



Convex hyperspaces of probability measures and extensors in the asymptotic category

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ABSTRACT

The objects of the Dranishnikov asymptotic category are proper metric spaces and the morphisms are asymptotically Lipschitz maps. In this paper we provide an example of an asymptotically zero-dimensional space (in the sense of Gromov) whose space of compact convex subsets of probability measures is not an absolute extensor in the asymptotic category in the sense of Dranishnikov.

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1. Introduction

The notion of absolute extensor plays an important role in different branches of mathematics. In asymptotic topology, the absolute extensors are used in constructing the homotopy theory and the asymptotic dimension theory. Among the two categories widely used in asymptotic category, the *Dranishnikov* and the *Roe* categories (see the definition below), it turns out that it is the Dranishnikov category (the category of proper metric spaces and the asymptotically Lipschitz maps) in which a richer extensor theory can be developed.

It was proved in [8] that in general, the space of probability measures of a metric space is not an absolute extensor for the Dranishnikov category. This provided a negative answer to a question formulated by Dranishnikov [2, Problem 12], in connection with existence of the homotopy extension theorem in this category. This leads to an open problem of searching functorial constructions that preserve the class of absolute extensors in the asymptotic categories.

In the present paper we deal with the hyperspaces of compact convex subsets of probability measures. Note that these hyperspaces play an important role in the decision theory, mathematical economics and finance, in particular, in the maximum (maxmin) expected utility theory (cf. e.g. [3]).

In the case of compact metric spaces as well as in the case of compact spaces of weight ω_1 , the hyperspaces of compact convex subsets of probability measures are known to be absolute extensors [1]. However, the extension properties of these hyperspaces in the asymptotic category remained unknown. Our aim is to demonstrate that the example presented in [8]

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also works for the hyperspaces compact convex subsets of probability measures. Thus the main result of this paper is that the spaces mentioned above are not in general, asymptotic extensors in the asymptotic category.

2. Preliminaries

2.1. Asymptotic category

Together with Roe's category of proper metric spaces and coarse maps [7], the asymptotic category \mathcal{A} introduced by Dranishnikov [2] turned out to be an important universe for developing asymptotic topology.

A typical metric will be denoted by d . A map $f : X \rightarrow Y$ between metric spaces is called (λ, ε) -Lipschitz for $\lambda > 0$, $\varepsilon \geq 0$ if $d(f(x), f(x')) \leq \lambda d(x, x') + \varepsilon$ for every $x, x' \in X$. A map is called *asymptotically Lipschitz* if it is (λ, ε) -Lipschitz for some $\lambda, \varepsilon > 0$. The $(1, 0)$ -Lipschitz maps are also called *short*. The set of all short functions on a metric space X is denoted by $\text{LIP}(X)$.

A metric space is *proper* if every closed ball in it is compact. A map of metric spaces is (metrically) *proper* if the preimages of the bounded sets are bounded. The objects of the category \mathcal{A} are the proper metric spaces and the morphisms are the proper asymptotically Lipschitz maps.

A metric space Y (not necessarily an object of \mathcal{A}) is an *absolute extensor* (AE) for the category \mathcal{A} if for every proper asymptotically Lipschitz map $f : A \rightarrow Y$ defined on a closed subset of a proper metric space X there exists a proper asymptotically Lipschitz extension $\tilde{f} : X \rightarrow Y$ of f .

2.2. Asymptotic dimension

The notion of asymptotic dimension was introduced by Gromov [4]. Let X be a metric space. A family \mathcal{C} of subsets of X is said to be *uniformly bounded* if there exists $M > 0$ such that $\text{diam } A \leq M$ for every $A \in \mathcal{C}$. Given $D > 0$, we say that a family \mathcal{C} of subsets of X is *D-disjoint* if $\inf\{d(a, a') \mid a \in A, a' \in A'\} > D$ for every $A, A' \in \mathcal{C}$, $A \neq A'$.

We say that the *asymptotic dimension* of X is $\leq n$ (written $\text{asdim } X \leq n$) if for every $D > 0$ there exists a cover \mathcal{U} of X such that $\mathcal{U} = \mathcal{U}^0 \cup \dots \cup \mathcal{U}^n$, where every family \mathcal{U}^i is D -discrete. If we require in the definition of the absolute extensor that $\text{asdim } X \leq n$, then the definition of the *absolute extensor in asymptotic dimension n* (briefly $\text{AE}(n)$) is obtained.

It is easy to see that for a proper metric space X , the inequality $\text{asdim } X \leq 0$ is equivalent to the condition that for every $C > 0$ the diameters of the C -chains in X (i.e. the sequences x_1, x_2, \dots, x_k with $d(x_i, x_{i+1}) \leq C$ for every $i = 1, 2, \dots, k-1$) are bounded from above.

2.3. Convex hyperspaces of probability measures

Let $P(X)$ denote the space of probability measures of compact supports on a metrizable space X . For any $x \in X$, we denote the Dirac measure concentrated at x by δ_x . If d is a metric on X , we denote by \hat{d} the Kantorovich metric generated by d ,

$$\hat{d}(\mu, \nu) = \sup \left\{ \left| \int \varphi d\mu - \int \varphi d\nu \right| \mid \varphi \in \text{LIP}(X) \right\}$$

(cf. e.g. [5]).

By $\text{cc}P(X)$ we denote the set of all nonempty compact convex subsets in $P(X)$; as usual, a subset $A \subset P(X)$ is convex if $t\mu + (1-t)\nu \in A$, for all $\mu, \nu \in P(X)$ and $t \in [0, 1]$. The set $\text{cc}P(X)$ is endowed with the Hausdorff metric, which we shall denote by \hat{d}_H :

$$\hat{d}_H(A, B) = \inf\{r > 0 \mid A \subset O_r(B), B \subset O_r(A)\}$$

(here $O_t(Y)$ stands for the t -neighborhood of $Y \subset P(X)$). Note that, clearly, the map $x \mapsto \{\delta_x\} : X \rightarrow \text{cc}P(X)$ is an isometric embedding.

Given a map $f : X \rightarrow Y$ of metric spaces, we define the map $P(f) : P(X) \rightarrow P(Y)$ as follows: $\int \varphi dP(f)(\mu) = \int \varphi f d\mu$. The map $\text{cc}P(f) : \text{cc}P(X) \rightarrow \text{cc}P(Y)$ is then defined by the formula:

$$\text{cc}P(f)(A) = \{P(f)(\mu) \mid \mu \in A\}.$$

It can be easily seen that the map $\text{cc}P(f)$ is short if such is f .

Let $b : P(\mathbb{R}^n) \rightarrow \mathbb{R}^n$ denote the barycenter map. Recall that this map assigns to every $\mu \in P(\mathbb{R}^n)$ the unique point $b(\mu)$ with the property that $L(b(\mu)) = \int L d\mu$, for every continuous linear functional L on \mathbb{R}^n . Since b is known to be continuous and linear, the image $b(A)$ of every $A \in \text{cc}P(\mathbb{R}^n)$ is a compact convex subset of \mathbb{R}^n , i.e., an element of the space $\text{cc}(\mathbb{R}^n)$ of compact convex subsets in \mathbb{R}^n endowed with the Hausdorff metric.

Let $p : \text{cc}(\mathbb{R}^n) \rightarrow \mathbb{R}^n$ denote the map defined by the condition:

$$y = p(A) \Leftrightarrow y \in A \quad \text{and} \quad \|y\| = \inf\{\|z\| \mid z \in A\}.$$

The proof of the following statement uses simple geometric arguments and will therefore be omitted.

Lemma 2.1. *The map π is well defined and short.*

3. The example

Our example described below is a modification of the second author’s example of a proper metric space whose space of probability measures is not an AE (even AE(0)) in the asymptotic category [9]. For the sake of completeness we shall provide here the details of the construction.

For every n , the Euclidean space \mathbb{R}^n can naturally be identified with the subspace $\{(x_i) \mid x_i = 0 \text{ for all } j > n\}$ of the space ℓ^2 . We endow the subspace $X' = \bigcup_{n \in \mathbb{N}} \{n^2\} \times \mathbb{R}^n \subset \mathbb{R} \times \ell^2$ with the metric

$$d((m, (x_i)), (n, (y_i))) = (|m - n|^2 + \|(x_i) - (y_i)\|^2)^{1/2}.$$

Obviously, X is a proper metric space. For every n we denote by $p_n : X' \rightarrow \mathbb{R}^n$ a map defined by the formula $p_n(m, (x_i)) = (x_1, \dots, x_n)$. Clearly, p_n is a short map.

It was shown in [6] (cf. Theorem 1.5 therein) that for any $n \geq 2$ there exists a metric space X_n which contains the Euclidean space \mathbb{R}^n as a metric subspace and such that there is no (λ, ε) -Lipschitz retraction from X_n onto \mathbb{R}^n with $\lambda < n^{1/4}$. In the sequel we shall need an explicit construction of these spaces. Following [6], for every natural k and natural $n \geq 2$ we define graphs $G_{n,k}$ as follows: the set of vertices $V(G_{n,k})$ is the union of $I(G_{n,k})$ and $T(G_{n,k})$, where

$$I(G_{n,k}) = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n \mid |x_i| = k \text{ for all } i\},$$

$$T(G_{n,k}) = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n \mid |x_i| = 2k \text{ for all } i\};$$

the set of edges $E(G_{n,k})$ is defined by the condition: $\{x, y\} \in E(G_{n,k})$ if and only if $x, y \in V(G_{n,k})$ and either $\|x - y\| = 2k$ or $y = 2x$ (we suppose that the spaces \mathbb{R}^n are endowed with the Euclidean metric).

The set $V(G_{n,k})$ is equipped with the metric $d = d_{n,k}$,

$$d(x, y) = \inf \left\{ \sum_{i=1}^l \|x_{i-1} - x_i\|_\infty \mid (x = x_0, x_1, \dots, x_l = y) \text{ is a path in } G_{n,k} \right\}$$

(as usual, $\|x\|_\infty$ denotes the max-norm of $x \in \mathbb{R}^n$).

Define spaces X and Y as follows

$$X = \bigcup_{n=2}^\infty \bigcup_{k=n}^\infty \{n^2\} \times T(G_{n^2, k^2}), \quad Y = \bigcup_{n=2}^\infty \bigcup_{k=n}^\infty \{n^2\} \times V(G_{n^2, k^2})$$

where the metric on X is inherited from X' and the metric on Y is the maximal metric that agrees with the already defined metric on X and the metric $d_{n,k}$ on every $V(G_{n,k})$. It easily follows from the construction that X and Y are proper metric spaces, i.e. objects of the category \mathcal{A} .

We are going to show that $\text{asdim } Y = 0$ (and consequently $\text{asdim } X = 0$). Let $C > 0$ and suppose that y_1, \dots, y_m is a C -chain in Y . Denote by k the minimal natural number such that $C < (k + 1)^2$. If

$$\{y_1, \dots, y_n\} \subset \bigcup_{j=2}^k \bigcup_{l=j}^k \{j^2\} \times V(G_{j^2, l^2})$$

then $\text{diam}\{y_1, \dots, y_n\} \leq \sqrt{(k^2) + (3k^2)^2} \leq \sqrt{10C}$. Otherwise

$$\{y_1, \dots, y_n\} \cap Y \setminus \bigcup_{j=2}^k \bigcup_{l=j}^k \{j^2\} \times V(G_{j^2, l^2}) \neq \emptyset$$

and $\{y_1, \dots, y_n\}$ is a singleton.

It was proved in [6] that the following holds for the spaces

$$X_{n^2} = \mathbb{R}^{n^2} \cup \bigcup_{k=n}^\infty \{n^2\} \times V(G_{n^2, k^2})$$

endowed with the maximal metric which agrees with the initial metric on \mathbb{R}^{n^2} and the metric on $\bigcup_{k=n}^\infty \{n^2\} \times V(G_{n^2, k^2})$ inherited from Y (note that these two metrics coincide on the intersection of their domains): there is no (λ, ε) -retraction of X_{n^2} to \mathbb{R}^{n^2} with $\lambda < \sqrt{n}$.

Now, let $f : X \rightarrow \text{ccP}(X)$ be the map that sends $x \in X$ to $\{\delta_x\} \in \text{ccP}(X)$. The map f is an isometric embedding and we are going to show that there is no asymptotically Lipschitz extension of f onto the whole space Y . Assume the contrary

and let $\bar{f} : Y \rightarrow ccP(X)$ be such an extension. We regard \bar{f} as a map into $F(X') \supset ccP(X)$. Then there exist $\lambda > 0$ and $\varepsilon > 0$ such that

$$d(\bar{f}(x), \bar{f}(x')) \leq \lambda d(x, x') + \varepsilon$$

for all $x, x' \in Y$.

Let $n > \lambda^2$. Since the maps $ccP(p_{n^2})$, $b : P(\mathbb{R}^{n^2}) \rightarrow \mathbb{R}^{n^2}$ and π are short, we conclude that the map

$$x \mapsto \pi(\{b(\mu) \mid \mu \in ccP(p_{n^2})(\bar{f}(x))\}) : X_{n^2} \rightarrow \mathbb{R}^{n^2}$$

is a (λ, ε) -Lipschitz retraction from X_{n^2} onto \mathbb{R}^{n^2} , which contradicts to the choice of λ . This demonstrates that the space $ccP(X)$ is not an $AE(0)$ for the asymptotic category \mathcal{A} .

4. Epilogue

We conjecture that the spaces of capacities (non-additive measures; cf. e.g. [10,11]) are always absolute extensors in the asymptotic category \mathcal{A} . Note that the scheme of our proof of the main result of this paper does not work for non-additive situation, because one does not have the “barycenter map” in this case.

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