \otimes f(1) = 0$. It is easy to show that a biaffine continuous map $f : (X, \bar{\oplus}, \otimes, \bar{\otimes}, \oplus) \to (X', \bar{\oplus}, \otimes, \bar{\otimes}, \oplus)$ preserves $\otimes$ iff it preserves the bottom element, and it preserves $\oplus$ if and only if it preserves the top element.

We present an example of (max, min)-idempotent biconvex compacta. Let A be a set and fix for each $a \in A$, a non-decreasing surjective map $\varphi_a : I \to I$. For $x, y \in I^A$, $x = (x_a)_{a\in A}$, $y = (y_a)_{a\in A}$, $\alpha \in I$ we put $x \bar{\oplus} y = (\max\{x_a, y_a\})_{a\in A}$, $x \bar{\otimes} y = (\min\{x_a, y_a\})_{a\in A}$, $\alpha \otimes x = (\min\{\varphi_a(\alpha), x_a\})_{a\in A}$, $\alpha \oplus x = (\max\{\varphi_a(\alpha), x_a\})_{a\in A}$. Then $(X, \bar{\oplus}, \otimes, \bar{\otimes}, \oplus)$ obviously satisfies the definition. In communication with M. Zarichnyi the following question arose:

Question 4.5 Does every (max, min)-idempotent biconvex compactum biaffinely embed into some I^A with the operations defined above?

Provided that the answer is positive, any biconvex map $f : (X, \bar{\oplus}, \otimes, \bar{\otimes}, \oplus) \to (X', \bar{\oplus}, \otimes, \bar{\otimes}, \oplus)$ algebraically (with preservation of idempotent and dual idempotent convex combinations) and topologically embeds into a biconvex map that is a projection of some I^A onto I^B, $B \subset A$ (operations on I^A and I^B are defined as above).

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