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RESEARCH ARTICLE

Characterizing compact Clifford semigroups that embed into convolution and functor-semigroups

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Abstract We study algebraic and topological properties of the convolution semigroup of probability measures on a topological groups and show that a compact Clifford topological semigroup S embeds into the convolution semigroup $P(G)$ over some topological group G if and only if S embeds into the semigroup $\exp(G)$ of compact subsets of G if and only if S is an inverse semigroup and has zero-dimensional maximal semilattice. We also show that such a Clifford semigroup S embeds into the functor-semigroup $F(G)$ over a suitable compact topological group G for each weakly normal monadic functor F in the category of compacta such that $F(G)$ contains a G-invariant element (which is an analogue of the Haar measure on G).

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1 Introduction

According to [7] (and [19]) each (commutative) semigroup S embeds into the global semigroup $\Gamma(G)$ over a suitable (abelian) group G. The global semigroup $\Gamma(G)$ over G is the set of all non-empty subsets of G endowed with the semigroup operation $(A, B) \mapsto AB = \{ab : a \in A,\ b \in B\}$. If G is a topological group, then the global semigroup $\Gamma(G)$ contains a subsemigroup $\exp(G)$ consisting of all non-empty compact subsets of G and carrying a natural topology which makes it a topological semigroup. This is the Vietoris topology generated by the sub-base consisting of the sets

$$U^+ = \{K \in \exp(G) : K \subset U\} \quad \text{and} \quad U^- = \{K \in \exp(G) : K \cap U \neq \emptyset\}$$

where U runs over open subsets of G. Endowed with the Vietoris topology the semigroup $\exp(G)$ will be referred to as the *hypersemigroup* over G (because its underlying topological space is the hyperspace $\exp(G)$ of G, see [17]). The problem of detecting topological semigroups embeddable into the hypersemigroups over topological groups has been considered in the literature, see [7].

This problem was resolved in [5] for the class of Clifford compact topological semigroups: such a semigroup S embeds into the hypersemigroup over a topological group if and only if the set E of idempotents of S is a zero-dimensional commutative subsemigroup of S. This characterization implies the result of [8] that the closed interval $[0, 1]$ with the operation of the minimum does not embed into the hypersemigroup over a topological group.

We recall that a semigroup S is *Clifford* if S is the union of its subgroups. We say that a topological semigroup S_1 embeds into another topological semigroup S_2 if there is a semigroup homomorphism $h : S_1 \to S_2$ which is a topological embedding.

In this paper we shall apply the already mentioned result of [5] and shall characterize Clifford compact semigroups embeddable into the convolution semigroups $P(G)$ over topological groups G. The convolution semigroup $P(G)$ consists of probability Radon measures on G and carries the $*$-weak topology generated by the sub-base $\{\mu \in P(G) : \mu(U) > a\}$ where $a \in \mathbb{R}$ and U runs over open subsets of G. A measure μ defined on the σ-algebra of Borel subsets of G is called *Radon* if for every $\varepsilon > 0$ there is a compact subset $K \subset G$ with $\mu(K) > 1 - \varepsilon$. The semigroup operation on $P(G)$ is given by the convolution measures. We recall that the *convolution* $\mu * \nu$ of two measures μ, ν is the measure assigning to each bounded continuous function $f : G \to \mathbb{R}$ the value of the integral $\int_{\mu*\nu} f = \int_\nu \int_\mu f(xy)dydx$. For more detail information on the convolution semigroups, see [12, 14].

The following theorem is the principal result of this paper.

Theorem 1.1 *For any Clifford compact topological semigroup S the following assertions are equivalent*:

(1) *S embeds into the hypersemigroup* $\exp(G)$ *over a topological group* G;
(2) *S embeds into the convolution semigroup* $P(G)$ *over a topological group* G;
(3) *The set* E *of idempotents of* S *is a zero-dimensional commutative subsemigroup of* S.

This theorem will be applied to a characterization of Clifford compact topological semigroups embeddable into the hyperpsemigroups or convolution semigroups over topological groups G belonging to certain varieties of topological groups. A class $\mathcal{G}$ of topological groups is called a *variety* if it is closed under arbitrary Tychonov products, and taking closed subgroups, and quotient groups by closed normal subgroups.

Theorem 1.2 *Let* $\mathcal{G}$ *be a non-trivial variety of topological groups. For a Clifford compact topological semigroup* S *the following assertions are equivalent*:

(1) *S embeds into the hypersemigroup* $\exp(G)$ *over a topological group* $G \in \mathcal{G}$;
(2) *S embeds into the convolution semigroup* $P(G)$ *over a topological group* $G \in \mathcal{G}$;
(3) *The set* E *of idempotents is a zero-dimensional commutative subsemigroup of* S *and all closed subgroups of* S *belong to the class* $\mathcal{G}$.

In fact, the equivalence of the first and last statements in Theorems 1.1 and 1.2 was proved in Theorems 3 and 4 of [5] so it remains to prove the equivalence of the assertions (1) and (2). This will be done in Proposition 1.3.

We recall that a semigroup S is called *regular* if each element $x \in S$ is *regular* in the sense that $xyx = x$ for some $y \in S$. An element $x \in S$ is called (*uniquely*) *invertible* if there is a (unique) element $x^{-1} \in S$ (called the *inverse* of x) such that $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$. A semigroup S is called *inverse* if each element of S is uniquely invertible. By [9, 1.17], [15, II.1.2] a semigroup S is inverse if and only if it is regular and the set E of idempotents of S is a commutative subsemigroup of S. An inverse semigroup S is Clifford if and only if $xx^{-1} = x^{-1}x$ for all $x \in S$. In this case S decomposes into the union $S = \bigcup_{e\in E} H_e$ of the maximal subgroups $H_e = \{x \in S : xx^{-1} = e = x^{-1}x\}$ of S parametrized by idempotents e of S.

We recall that a topological semigroup S is called a *topological inverse semigroup* if S is an inverse semigroup and the inversion map $(\cdot)^{-1} : S \to S$, $(\cdot)^{-1} : x \mapsto x^{-1}$ is continuous. The set E of idempotents of a topological inverse semigroup S is a closed commutative subsemigroup of S called the *idempotent semilattice* of S. We say that two idempotents $e, f \in E$ are *incomparable* if their product ef differs from e and f. Two elements x, y of an inverse semigroup S are called *conjugate* if $x = zyz^{-1}$ and $y = z^{-1}xz$ for some element $z \in S$. For any idempotent $e \in E$ let $\uparrow e = \{f \in E : ef = e\}$ denote the principal filter of e. A topological space X is called *totally disconnected* if for any distinct points $x, y \in X$ there is a closed-and-open subset $U \subset X$ containing x but not y.

The following proposition shows that the semigroups $\exp(G)$ and $P(G)$ over a topological group G have the same regular subsemigroups (which are necessarily topological inverse semigroups). Moreover, regular subsemigroups of $\exp(G)$ or $P(G)$ have many specific topological and algebraic features.

Proposition 1.3 *Let* G *be a topological group. A topological regular semigroup* S *embeds into* $P(G)$ *if and only if* S *embeds into* $\exp(G)$. *If the latter happens, then*

(1) *S is a topological inverse semigroup*;
(2) *The idempotent semilattice E of S has totally disconnected principal filters $\uparrow e$, $e \in E$*;
(3) *An element $x \in S$ is an idempotent if and only if x^2x^{-1} is an idempotent*;
(4) *Any distinct conjugated idempotents of S are incomparable.*

This proposition allows one to construct many examples of topological regular semigroups non-embeddable into the hypersemigroups or convolution semigroups over a topological groups. The first two assertions of this proposition imply the result of [8] to the effect that non-trivial semigroups of left (or right) zeros as well as connected topological semilattices do not embed into the hypersemigroup $\exp(G)$ over a topological group G. The last two assertions imply that the semigroups $\exp(G)$ and $P(G)$ do not contain Brandt semigroups and bicyclic semigroups. By a *Brandt semigroup* we understand a semigroup of the form $B(H, I) = I \times H \times I \cup \{0\}$ where H is a group, I is a non-empty set, and the product $(\alpha, h, \beta) * (\alpha', h', \beta')$ of two non-zero elements of $B(H, I)$ is equal to (α, hh', β') if $\beta = \alpha'$ and 0 otherwise. A *bicyclic semigroup* is a semigroup generated by two elements p, q with the relation $qp = 1$. Brandt semigroups and bicyclic semigroups play an important role in the structure theory of inverse semigroups, see [15].

In fact, the semigroups $\exp(G)$ and $P(G)$ are special cases of the so-called functor-semigroups introduced by Teleiko and Zarichnyi [17]. They observed that any weakly normal monadic functor $F : \mathcal{C}omp \to \mathcal{C}omp$ in the category of compact Hausdorff spaces lifts to the category of compact topological semigroups, which means that for any compact topological semigroup X the space FX possesses a natural semigroup structure. The semigroup operation $*$ on FX can be defined by the following formula

$$a * b = Fp(a \otimes b) \quad \text{for } a, b \in FX$$

where $p : X \times X \to X$ is the semigroup operation of X and $a \otimes b \in F(X \times X)$ is the tensor product of the elements $a, b \in FX$, see [17, §3.4].

Therefore we actually consider in this paper the following general problem:

Problem 1.4 Given a weakly normal monadic functor $F : \mathcal{C}omp \to \mathcal{C}omp$, find a characterization of compact (regular, inverse, Clifford) topological semigroups embeddable into the semigroup FX over a compact topological group X. Given a compact topological group X describe invertible elements and idempotents of the semigroup FX.

Observe that for the functors exp and P the answer to the first part of this problem is given in Theorem 1.1. Functor-semigroups induced by the functors G of inclusion hyperspaces and λ of superextension have been studied in [2–4, 6, 11].

In fact, Theorem 1.2 also can be partly generalized to some monadic functors F (including the functors exp, P, G and λ). Given a compact topological group G let us define an element $a \in F(G)$ to be *G-invariant* if $g * a = a = a * g$ for every $g \in G$. Here we identify G with a subspace of $F(G)$ (which is possible because F, being weakly normal, preserves singletons). A G-invariant element in $F(G)$ exists for the functors exp, P, λ, and G. For the functors exp and P a G-invariant element on $F(G)$ is unique: it is $G \in \exp(G)$ and the Haar measure on G, respectively.

Theorem 1.5 *Let $F : \mathcal{C}omp \to \mathcal{C}omp$ be a weakly normal monadic functor such that for every compact topological group G the semigroup $F(G)$ contains a G-invariant element. Each Clifford compact topological inverse semigroup S with zero-dimensional idempotent semilattice E embeds into the functor-semigroup $F(G)$ over the compact topological group $G = \prod_{e\in E} \widetilde{H}_e$ where each $\widetilde{H}_e$ is a non-trivial compact topological group containing the maximal subgroup $H_e \subset S$ corresponding to an idempotent $e \in E$ of S.*

Proof By Theorem 3 of [5] (see also [13]), each Clifford compact topological inverse semigroup S with zero-dimensional idempotent semilattice E embeds into the product $\prod_{e\in E} H_e^0$, where H_e^0 stands for the extension of the maximal subgroup H_e by an isolated point $0 \notin H_e$ such that $x0 = 0x = 0$ for all $x \in H_e$. For every idempotent $e \in E$, fix a non-trivial compact topological group $\widetilde{H}_e$ containing H_e. By our hypothesis, the space $F(\widetilde{H}_e)$ contains an $\widetilde{H}_e$-invariant element $z_e \in F(\widetilde{H}_e)$. Then H_e^0 can be identified with the closed subsemigroup $H_e \cup \{z_e\}$ of $F(\widetilde{H}_e)$ and the product $\prod_{e\in E} H_e^0$ can be identified with a subsemigroup of the product $\prod_{e\in E} F(\widetilde{H}_e)$. By [17, p. 126], the latter product can be identified with a subspace (actually a subsemigroup) of $F(\prod_{e\in E} \widetilde{H}_e) = F(G)$, where $G = \prod_{e\in E} \widetilde{H}_e$. In this way, we obtain an embedding of S into $F(G)$. □

As we have said, the functors λ of superextension and G of inclusion hyperspaces satisfy the hypothesis of Theorem 1.5. However, Proposition 1.3 is specific for the functor P and cannot be generalized to the functors λ or G.

Indeed, for the 4-element cyclic group C_4 the semigroup $\lambda(C_4)$ is isomorphic to the commutative inverse semigroup $C_4 \oplus C_2^1$, where $C_2^1 = C_2 \cup \{1\}$ is the result of attaching an external unit to the 2-element cyclic group C_2, (see [6]). On the other hand, the 12-element semigroup $C_4 \oplus C_2^1$ cannot be embedded into $\exp(C_4)$ because the set of regular elements of $\exp(C_4)$ consists of 7 elements (which are shifted subgroups of C_4). Also the commutative inverse semigroup $\lambda(C_4) \cong C_4 \oplus C_2^1$ can be embedded into $G(C_4)$ (because λ is a submonad of G) but cannot embed into $\exp(C_4)$.

2 Idempotents and invertible elements of the convolution semigroups

In this section we prove Proposition 1.3. For each topological group G the semigroups $P(G)$ and $\exp(G)$ are related via the map of the support. We recall that the *support* of a Radon measure $\mu \in P(G)$ is the closed subset

$$S_\mu = \{x \in G : \mu(Ox) > 0 \text{ for each neighborhood } Ox \text{ of } x\}$$

of G. Let 2^G denote the semigroup of all non-empty closed subsets of G endowed with the semigroup operation $A * B = \overline{AB}$. By

$$\mathrm{supp} : P(G) \to 2^G, \qquad \mathrm{supp} : \mu \mapsto S_\mu$$

we denote the support map.

The following proposition is well-known, see (the proof of) Theorem 1.2.1 in [12].

Proposition 2.1 *Let G be a topological group. For any measures $\mu, \nu \in P(G)$ the following holds*: $S_{\mu*\nu} = \overline{S_\mu \cdot S_\nu}$. *This means that the support map* $\mathrm{supp} : P(G) \to 2^G$ *is a semigroup homomorphism.*

We shall show that for any regular element μ of the convolution semigroup $P(G)$ the support S_μ is compact and thus belongs to the subsemigroup $\exp(G)$ of 2^G. First, we characterize idempotent measures on a topological group G.

A measure $\mu \in P(G)$ is called an *idempotent measure* if $\mu * \mu = \mu$. In 1954 Wendel [20] proved that each idempotent measure on a compact topological group coincides with the Haar measure of some compact subgroup. Later, Wendel's result was generalized to locally compact groups by Pym [16] and to all topological groups by Tortrat [18]. By the *Haar measure* on a compact topological group G we understand the unique G-invariant probability measure on G. It is a classical result that such a measure exists and is unique. Thus we have the following characterization of idempotent measures on topological groups:

Proposition 2.2 *A probability Radon measure $\mu \in P(G)$ on a topological group G is an idempotent of the semigroup $P(G)$ if and only if μ is the Haar measure of some compact subgroup of G.*

We shall use this proposition to describe regular elements of the convolution semigroups. To this end we apply Proposition 4 of [5] that describes regular elements of the hypersemigroups over topological groups:

Proposition 2.3 (Banakh-Hryniv) *For a compact subset $K \in \exp(G)$ of a topological group G the following assertions are equivalent*:

(1) *K is a regular element of the semigroup* $\exp(G)$;
(2) *K is uniquely invertible in* $\exp(G)$;
(3) *$K = Hx$ for some compact subgroup H of G and some $x \in G$.*

A similar description of regular elements holds for the convolution semigroup:

Proposition 2.4 *For a measure $\mu \in P(G)$ on a topological group G the following assertions are equivalent*:

(1) *μ is a regular element of the semigroup $P(G)$*;
(2) *μ uniquely invertible in $P(G)$*;
(3) *$\mu = \lambda * x$ for some idempotent measure $\lambda \in P(G)$ and some element $x \in G$.*

Proof Assume that μ is a regular element of $P(G)$ and $\nu \in P(G)$ is a measure such that $\mu * \nu * \mu = \mu$. The measure $\mu * \nu$, being an idempotent of $P(G)$ coincides with the Haar measure λ on some compact subgroup H of G. It follows that $\overline{S_\mu \cdot S_\nu} = S_{\mu*\nu} = S_\lambda = H$ and hence S_μ and S_ν are compact subsets of the group G. Since $\mathrm{supp} : P(G) \to 2^G$ is a semigroup homomorphism, we get $S_\mu * S_\nu * S_\mu = S_\mu$, which means that S_μ is a regular element of the semigroup $\exp(G)$ and hence $S_\mu = \tilde{H}x$ for some compact subgroup $\tilde{H}$ and some element $x \in G$ according to Proposition 2.3.

We claim that $\tilde{H} = H$. Indeed, $H\tilde{H}x = S_\lambda S_\mu = S_{\mu*\nu} S_\mu = S_{\mu*\nu*\mu} = S_\mu = \tilde{H}x$ implies that $H \subset \tilde{H}$. Next, for any point $y \in S_\nu$ we get

$$\tilde{H}xy \subset \tilde{H}xS_\nu = S_\mu S_\nu = S_\lambda = H \subset \tilde{H}$$

which yields $xy \in \tilde{H}$ and finally $H = \tilde{H}$.

Next, we show that $\mu = \lambda * x$, which is equivalent to $\lambda = \mu * x^{-1}$. Observe that $S_{\mu * x^{-1}} = S_\mu x^{-1} = Hxx^{-1} = H$. Now the equality $\mu * x^{-1} = \lambda$ will follow as soon as we check that the measure $\mu * x^{-1}$ is H-invariant. Take any point $y \in H$ and note that

$$y * \mu * x^{-1} = y * \mu * \nu * \mu * x^{-1} = y * \lambda * \mu * x^{-1} = \lambda * \mu * x^{-1} = \mu * x^{-1},$$

which means that the measure $\mu * x^{-1}$ on H is left-invariant. Since H possesses a unique left-invariant probability measure λ, we conclude that $\mu = \lambda * x$.

Finally, we show that μ is uniquely invertible in $P(G)$. It suffices to check that the measure ν is equal to $x^{-1} * \lambda$ provided $\nu = \nu * \mu * \nu$. For this just observe that S_ν being a unique inverse of S_μ is equal to $x^{-1}H$. Then $S_{x*\nu} = xS_\nu = xx^{-1}H$. Finally, noticing that for every $y \in H$ we get

$$x * \nu * y = x * \nu * \mu * \nu * y = x * \nu * \lambda * y = x * \nu * \lambda = x * \nu,$$

which means that $x * \nu$ is a right invariant measure on H. Since λ is the unique right-invariant measure on H we also get $x * \nu = \lambda$ and hence $\nu = x^{-1} * \lambda$. □

Given a semigroup S we denote the set of regular elements of S by $\mathrm{Reg}(S)$.

Proposition 2.5 *For any topological group G, the support map*

$$\mathrm{supp} : \mathrm{Reg}(P(G)) \to \mathrm{Reg}(\exp(G))$$

is a homeomorphism.

Proof The preceding proposition implies that the map

$$\mathrm{supp} : \mathrm{Reg}(P(G)) \to \mathrm{Reg}(\exp(G))$$

is bijective. In order to check the continuity of this map, we must prove that for any open set $U \subset G$ the preimages

$$\mathrm{supp}^{-1}(U^+) = \{\mu \in \mathrm{Reg}(P(G)) : \mathrm{supp}(\mu) \subset U\} \quad \text{and}$$

$$\mathrm{supp}^{-1}(U^-) = \{\mu \in \mathrm{Reg}(P(G)) : \mathrm{supp}(\mu) \cap U \neq \emptyset\}$$

are open in $P(G)$. The openness of $\mathrm{supp}^{-1}(U^-)$ follows from the observation that $\mathrm{supp}(\mu) \cap U \neq \emptyset$ if and only if $\mu(U) > 0$. To see that $\mathrm{supp}^{-1}(U^+)$ is open, fix any measure $\mu \in \mathrm{Reg}(P(G))$ with $\mathrm{supp}(\mu) \subset U$. By Proposition 2.4, $\mathrm{supp}(\mu) = Hx$ for some compact subgroup H of G and some $x \in G$. The compactness of H allows us to find an open neighborhood V of the neutral element of G such that $HV^2HV^{-2}HV \subset Ux^{-1}$. Now consider the open neighborhood $W = \{\nu \in \mathrm{Reg}(P(G)) : \nu(HVx) > \frac{1}{2}\}$ of the measure μ. We claim that $W \subset \mathrm{supp}^{-1}(U^+)$. Indeed, given any measure $\nu \in W$ we can apply Proposition 2.4 to find an idempotent measure λ and $y \in G$ such that $\nu = \lambda * y$. Then $\frac{1}{2} < \nu(HVx) = \lambda(HVxy^{-1})$. We claim that $S_\lambda \subset HVVH$. Indeed, given an arbitrary point $z \in S_\lambda$ use the S_λ-invariance of λ to conclude that $\lambda(zHVxy^{-1}) = \lambda(HVxy^{-1}) > 1/2$, which implies that the intersection $zHVxy^{-1} \cap HVxy^{-1}$ is non-empty which yields $z \in HVxy^{-1}(HVxy^{-1})^{-1} = HVVH$. The inequality $\lambda(HVxy^{-1}) > 1/2$ implies that $HVxy^{-1}$ intersects S_λ and hence the set $HVVH$. Then $y \in HV^{-2}HHVx$ and $S_\nu = S_\lambda * y \subset HV^2HHV^{-2}HVx \subset Ux^{-1}x = U$, which implies that $\nu \in \mathrm{supp}^{-1}(U^+)$. This completes the proof of the continuity of the map $\mathrm{supp} : \mathrm{Reg}(P(G)) \to \mathrm{Reg}(\exp(G))$.

The proof of the continuity of the inverse map

$$\mathrm{supp}^{-1} : \mathrm{Reg}(\exp(G)) \to \mathrm{Reg}(P(G))$$

is even more involved. Assume that supp^{-1} is discontinuous at some point $K_0 \in \mathrm{Reg}(\exp(G))$. By Proposition 2.3, K_0 is a coset of some compact subgroup of G. After a suitable shift, we can assume that K_0 is a compact subgroup of G and then $\mu_0 = \mathrm{supp}^{-1}(K_0)$ is the unique Haar measure on K_0.

Since supp^{-1} is discontinuous at K_0, there is a neighborhood $O(\mu_0) \subset P(G)$ of μ_0 such that $\mathrm{supp}^{-1}(O(K_0)) \not\subset O(\mu_0)$ for any neighborhood $O(K_0) \subset \mathrm{Reg}(\exp(G))$ of K_0 in $\mathrm{Reg}(\exp(G))$.

It is well-known that the topology of G is generated by the left uniform structure, which is generated by bounded left-invariant pseudometrics. Each bounded left-invariant pseudometric ρ on G induces a pseudometric $\hat{\rho}$ on $P(G)$ defined by

$$\hat{\rho}(\mu_1, \mu_2) = \inf\{\mu(\rho) : \mu \in P(G \times G)\ \ P\mathrm{pr}_1(\mu) = \mu_1,\ \ P\mathrm{pr}_2(\mu) = \mu_2\}$$

where $P\mathrm{pr}_i : P(G \times G) \to P(G)$ is the map induced by the projection $\mathrm{pr}_i : G \times G \to G$ onto the ith coordinate. By [1, §4] or [10, 3.10], the topology of the space $P(G)$ is generated by the pseudometrics $\hat{\rho}$ where ρ runs over all bounded left-invariant continuous pseudometrics on G.

Consequently, we can find a left-invariant continuous pseudometric ρ on G such that the neighborhood $O(\mu_0)$ contains the ε_0-ball $B(\mu_0, \varepsilon_0) = \{\mu \in P(G) : \hat{\rho}(\mu, \mu_0) < \varepsilon_0\}$ for some $\varepsilon_0 > 0$. Replacing ρ by a larger left-invariant pseudometric, we can additionally assume that for the pseudometric space $G_\rho = (G, \rho)$ the map $\gamma : G_\rho \times G_\rho \to G_\rho, \gamma : (x, y) \mapsto xy^{-1}$, is continuous at each point $(x, y) \in K_0 \times K_0$ (this follows from the fact that for each continuous left-invariant pseudometric ρ_1 on G we can find a continuous left-invariant pseudometric ρ_2 on G such that the map $\gamma : G_{\rho_2} \times G_{\rho_2} \to G_{\rho_1}$ is continuous at points of the compact subset $K_0 \times K_0$).

The continuity and the left-invariance of the pseudometric ρ implies that the set $G_0 = \{x \in G : \rho(x, 1) = 0\}$ is a closed subgroup of G. Let $G' = \{xG_0 : x \in G\}$ be the left coset space of G by G_0 and $q : G \to G'$, $q : x \mapsto xG_0$, be the quotient projection. The space $G' = G/G_0$ will be considered as a G-space endowed with the natural left action of the group G. The pseudometric ρ induces a continuous left-invariant metric ρ' on G' such that $\rho(x, y) = \rho'(q(x), q(y))$ for all $x, y \in G$. So, $q : (G, \rho) \to (G', \rho')$ is an isometry. The pseudometrics ρ and ρ' induce the Hausdorff pseudometrics ρ_H and ρ'_H on the hyperspaces $\exp(G)$ and $\exp(G')$ such that the map $\exp q : \exp(G) \to \exp(G')$ is an isometry. Also these pseudometrics induce the pseudometrics $\hat{\rho}$ and $\hat{\rho}'$ on the spaces of measures $P(G)$, $P(G')$ such that the map $Pq : (P(G), \hat{\rho}) \to (P(G'), \hat{\rho}')$ is an isometry. The continuity of the map $\gamma : G^2_\rho \to G_\rho$ at K_0^2 implies that (K_0, ρ) is a (not necessarily separated) topological group, $K_0 \cap G_0$ is a closed normal subgroup of K_0 and hence $K'_0 = q(K_0) = K_0/K_0 \cap G_0$ has the structure of topological group. Then $\mu'_0 = Pq(\mu_0)$ is a Haar measure in K'_0.

By the choice of the neighborhood $O(\mu_0)$, for every $n \in \mathbb{N}$ we can find a compact set $K_n \in \mathrm{Reg}(\exp(G))$ such that the measure $\mu_n = \mathrm{supp}^{-1}(K_n)$ does not belong to $O(\mu_0)$. Then $\hat{\rho}(\mu_n, \mu_0) \geq \varepsilon_0$ by the choice of the pseudometric ρ.

For every $n \in \mathbb{N}$ let $\mu'_n = Pq(\mu_n) \in P(G')$, and $K'_n = q(K_n) \in \exp(G')$. The convergence of the sequence (K_n) to K_0 in the pseudometric space $(\exp(G), \rho_H)$ implies the convergence of the sequence (K'_n) to K'_0 in the metric space $(\exp(G'), \rho'_H)$, which implies that the union $K' = \bigcup_{n\in\omega} K'_n$ is compact in the metric space (G', ρ'). Then the subspace $P(K')$ is compact in the metric space $(P(G), \hat{\rho}')$ and hence the sequence $(\mu'_n)_{n\in\mathbb{N}}$ contains a subsequence that converges to some measure μ' in $(P(G'), \hat{\rho}')$. We lose no generality assuming that whole sequence $(\mu'_n)_{n\in\mathbb{N}}$ converges to μ'. Since $\varepsilon_0 \leq \hat{\rho}(\mu_n, \mu_0) = \hat{\rho}'(\mu'_n, \mu'_0)$, we conclude that $\mu' \neq \mu'_0$. We shall derive a contradiction (with the uniqueness of a left-invariant probability measure on compact groups) by showing that μ' is a left-invariant measure on K'_0, distinct from the Haar measure μ'_0.

The $\hat{\rho}'$-convergence $\mu'_n \to \mu'$ and ρ'_H-convergence $\mathrm{supp}(\mu'_n) = K'_n \to K'_0$ imply that $\mathrm{supp}(\mu') \subset K'_0$ and thus μ' is a probability measure on the compact topological group K'_0. It remains to check that the measure μ' is left-invariant. Assuming the converse, we can find a point $a \in K'_0$ such that $a * \mu' \neq \mu'$ and thus $\varepsilon = \hat{\rho}'(\mu', a * \mu') > 0$. Since the map $\gamma : G_\rho \times G_\rho \to G_\rho$ is continuous at each point $(x, y) \in K_0 \times K_0$, we can find a positive $\delta < \frac{\varepsilon}{4}$ so small that $\rho(xy, x'y) < \frac{\varepsilon}{4}$ for any $x, y \in K_0$ and $x' \in G$ with $\rho(x', x) < \delta$. Since $\rho_H(K_n, K_0) \to 0$ and $\hat{\rho}'(\mu'_n, \mu') \to 0$, there is a number $n \in \mathbb{N}$ and a point $a_n \in K_n$ such that $\rho(a, a_n) < \delta$ and $\hat{\rho}'(\mu'_n, \mu') \leq \varepsilon/4$. Consider two left shifts $l_a : G \to G$, $l_a : x \mapsto ax$, and $l_{a_n} : G \to G$. The choice of δ guarantees that $\rho_{K_0}(l_a, l_{a_n}) = \sup_{x\in K_0} \rho(l_a(x), l_{a_n}(x)) \leq \frac{\varepsilon}{4}$. Then

$$\hat{\rho}'(a * \mu', a_n * \mu') = \hat{\rho}'(Pl_a(\mu'), Pl_{a_n}(\mu')) \leq \frac{\varepsilon}{4}.$$

The left shift $l_{a_n} : G \to G$, being an isometry of the pseudometric space (G, ρ), induces an isometry $l'_{a_n} : G' \to G'$ of the metric space (G', ρ'), which induces the isometry $Pl'_{a_n} : P(G') \to P(G')$ of the corresponding space of measures. So, $\hat{\rho}'(a_n * \mu', a_n * \mu'_n) = \hat{\rho}'(Pl'_{a_n}(\mu'), Pl'_{a_n}(\mu'_n)) = \hat{\rho}'(\mu', \mu'_n) \leq \frac{\varepsilon}{4}$. The compact set K_n, being a regular element of the semigroup $\exp(G)$ is equal to $H_n x_n$ for some compact subgroup $H_n \subset G$ and some point $x_n \in G$ according to Proposition 2.3. Then $\mu_n = \mathrm{supp}^{-1}(K_n)$ is equal to $\lambda_n * x_n$ where λ_n is the Haar measure on the group H_n. Since λ_n is left-invariant, $a_n * \mu_n = a_n * \lambda_n * x_n = \lambda_n * x_n = \mu_n$ and hence $a_n * \mu'_n = \mu'_n$.

Now we see that

$$\begin{aligned}\hat{\rho}'(\mu', a * \mu') &\le \hat{\rho}'(\mu', \mu'_n) + \hat{\rho}'(\mu'_n, a_n{*}\mu'_n) + \hat{\rho}'(a_n{*}\mu'_n, a_n{*}\mu') + \hat{\rho}'(a_n{*}\mu', a{*}\mu') \\ &\le \frac{\varepsilon}{4} + 0 + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} < \varepsilon = \hat{\rho}'(\mu', a * \mu'),\end{aligned}$$

which is a desired contradiction. □

The following corollary establishes the first part of Proposition 1.3. The second part of that proposition follows from Theorem 2 of [5].

Corollary 2.6 *Let G be a topological group. Then a topological regular semigroup S can be embedded into the hypersemigroup* $\exp(G)$ *if and only if S can be embedded into the convolution semigroup $P(G)$.*

Proof If $S \subset \exp(G)$ is a regular subsemigroup, then $S \subset \mathrm{Reg}(\exp(G))$ and $\mathrm{supp}^{-1}(S)$ is an isomorphic copy of S in $P(G)$ according to Propositions 2.5. Conversely, if $S \subset P(G)$ is a regular subsemigroup, then its image $\mathrm{supp}(S)$ is an isomorphic copy of S in $\exp(G)$. □

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References

1. Banakh, T.: Topology of spaces of probability measures, II. Mat. Stud. **5**, 88–106 (1995) (in Russian)
2. Banakh, T., Gavrylkiv, V.: Algebra in superextensions of groups, II: cancelativity and centers. Algebra Discrete Math. **4**, 1–14 (2008)
3. Banakh, T., Gavrylkiv, V.: Algebra in the superextensions of groups: minimal left ideals. Mat. Stud. **31**(2), 142–148 (2009)
4. Banakh, T., Gavrylkiv, V.: Algebra in the superextensions of twinic groups. Diss. Math. **473** (2010), 74 p.
5. Banakh, T., Hryniv, O.: Embedding topological semigroups into the hyperspaces over topological groups. Acta Univ. Carol. Math. Phys. **48**(2), 3–18 (2007)
6. Banakh, T., Gavrylkiv, V., Nykyforchyn, O.: Algebra in superextensions of groups. I: zeros and commutativity. Algebra Discrete Math. **3**, 1–29 (2008)
7. Bershadskii, S.G.: Imbeddability of semigroups in a global supersemigroup over a group. In: Semigroup Varieties and Semigroups of Endomorphisms, pp. 47–49. Leningrad. Gos. Ped. Inst., Leningrad (1979)
8. Bilyeu, R.G., Lau, A.: Representations into the hyperspace of a compact group. Semigroup Forum **13**, 267–270 (1977)
9. Clifford, A.H., Preston, G.B.: The Algebraic Theory of Semigroups, vol. 1. Math. Surv., No. 7, Amer. Math. Soc., Providence (1964)
10. Fedorchuk, V.V.: Functors of probability measures in topological categories. J. Math. Sci. **91**(4), 3157–3204 (1998)
11. Gavrylkiv, V.: Right-topological semigroup operations on inclusion hyperspaces. Mat. Stud. **29**(1), 18–34 (2008)
12. Heyer, H.: Probability Measures on Locally Compact Groups. Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 94. Springer, Berlin (1977)

13. Hryniv, O.: Universal objects in some classes of Clifford topological inverse semigroups. Semigroup Forum **75**(3), 683–689 (2007)
14. Parthasarathy, K.R.: Probability Measures on Metric Spaces. Amer. Math. Soc., Providence (2005)
15. Petrich, M.: Introduction to Semigroups. Charles E. Merrill Publishing, Columbus (1973)
16. Pym, J.S.: Idempotent measures on semigroups. Pac. J. Math. **12**, 685–698 (1962)
17. Teleiko, A., Zarichnyi, M.: Categorial Topology of Compact Hausdorff Spaces. VNTL Publ., Lviv (1999)
18. Tortrat, A.: Lois de probabilité sur un espace topologique complètement régulier et produits infinis à termes indépendants dans un groupe topologique. Ann. Inst. H. Poincaré Sect. B **1**, 217–237 (1964/1965)
19. Trnkova, V.: On a representation of commutative semigroups. Semigroup Forum **10**(3), 203–214 (1975)
20. Wendel, J.G.: Haar measure and the semigroup of measures on a compact group. Proc. Am. Math. Soc. **5**, 923–929 (1954)