

Semigroup closures of finite rank symmetric inverse semigroups

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Abstract We introduce the notion of semigroup with a tight ideal series and investigate their closures in semitopological semigroups, particularly inverse semigroups with continuous inversion. As a corollary we show that the symmetric inverse semigroup of finite transformations \mathcal{F}_λ^n of the rank $\leq n$ is algebraically closed in the class of (semi)topological inverse semigroups with continuous inversion. We also derive related results about the nonexistence of (partial) compactifications of classes of semigroups that we consider.

Keywords Topological semigroup · Semitopological semigroup · Topological inverse semigroup · Symmetric inverse semigroup of finite transformations · Algebraically closed semigroup · ω -unstable set · Semigroup with a tight ideal series

Introduction

A partial one-to-one transformation on a set X is a one-to-one function with domain and range subsets of X (including the empty transformation with empty domain).

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There is a natural associative operation of composition on these transformations, $ab(x) = a(b(x))$ wherever defined, and the resulting semigroup is called the *symmetric inverse semigroup* \mathcal{F}_X [10, 11]. The symmetric inverse semigroup was introduced by V.V. Wagner [35] and it plays a major role in the theory of semigroups. If the domain is finite and has cardinality n , which is then also the cardinality of the range, the transformation is said to be of *rank* n . For each $n \geq 0$, the members of \mathcal{F}_X of rank less than or equal to n form an ideal of \mathcal{F}_X , denoted \mathcal{F}_X^n . (Recall that a nonempty subset I of a semigroup S is an *ideal* if $SI \cup IS \subseteq I$.) If X and Y have the same cardinality, then these respective semigroups are isomorphic, and thus we may restrict our attention to a canonical one from that class, which we take to be the one arising by taking X to be the cardinal λ . We thus label the corresponding semigroups as \mathcal{F}_λ and \mathcal{F}_λ^n respectively. The semigroups \mathcal{F}_λ^n for λ infinite form a motivating example for the considerations and developments of this paper.

Many topologists have studied topological properties of topological spaces of partial continuous maps $\mathcal{P}\mathcal{C}(X, Y)$ from a topological space X into a topological space Y with various topologies such as the Vietoris topology, generalized compact-open topology, graph topology, τ -topology, and others (see [1, 7, 12, 14, 19, 20, 23, 24]). Since the set of all partial continuous self-transformations $\mathcal{P}\mathcal{C}\mathcal{T}(X)$ of the space X with the operation composition is a semigroup, many semigroup theorists have considered the semigroup of continuous transformations (see surveys [25] and [16]), or the semigroup of partial homeomorphisms of an arbitrary topological space (see [2–5, 15, 26, 31, 36]). Beřda [6], Orlov [27, 28], and Subbiah [34] have considered semigroup and inverse semigroup topologies of semigroups of partial homeomorphisms of some classes of topological spaces. In this context the results of our paper yield some notable results about the topological behavior of the finite rank symmetric inverse semigroups sitting inside larger function space semigroups, or larger semigroups in general. For example, under reasonably general conditions, the inverse semigroup of partial finite bijections \mathcal{F}_λ^n of rank $\leq n$ is a closed subsemigroup of a topological semigroup which contains \mathcal{F}_λ^n as a subsemigroup.

The class of semigroups that we consider in this paper is a general class of semigroups that is modeled on the semigroup \mathcal{F}_λ^n and includes it as a special case. In Sect. 2 we take a closer look at this class including some categorical properties and a more general example than \mathcal{F}_λ^n .

A question of interest over the years has been to identify classes of semigroups that can be embedded in compact semigroups and classes that resist such embeddings. In Sect. 3 we consider this question in the context of the class of semigroups we are considering.

In this paper all topological spaces will be assumed to be Hausdorff. We shall follow the terminology of [8–11, 13, 29, 30]. If A is a subset of a topological space X , then we denote the closure of the set A in X by $\text{cl}_X(A)$, or simply \overline{A} if X is obvious from context. By

$$\begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n \end{pmatrix}$$

we denote a partial one-to-one transformation which maps x_1 onto y_1 , x_2 onto y_2 , \dots , and x_n onto y_n , and by 0 the empty transformation.

1 Semigroup closures

Definition 1 A subset D of a semigroup S is said to be ω -unstable if D is infinite and for any $a \in D$ and infinite subset $B \subseteq D$, we have $aB \cup Ba \not\subseteq D$.

The next lemma gives a basic example of ω -unstable sets.

Lemma 2 For λ infinite, $D := \mathcal{F}_\lambda^n \setminus \mathcal{F}_\lambda^{n-1}$ is an ω -unstable subset of \mathcal{F}_λ^n . Indeed for $a \in D$ and $B \subseteq D$ of cardinality at least $n! + 1$, $aB \cup Ba \not\subseteq D$.

Proof In order for aB to be contained in D , it must be the case that the range of each member of B equals the domain of a and in order for Ba to be contained in D , it must be the case the every member of B have the same domain as the range a . Thus there are only $n!$ possibilities for members of B . \square

Recall that a semigroup S is a *semitopological semigroup* if it is equipped with a Hausdorff topology for which all left translation maps λ_s and all right translation maps ρ_s are continuous [30]. In this case we say equivalently that the multiplication is *separately continuous*. We come now to the crucial lemma for all that follows.

Lemma 3 Let S be a *semitopological semigroup*, let T be a *subsemigroup*, let I be an *ideal* of T , and assume $D := T \setminus I$ is ω -unstable. If $s \in S$ is a *limit point* of T , then for each $t \in \overline{T}$, either $st \in \overline{T}$ or $ts \in \overline{T}$.

Proof We have $s \in \overline{T} = \overline{D} \cup \overline{T}$. If $s \in \overline{T}$, then by separate continuity of multiplication for each $t \in T$, $st \in \overline{T} \subseteq \overline{D} \cup \overline{T}$. Thus $sT \subseteq \overline{T}$, and by continuity of left translation by s , $s\overline{T} \subseteq \overline{T}$. Similarly $\overline{T}s \subseteq \overline{T}$.

For the case that $s \in \overline{D}$, but $s \notin \overline{T}$, suppose for some $t \in T$ that $st, ts \in W := S \setminus \overline{T}$. Then $t \notin I$, for otherwise $st, ts \in \overline{T}$, and therefore $t \in D$. Using the continuity of left and right translation by t , we find an open set U containing s such that $Ut \cup tU \subseteq W$. Since s is a limit point of T and S is Hausdorff, the set $B := T \cap (U \setminus \overline{T})$ is infinite. Since $B \subseteq T \setminus \overline{T} \subseteq D$ and D is ω -unstable, either Bt or tB meets I . But $tB \cup Bt \subseteq tU \cup Ut \subseteq W$, and W misses I , a contradiction. We conclude that for all $t \in T$, either $st \in \overline{T}$ or $ts \in \overline{T}$. Thus the closed set $(\lambda_s)^{-1}(\overline{T}) \cup (\rho_s)^{-1}(\overline{T})$ contains T and hence \overline{T} , which completes the proof. \square

Corollary 4 Under the hypotheses of Lemma 3, if $x, y \in \overline{T}$ and $xyx = x$, then either $x \in \overline{T}$ or $x \in T$ and x is an *isolated point* of \overline{T} .

Proof Suppose that $x \notin \overline{T}$. If $x \notin T$, then x must be a limit point of T . By Lemma 3 either $xy \in \overline{T}$ or $yx \in \overline{T}$. But then $x = (xy)x = x(yx) \in \overline{T}$, since \overline{T} is an ideal of \overline{T} by separate continuity. This contradicts our assumption that $x \notin \overline{T}$. Thus $x \in T \setminus \overline{T}$. If x is a limit point of T , then Lemma 3 would again imply that $x = (xy)x = x(yx) \in \overline{T}$, a contradiction. Thus x must be an isolated point of T and hence also of \overline{T} . \square

Definition 5 An *ideal series* (see, for example, [10, 11]) for a semigroup S is a chain of ideals

$$I_0 \subseteq I_1 \subseteq I_2 \subseteq \cdots \subseteq I_m = S.$$

We call the ideal series *tight* if I_0 is a finite set and $D_k := I_k \setminus I_{k-1}$ is an ω -unstable subset for each $k = 1, \dots, m$.

Example 6 It follows from Lemma 2 that for an infinite cardinal λ , $\{\emptyset\} \subseteq \mathcal{F}_\lambda^1 \subseteq \cdots \subseteq \mathcal{F}_\lambda^m$ is a tight ideal series for $S := \mathcal{F}_\lambda^m$.

Recall that an element x of a semigroup S is regular if there exists $y \in S$ such that $xyx = x$ and that S is regular if every element is regular [10, 11]. If $xyx = x$, then it is straightforward to verify that $x' = yxy$ is an inverse for x , i.e., $xx'x = x$ and $x'xx' = x'$. If x belongs to an ideal I , then $x' = yxy$ also belongs to the ideal, and it follows that an ideal of a regular semigroup is regular.

Proposition 7 Let S be a semitopological regular semigroup that admits a tight ideal series $I_0 \subseteq \cdots \subseteq I_m = S$. Then each I_k is closed in S and each member of $S \setminus I_{m-1}$ is an isolated point of S .

Proof We first prove by finite induction that I_k is closed in S for each k . First note that I_0 is closed since it is finite and S is Hausdorff. Assume that I_{k-1} is closed for some k , $1 \leq k \leq m$. If $s \in \overline{I_k}$, then by regularity of the ideal $\overline{I_k}$ and Corollary 4 $s \in \overline{I_{k-1}} = I_{k-1}$ or $s \in I_k$. In either case $s \in I_k$, so I_k is closed. By induction I_k is closed for all k , $0 \leq k \leq m$. The last assertion now follows from Corollary 4. \square

By Example 6 and Proposition 7 we have the following

Corollary 8 Let $\lambda \geq \omega$ and let n be any positive integer. If τ is a topology on \mathcal{F}_λ^n such that $(\mathcal{F}_\lambda^n, \tau)$ is a semitopological semigroup, then every element $\alpha \in \mathcal{F}_\lambda^n \setminus \mathcal{F}_\lambda^{n-1}$ is an isolated point of the topological space $(\mathcal{F}_\lambda^n, \tau)$.

The next proposition establishes that semigroups with tight ideal series are restricted in regard to the type of semigroups in which they may be densely embedded.

Proposition 9 Let S be a semitopological semigroup, let T be a subsemigroup, and let \overline{T} be its closure in S . If T admits a tight ideal series, then any regular element of \overline{T} , in particular any idempotent, must already be in T .

Proof Let x be a regular element of \overline{T} , and let $I_0 \subseteq \cdots \subseteq I_m = T$ be a tight ideal series for T . Let k be the smallest index such that $x \in \overline{I_k}$. We claim $x \in I_k$. If $k = 0$, then we are done since I_0 is finite, hence closed. So we assume $k \geq 1$. By hypothesis there exists $y \in \overline{T}$ such that $xyx = x$. Then $x' := yxy$ satisfies $xx'x = x$ and $x' = yxy \in \overline{I_k}$, since $\overline{I_k}$ is an ideal of \overline{T} . We apply Corollary 4 (with $T = I_k$ and $I = I_{k-1}$) and conclude that $x \in I_k \subseteq T$. \square

Recall that a semigroup is an *inverse semigroup* if it is a regular semigroup in which each element x has a unique inverse x' [29]. The semigroups \mathcal{F}_λ and \mathcal{F}_λ^n are inverse semigroups.

Proposition 10 *Let S be a semitopological inverse semigroup for which the inversion map $x \mapsto x'$ is continuous. If T is an inverse subsemigroup that admits a tight ideal series, then T is closed in S .*

Proof By separate continuity \overline{T} is a subsemigroup. Since T is closed under the inversion map and the inversion map is continuous, one readily sees that \overline{T} is closed under inversion, i.e., \overline{T} is an inverse, hence regular, subsemigroup. Proposition 9 then yields that $\overline{T} \subseteq T$, i.e., T is closed. \square

Proposition 10 applies directly to the symmetric inverse semigroup \mathcal{F}_λ^n for λ infinite and n a positive integer and yields the following corollary.

Corollary 11 *Let S be a semitopological inverse semigroup for which the inversion map $x \mapsto x'$ is continuous. If (an isomorphic copy of) \mathcal{F}_λ^n is a subsemigroup of S , then it is a closed subset of S .*

Definition 12 [17, 32] Let \mathfrak{S} be a class of topological semigroups. A topological semigroup $S \in \mathfrak{S}$ is called *H -closed in the class \mathfrak{S}* if S is a closed subsemigroup of any topological semigroup $T \in \mathfrak{S}$ which contains S as a subsemigroup. If \mathfrak{S} coincides with the class of all topological semigroups, then the semigroup S is called *H -closed*.

We remark that in [32] the H -closed semigroups are called *maximal*.

Definition 13 [17, 33] Let \mathfrak{S} be a class of topological semigroups. A semigroup S is called *algebraically closed in the class \mathfrak{S}* if for any topology τ on S such that $(S, \tau) \in \mathfrak{S}$ we have that (S, τ) is an H -closed topological semigroup in the class \mathfrak{S} . If \mathfrak{S} coincides with the class of all topological semigroups, then the semigroup S is called *algebraically closed*.

We have immediately from Corollary 11 the following corollaries.

Corollary 14 *For any infinite cardinal λ and positive integer n , the semigroup \mathcal{F}_λ^n is algebraically closed in the class of topological inverse semigroups (inverse semigroups that are topological semigroups with continuous inversion).*

Corollary 15 *Let n be any positive integer and let τ be any inverse semigroup topology on \mathcal{F}_λ^n . Then $(\mathcal{F}_\lambda^n, \tau)$ is an H -closed topological inverse semigroup in the class of topological inverse semigroups.*

The following example implies that for all $\lambda \geq \omega$, the semigroup \mathcal{F}_λ^k with the discrete topology is not H -closed in the class of all locally compact topological semigroups, for any positive integer k .

Example 16 We fix any positive integer k . Let $a \notin \mathcal{F}_\omega^k$. Let $S = \mathcal{F}_\omega^k \cup \{a\}$. We put

$$a \cdot a = a \cdot x = x \cdot a = 0$$

for all $x \in \mathcal{F}_\omega^k$.

We further enumerate the elements of the set ω by natural numbers. Let

$$A_m = \left\{ \binom{2l-1}{2l} \mid l \geq m \right\}$$

for each positive integer m . A topology τ on S is now defined as follows:

- (1) all points of \mathcal{F}_ω^k are isolated in S ; and
- (2) $\mathcal{B}(a) = \{U_n(a) = \{a\} \cup A_n \mid n = 1, 2, 3, \dots\}$ is the base of the topology τ at the point $a \in S$.

Then

- (a) for all $\begin{pmatrix} x_1 & x_2 & \dots & x_i \\ y_1 & y_2 & \dots & y_i \end{pmatrix} \in \mathcal{F}_\omega^k$ and $n \geq \max\{x_1, x_2, \dots, x_i, y_1, y_2, \dots, y_i\}$ we have

$$\begin{pmatrix} x_1 & x_2 & \dots & x_i \\ y_1 & y_2 & \dots & y_i \end{pmatrix} \cdot U_n(a) = U_n(a) \cdot \begin{pmatrix} x_1 & x_2 & \dots & x_i \\ y_1 & y_2 & \dots & y_i \end{pmatrix} = \{0\};$$

- (b) $U_n(a) \cdot U_n(a) = U_n(a) \cdot \{0\} = \{0\} \cdot U_n(a) = \{0\}$ for any positive integer n ; and
- (c) $U_n(a)$ is a compact subset of S for each positive integer n .

Therefore (S, τ) is a locally compact topological semigroup. Obviously \mathcal{F}_ω^k is not a closed subset of (S, τ) .

The following example shows that for all $\lambda \geq \omega$, the semigroup $\mathcal{F}_\lambda^\infty := \bigcup_n \mathcal{F}_\lambda^n$ with the discrete topology is not H -closed in the class of all topological inverse semigroups.

Example 17 Let $\lambda \geq \omega$ and let τ_d be the discrete topology on the semigroup $\mathcal{F}_\lambda^\infty$.

For any $\varepsilon \in E(\mathcal{F}_\lambda^\infty)$ we define

$$M(\varepsilon) = \{\chi \in \mathcal{F}_\lambda^\infty \mid \varepsilon\chi = \chi\varepsilon = \varepsilon\}.$$

Let S be the semigroup $\mathcal{F}_\lambda^\infty$ with the adjoined identity ι . We now define a topology τ_S on the semigroup S as follows:

- (i) χ is an isolated point in S for all $\chi \in \mathcal{F}_\lambda^\infty$; and
- (ii) the family

$$\mathcal{B}(\iota) = \{U_\varepsilon(\iota) = \{\iota\} \cup M(\varepsilon) \mid \varepsilon \in E(\mathcal{F}_\lambda^\infty)\}$$

is the base of the topology τ_S at the point ι .

The definition of the family $\mathcal{B}(\iota)$ implies that ι is not an isolated point of a topological space (S, τ_S) and the restriction of the topology τ_S on the set $\mathcal{F}_\lambda^\infty$ coincides with the topology τ_d .

Obviously, this is sufficient to show that the semigroup operation on (S, τ_S) is continuous in the following cases:

- (i) $u = \iota$; and
- (ii) $\iota\chi = \chi\iota = \chi$ for all $\chi \in \mathcal{F}_\lambda^\infty$.

In case (i) we have

$$U_\varepsilon(\iota) \cdot U_\varepsilon(\iota) \subseteq U_\varepsilon(\iota).$$

In case (ii) we denote

$$\chi = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n \end{pmatrix}.$$

Then we put

$$K = \{x_1\} \cup \{x_2\} \cup \cdots \cup \{x_n\} \cup \{y_1\} \cup \{y_2\} \cup \cdots \cup \{y_n\}.$$

Let be $K = \{a_1, a_2, \dots, a_k\}$. Obviously $k \leq n$. We define

$$\varepsilon = \begin{pmatrix} a_1 & a_2 & \cdots & a_k \\ a_1 & a_2 & \cdots & a_k \end{pmatrix}.$$

Then we have $\chi\varepsilon = \varepsilon\chi = \chi$ and hence

$$U_\varepsilon(\iota) \cdot \chi = U_\varepsilon(\iota) \cdot \chi = \{\chi\}.$$

Since $(U_\varepsilon(\iota))^{-1} = U_\varepsilon(\iota)$, we have that (S, τ_S) is a topological inverse semigroup which contains $\mathcal{F}_\lambda^\infty$ as dense inverse subsemigroup.

2 Semigroups with tight ideal series

We have seen in the previous section that semigroups admitting a tight ideal series have interesting closure properties in larger semigroups. In this section we take a brief closer look at this class of semigroups, primarily to see that such semigroups extend significantly beyond the finite rank symmetric inverse semigroups.

Lemma 18 *The class of semigroups admitting a tight ideal series is closed under finite products.*

Proof It suffices to check for the case $n = 2$. Let S have a tight ideal series $I_0 \subseteq \cdots \subseteq I_m = S$ and T have a tight ideal series $J_0 \subseteq \cdots \subseteq J_n = T$. Set $K_i = I_i \times J_0$ for $0 \leq i \leq m$ and $K_i = S \times J_{i-m}$ for $m < i \leq n + m$. Then any infinite $B \subseteq K_{i+1} \setminus K_i$ has an infinite projection into either S for $i \leq m$ and into T for $i > m$, and since multiplication is coordinatewise, it directly follows that $aB \cup Ba$ meets K_i . \square

Lemma 19 *Let $h: S \rightarrow T$ be a surjective semigroup homomorphism such that each point inverse $h^{-1}(t)$ is finite. If S has a tight ideal series, then so does T .*

Proof It is easy to see that $h^{-1}(I_0) \subseteq \cdots \subseteq h^{-1}(I_m) = S$ is a tight ideal series for S if $I_0 \subseteq \cdots \subseteq I_m$ is one for T . \square

Example 20 Let $\{X_i \mid i \in I\}$ be a collection of finite, pairwise disjoint sets, and let $X = \bigcup_{i \in I} X_i$. We consider the semigroup $\mathcal{PT}(X, I)$ of partial functions on X defined in the following way. First choose a subset $J \subset I$ and let $\alpha: J \rightarrow K$ be a bijection to another subset of I . A function $f: \bigcup_{i \in J} X_i \rightarrow \bigcup_{i \in K} X_i$ is by definition in $\mathcal{PT}(X, I)$ if and only if $x \in X_i$ implies $f(x) \in X_{\alpha(i)}$. All such partial functions ranging over all such α and corresponding f form the subsemigroup $\mathcal{PT}(X, I)$ under composition, a subsemigroup of the semigroup of all partial functions on X . We define such a function to have rank n if the cardinality of the range (and hence domain) of α has cardinality n and $\mathcal{PT}(X, I)^n$ to be all functions of rank less than or equal to n . We then have an ideal series for the semigroup $\mathcal{PT}(X, I)^n$ defined by $I_k = \mathcal{PT}(X, I)^k$ for $0 \leq k \leq n$.

There is a surjective homomorphism h from $\mathcal{PT}(X, I)^n$ to \mathcal{S}_I^n , which assigns to $f \in \mathcal{PT}(X, I)^n$ the corresponding $\alpha: J \rightarrow K$ between the index sets. One checks directly that this homomorphism has finite point inverses, and hence it follows from Lemma 19 that the ideal series $I_k = \mathcal{PT}(X, I)^k$ for $0 \leq k \leq n$ is tight.

We remark that the semigroup $\mathcal{PT}(X, I)^n$ is regular, but not an inverse semigroup. It is well known that the semigroup of all transformations $\mathcal{T}(X)$ is regular [10, 11], and essentially the same proof yields the regularity of $\mathcal{PT}(X, I)^n$. Thus the principal results of Sect. 1 may be applied to the semigroup $\mathcal{PT}(X, I)^n$.

3 Compact embeddings

Several authors have considered the problem of showing that various specific semigroups or classes of semigroups do or do not embed into compact semigroups. Indeed such considerations go back to very early work in the area of compact semigroups such as that of R.J. Koch and A.D. Wallace [21, 22]. For example one can use the Swelling Lemma to show that the bicyclic semigroup does not admit an embedding into a compact topological semigroup; see [8, 9] for a convenient reference to such results. Closer to our current investigations, it was shown by Gutik and Pavlyk in [18] that an infinite topological semigroup of $\lambda \times \lambda$ -matrix units B_λ does not embed into a compact topological semigroup, every non-zero element of B_λ is an isolated point of B_λ , and B_λ is algebraically closed in the class of topological inverse semigroups. (This is essentially a special case of results of this paper for \mathcal{S}_λ^1 .) However, we add a new wrinkle to earlier investigations by showing that certain partially compact embeddings do not exist, more precisely that the closure of certain embedded \mathcal{D} -classes cannot be compact.

Recall the Green's relations on a semigroup S . Two elements are \mathcal{L} -equivalent if they generate the same principal left ideal, i.e., $s\mathcal{L}t$ if $\{s\} \cup Ss = \{t\} \cup St$, and \mathcal{R} related if they generate the same principal right ideal. (In the case that s is regular the principal left ideal reduces to Ss since $s = ss's \in sS$.) The join of the equivalence relations \mathcal{L} and \mathcal{R} is denoted \mathcal{D} . It is a standard semigroup result that \mathcal{D} is alternatively given by the relational compositions $\mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$ [10, 11, Sect. 2.1]. A \mathcal{D} -equivalence class D is called a *regular \mathcal{D} -class* if it contains a regular element. This is the case if and only if each \mathcal{L} -class and each \mathcal{R} -class contained in D contains at least one idempotent if and only if every element of D is regular [10, 11, Chap. 2.3]. Furthermore, each inverse of a member of D is back in D [10, 11, Chap. 2.3].

We recall a useful fact about regular \mathcal{D} -classes.

Lemma 21 *Let $a, c \in D$, a regular \mathcal{D} -class in a semigroup S . Then there exist $s, t \in D$ such that $c = sat$.*

Proof Since $\mathcal{D} = \mathcal{R} \circ \mathcal{L}$, we may pick $b \in D$ such that $a\mathcal{R}b$ and $b\mathcal{L}c$. Pick an idempotent e in the \mathcal{R} -class of a and $u \in S$ such that $au = e$. Since $eS = aS = bS$, we have also $ea = a$ and $eb = b$. For $t = ub$, we have $at = aub = eb = b$. Furthermore, $t = ub \in Sb$ and $b = at \in St$, so $t\mathcal{L}b$, and thus $t \in D$. In a similar fashion one finds $s \in D$ such that $c = sb$. Then $c = sb = sat$ and $s, t \in D$. □

The next theorem is our main one on the non-existence of compact embeddings of certain \mathcal{D} -classes.

Theorem 22 *Let S be a topological semigroup and let T be a subsemigroup having a tight ideal series $I_0 \subseteq \dots \subseteq I_m$. If $D := I_{k+1} \setminus I_k$ is a regular \mathcal{D} -class, then $\overline{D} = \text{cl}_S(D)$ is not compact.*

Proof Suppose the contrary. Then the infinite set D has a limit point x in the compact set \overline{D} . Let x_α denote a net in $D \setminus \{x\}$ converging to x . For each α , pick an inverse x'_α , which must again be in D . By compactness of \overline{D} , some subnet of x'_α (which we again label x'_α) must converge to some $y \in \overline{D}$. By continuity of multiplication $x_\alpha = x_\alpha x'_\alpha x_\alpha \rightarrow xyx$, and by uniqueness of limits $x = xyx$. Thus x is a regular element.

Fix some $a \in D$. By Lemma 21 for each α , there exists $s_\alpha, t_\alpha \in D$ such that $s_\alpha x_\alpha t_\alpha = a$. Again passing to convergent subnets, we have $s_\alpha \rightarrow s \in \overline{D}$, $t_\alpha \rightarrow t \in \overline{D}$, and $a = s_\alpha x_\alpha t_\alpha \rightarrow sxt$. Therefore $a = sxt$. In a similar fashion, we can write $x_\alpha = u_\alpha a v_\alpha$ and conclude $x = uav$ for $u, v \in \overline{D}$.

It follows from Corollary 4 that $x \in \overline{I_k}$. Let j be the smallest index such that $x \in \overline{I_j}$. Then $j \neq 0$, for otherwise since I_0 is finite, hence closed, $x \in I_0$, and hence $a = sxt \in I_0$, contradicting the fact that $a \notin I_k$, which contains I_0 . Thus $j \geq 1$. If $x \notin I_j$, then x would be a limit point of I_j , and hence a limit point of $I_j \setminus I_{j-1}$, since $x \notin \overline{I_{j-1}}$. Again by Corollary 4 it would follow that $x \in \overline{I_{j-1}}$, a contradiction. We conclude that $x \in I_j$, and then that $a = sxt \in \overline{I_j}$, since $\overline{I_j}$ is an ideal of \overline{T} . Applying Corollary 4 to the regular element a , we conclude either that $a \in I_j$, an impossibility since $I_j \subseteq I_k$, or $a \in \overline{I_{j-1}}$. The latter would imply $x = uav \in \overline{I_{j-1}}$, which we have just seen is not the case. Thus we have reached a contradiction to our assumption that \overline{D} is compact. □

Since it is well-known and direct to verify that the sets $D = \mathcal{F}_\lambda^k \setminus \mathcal{F}_\lambda^{k-1}$ are \mathcal{D} -classes for $1 \leq k \leq n$ in the semigroup \mathcal{F}_λ^n , we have the following corollary, which generalizes the previously mentioned result of Gutik and Pavlyk in [18].

Corollary 23 *For an infinite cardinal λ and positive integer n , if \mathcal{F}_λ^n is a subsemigroup of a topological semigroup S , it cannot be the case that $\text{cl}_S(\mathcal{F}_\lambda^k \setminus \mathcal{F}_\lambda^{k-1})$ is compact for $1 \leq k \leq n$.*

We close with a theorem on $\mathcal{F}_\lambda^\infty$.

Theorem 24 *For any infinite cardinal λ there exists no topology τ on $\mathcal{F}_\lambda^\infty$ such that $(\mathcal{F}_\lambda^\infty, \tau)$ is a compact semitopological semigroup.*

Proof Suppose to the contrary, that there exists a topology τ on $\mathcal{F}_\lambda^\infty$ such that $(\mathcal{F}_\lambda^\infty, \tau)$ is a compact semitopological semigroup. The definition of the semigroup $\mathcal{F}_\lambda^\infty$ implies that for any idempotent $\varepsilon \in \mathcal{F}_\lambda^\infty$ there exists an idempotent $\phi \in \mathcal{F}_\lambda^\infty$ such that $\varepsilon\phi = \phi\varepsilon = \varepsilon$ and $\phi \neq \varepsilon$, i.e. $\varepsilon < \phi$. Therefore there exist subsets of idempotents $A = \{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n, \dots\}$ in $\phi \in \mathcal{F}_\lambda^\infty$ such that

$$0 < \varepsilon_1 < \varepsilon_2 < \dots < \varepsilon_n < \dots.$$

Without loss of generality we can assume that $\varepsilon_k \in \mathcal{F}_\lambda^k \setminus \mathcal{F}_\lambda^{k-1}$, for any $k = 2, 3, 4, \dots$ and $\varepsilon_1 \in \mathcal{F}_\lambda^1 \setminus \{0\}$. Then Corollary 8 implies that the idempotent ε_k has an open neighbourhood $U(\varepsilon_k)$ such that $U(\varepsilon_k) \cap \mathcal{F}_\lambda^k = \{\varepsilon_k\}$ for all $k = 1, 2, 3, \dots$. Since the translations in $(\mathcal{F}_\lambda^n, \tau)$ are continuous maps, the set

$$U_l(\varepsilon_k) = \{\beta \in \mathcal{F}_\lambda^\infty \mid \beta\varepsilon_k = \varepsilon_k\}$$

is clopen in the topological space $(\mathcal{F}_\lambda^\infty, \tau)$ for all $k = 1, 2, 3, \dots$. We define the family

$$\mathbb{O} = \{O_k \mid k = 1, 2, 3, \dots\}$$

as follows:

- (i) $O_1 = \mathcal{F}_\lambda^\infty \setminus U_l(\varepsilon_1)$; and
- (ii) $O_k = U_l(\varepsilon_{k-1}) \setminus U_l(\varepsilon_k)$ for all $k = 2, 3, 4, \dots$

Obviously, the family \mathbb{O} is a clopen cover of the topological space $(\mathcal{F}_\lambda^\infty, \tau)$, which does not contain a finite subcover, a contradiction. \square

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