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ON THE ASYMPTOTIC EXTENSION DIMENSION

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We introduce an asymptotic counterpart of the extension dimension defined by Dranishnikov. The main result establishes the relationship between the asymptotic extensional dimension of a proper metric space and the extension dimension of its Higson corona.

1. Introduction

The asymptotic dimension of metric spaces was first defined by Gromov [1] for finitely generated groups. Since that time, this dimension is an object of study in numerous publications (see an expository paper [2]).

A metric space (X, d) is of asymptotic dimension $\leq n$ (written $\operatorname{asdim} X \leq n$) if, for every $D > 0$, there exists a uniformly bounded cover $\mathcal{U}$ of X such that $\mathcal{U} = \mathcal{U}^0 \cup \ldots \cup \mathcal{U}^n$, where every family $\mathcal{U}^i$ is D-disjoint, $i = 0, 1, \ldots, n$. Recall that a family $\mathcal{A}$ of subsets of X is *uniformly bounded* if

$$\operatorname{mesh} \mathcal{A} = \sup\{\operatorname{diam} A \mid A \in \mathcal{A}\} < \infty$$

(as usual, $\operatorname{diam} A = \sup\{d(x, y) \mid x, y \in A\}$ is the *diameter* of a subset A in a metric space (X, d)) and is called *D-disjoint* if

$$\inf\{d(a, a') \mid a \in A,\ a' \in A'\} > D$$

for all distinct $A, A' \in \mathcal{A}$.

The asymptotic dimension can be characterized in different ways and, in particular, in terms of the extensions of maps into Euclidean spaces [3]: A proper metric space X is of asymptotic dimension $\leq n$ if and only if any proper asymptotically Lipschitz map $f\colon A \to \mathbb{R}^{n+1}$ (see the definition in what follows) defined on a closed subset A of X admits a proper asymptotically Lipschitz extension over X. This result corresponds to the Aleksandrov theorem in the classical dimension theory: For any metric space X, $\dim X \leq n$, where dim stands for the covering dimension, if and only if any continuous map $f\colon A \to S^n$ defined on a closed subset A of X admits a continuous extension over X.

In [3, 4] Dranishnikov introduced the notion of extension dimension. This dimension takes its values in the so-called dimension types of CW-complexes. The aim of the present paper is to develop an asymptotic counterpart of the extension dimension. Our main result is a generalization of the well-known result due to Dranishnikov [3] on the equality, for the spaces of finite asymptotic dimensions, of the asymptotic dimension of a proper metric space and the dimension of the Higson corona of this space.

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2. Preliminaries

A typical metric is denoted by d. By $N_r(x)$ we denote an open ball of radius r centered at a point x of a metric space.

2.1. Asymptotic Category. A map $f: X \to Y$ between metric spaces is called (λ, ε)*-Lipschitz* for $\lambda > 0$, $\varepsilon \geq 0$ if

$$d(f(x), f(x')) \leq \lambda d(x, x') + \varepsilon$$

for any $x, x' \in X$. A map is called *asymptotically Lipschitz* if it is (λ, ε)-Lipschitz for some $\lambda, \varepsilon > 0$.

The $(\lambda, 0)$-Lipschitz maps are also called λ*-Lipschitz* and the $(1, 0)$-Lipschitz maps are also called *short*.

A metric space X is called *proper* if every closed ball is compact in X.

The *asymptotic category* $\mathcal{A}$ was introduced by A. Dranishnikov [3]. The objects of $\mathcal{A}$ are proper metric spaces and the morphisms are proper asymptotically Lipschitz maps. Recall that a map is called *proper* if the preimage of every compact set is compact.

We also need the notion of *coarse map*. A map between proper metric spaces is called *coarse uniform* if, for any $C > 0$, one can find $K > 0$ such that, for every $x,\ x' \in X$ with $d(x, x') < C$, we have $d(f(x), f(x')) < K$. A map $f: X \to Y$ is called *metric proper* if the preimage $f^{-1}(B)$ is bounded for every bounded set $B \subset Y$. A map is *coarse* if it is both metric proper and coarse uniform.

2.2. Higson Compactification and Higson Corona. Let $\varphi: X \to \mathbb{R}$ be a function defined on a metric space X. For every $x \in X$ and every $r > 0$, let

$$\mathrm{Var}_r \varphi(x) = \sup\{|\varphi(y) - \varphi(x)| \mid y \in N_r(x)\}.$$

A function φ is called *slowly oscillating* if, for any $r > 0$, we have $\mathrm{Var}_r \varphi(x) \to 0$ as $x \to \infty$ (this means that, for any $\varepsilon > 0$, there exists a compact subspace $K \subset X$ such that $|\mathrm{Var}_r \varphi(x)| < \varepsilon$ for all $x \in X \setminus K$. Let $\bar{X}$ be the compactification of X corresponding to the family of all continuous bounded slowly oscillating functions. The *Higson corona* of X is the remainder $\nu X = \bar{X} \setminus X$ of this compactification.

It is known that the Higson corona is a functor from the category of proper metric spaces and coarse maps into the category of compact Hausdorff spaces. In particular, if $X \subset Y$, then $\nu X \subset \nu Y$.

For any subset A of X, by A' we denote its trace on νX, i.e., the intersection of the closure of A in $\bar{X}$ with νX. Obviously, the set A' coincides with the Higson corona νA.

2.3. Cone. Let X be a metric space of diameter ≤ 1. The *open cone* of X is a set $\mathcal{O}X = (X \times \mathbb{R}_+)/(X \times \{0\})$ endowed with the metric (by $[x, t]$ we denote the equivalence class of $(x, t) \in X \times \mathbb{R}_+$):

$$d([x_1, t_1], [x_2, t_2]) = |t_1 - t_2| + \min\{t_1, t_2\} d(x_1, x_2).$$

For a map $f: X \to Y$ of metric spaces, by $\mathcal{O}f: \mathcal{O}X \to \mathcal{O}Y$ we denote the map defined as $\mathcal{O}f([x, t]) = [f(x), t]$.

Proposition 2.1. *If $f: X \to Y$ is a Lipschitz map, then $\mathcal{O}f$ is an asymptotically Lipschitz map.*

Proof. Suppose that a map $f: X \to Y$ is λ-Lipschitz. Then, for any $[x_1, t_1], [x_2, t_2] \in \mathcal{O}X$, we have

$$
\begin{aligned}
d(\mathcal{O}f([x_1,t_1]),\mathcal{O}f([x_2,t_2])) &= d([f(x_1),t_1],[f(x_2),t_2]) \\
&= |t_1-t_2| + \min\{t_1,t_2\}d(f(x_1),f(x_2)) \\
&\le \lambda'(|t_1-t_2| + \min\{t_1,t_2\}d(x_1,x_2)),
\end{aligned}
$$

where $\lambda' = \max\{\lambda, 1\}$.

Proposition 2.1 is proved.

The open cone of a finite CW-complex is a coarse CW-complex in a sense of [5].

Denote by $\alpha_L\colon \mathcal{O}L \to \mathbb{R}$ the function defined as $\alpha_L([x,t]) = t$. Obviously, α_L is a short function.

Let $\tilde{\mathcal{O}}L = \{[x,t] \in \mathcal{O}L \mid t \ge 1\}$. Denote by $\beta_L\colon \tilde{\mathcal{O}}L \to L$ the map $\beta_L([x,t]) = x$.

Lemma 2.1. *The map β_L is slowly oscillating.*

Proof. For $R > 0$, the R-ball centered at $[x,0]$ is $\{[x,t] \mid t < R\}$. If

$$d([x,t],[x_1,t_1]) < K < R,$$

then

$$|t-t_1| + \min\{t,t_1\}d(x,x_1) < K,$$

i.e., $(t-R)d(x,x_1) < R$ and $d(x,x_1) < K/(t-K)$. Therefore,

$$d(\beta_L(x),\beta_L(x_1)) < K/(R-K) \to 0 \qquad \text{as} \quad R \to \infty.$$

Lemma 2.1 is proved.

Let $\bar{\beta}_L\colon \tilde{\mathcal{O}}L \to L$ be the (unique) extension of the map β_L. By $\eta_L\colon \nu\tilde{\mathcal{O}}L \to L$ we denote the restriction of β_L.

Proposition 2.2. *Let $f\colon A \to \mathcal{O}L$ be a proper asymptotically Lipschitz map defined on a proper closed subset A of a proper metric space X. There exists a neighborhood W of A in X and a proper asymptotically Lipschitz map $g\colon W \to \mathcal{O}L$ with the following property: one can find constants $\lambda, s > 0$ such that*

$$\alpha_L(g(a)) \le \lambda d(a, X \setminus W) + s.$$

Proof. We can assume that L is a subset of I^n for some n and there exists a Lipschitz retraction $r\colon U \to L$ of a neighborhood U of L in I^n. Since $\mathcal{O}I^n$ is Lipschitz equivalent to $\mathbb{R}^{n+1}_+$, there exists a (λ',s')-Lipschitz extension $\tilde{g}\colon X \to \mathcal{O}I^n$ of g.

We set $W = \tilde{g}^{-1}(\mathcal{O}U)$ and $\bar{g} = \tilde{g}|W$. For every $a \in A$ and $w \in X \setminus W$, we have

$$d(g(a),\tilde{g}(w) \le \lambda' d(a,w) + s' \le \lambda' d(a, X \setminus W) + s.$$

Suppose that $d(L, I^n \setminus U) = c > 0$. Thus, since $\tilde{g}(w) \notin CU$, we get

$$d(g(a),\tilde{g}(w)) = |\alpha_L(g(a)) - \alpha_L(\tilde{g}(w))| + \min\{\alpha_L(g(a)),\alpha_L(\tilde{g}(w))\}d(\beta_L(g(a)),\beta_L(\tilde{g}(w)))$$

$$\geq |\alpha_L(g(a)) - \alpha_L(\tilde{g}(w))| + c\min\{\alpha_L(g(a)),\alpha_L(\tilde{g}(w))\} \geq c'\alpha_L(g(a)),$$

where $c' = \min\{c, 1\}$. Hence, $\alpha_L(g(a)) \leq \lambda d(a, X \setminus W) + s$, where $\lambda = \lambda'/c'$, $s = s'/c'$.

Proposition 2.2 is proved.

3. Auxiliary Results

In the present section, we collect some results required in the proof of the main result. They are proved in [3]. However, it turns out that we have also covered the case of functions with infinite values.

A map $f: X \to \mathbb{R}_+ \cup \{\infty\}$ is said to be *coarsely proper* if the preimage $f^{-1}([0, c])$ is bounded for every $c \in \mathbb{R}_+$.

Lemma 3.1. *For any function $\varphi: X \to \mathbb{R}_+$ with $\varphi(x) \to 0$ as $x \to \infty$, the function $1/\varphi: X \to \mathbb{R}_+ \cup \{\infty\}$ is coarsely proper.*

Proposition 3.1. *Let $f: X \to \mathbb{R}_+ \cup \{\infty\}$ be a coarsely proper function. Then there exists an asymptotically Lipschitz proper function $q: X \to \mathbb{R}_+$ with $q \leq f$.*

Proof. This was proved in [3] for the case of $f: X \to \mathbb{R}_+$ (see Proposition 3.5). This proof also works in our case.

Proposition 3.2. *Let $f_n: X \to \mathbb{R}_+ \cup \{\infty\}$ be a sequence of coarsely proper functions. Then there exists a filtration $X = \cup_{n=1}^{\infty} A_n$ and a coarsely proper function $f: X \to \mathbb{R}_+$ with $f|A_n \leq n$ and $f|(X \setminus A_n) \leq f_n$ for every n.*

Proof. Let $B_n = \bigcup_{i=1}^{n} f_i^{-1}([0, n])$. The sets B_i are bounded and $B_1 \subset B_2 \subset \ldots$. Therefore, there exist bounded subsets $A_1 \subset A_2 \subset \ldots$ such that $A_n \cap \left(\bigcup_{i=1}^{\infty} B_i\right) = B_n$ and $\bigcup_{i=1}^{\infty} A_i = X$. For $x \in A_n \setminus A_{n-1}$, we set $f(x) = n$. Obviously, f is coarsely proper and $f|A_n \leq n$. We now suppose that $x \notin A_n$. Then $x \notin B_n$ and, therefore, $x \notin f_n^{-1}([0, n])$, i.e., $f_n(x) > n \geq f|(X \setminus A_n)$.

Proposition 3.2 is proved.

The following assertion is an evident modification of Lemma 3.6 from [3] and its proof also works in the analyzed case.

Lemma 3.2. *Suppose that $f: A \to \mathbb{R}_+ \cup \{\infty\}$ is a coarsely proper map defined on a closed subset A of a proper metric space X and $g: W \to \mathbb{R}_+$ is a proper asymptotically Lipschitz map such that $g \leq f|W$ and there exist λ and s such that $\lambda d(a, X \setminus W) + s \geq g(a)$ for every $a \in A$. Then there exists a proper asymptotically Lipschitz map $\bar{g}: X \to \mathbb{R}_+$ for which $\bar{g} \leq f$ and $\bar{g}|A = g$.*

3.1. Almost Geodesic Spaces. A metric space X is said to be *almost geodesic* if there exists $C > 0$ such that, for any two points $x, y \in X$ there is a short map $f: [0, Cd(x, y)] \to X$ with $f(0) = x$ and $f(Cd(x, y)) = y$. If, in this definition $C = 1$, then we come to the well-known notion of *geodesic* space.

We are now going to describe the construction of embedding of a discrete metric space X into an almost geodesic space of asymptotic dimension $\min\{\text{asdim} X, 1\}$.

For an unbounded discrete metric space X with base point x_0, we define a function $f\colon X \to [0,\infty)$ by the formula $f(x) = d(x, x_0)$. Further, we choose a sequence $0 = t_0 < t_1 < t_2 < \ldots$ in $f(X)$ such that $t_{i+1} > 2t_i$ for any i. To every pair of points $x, y \in f^{-1}([t_i, t_{i+1}])$ for some i, we attach a line segment $[0, d(x, y)]$ with the indicated endpoints. Let $\hat{X}$ be the union of X and all attached segments. We endow $\hat{X}$ with the maximum metric that agrees with the initial metric on X and the standard metric on every attached segment.

Note that since X is discrete and proper, every set $f^{-1}([t_i, t_{i+1}])$ is finite and, therefore, $\hat{X}$ is a proper metric space.

Proposition 3.3. *The space $\hat{X}$ is almost geodesic.*

Proof. Suppose that $x, y \in \hat{X}$. Then $x \in [x_1, x_2]$ and $y \in [y_1, y_2]$, where $x_1, x_2, y_1, y_2 \in X$ and $[x_1, x_2]$, $[y_1, y_2]$ are attached segments. We may suppose that

$$d(x, y) = d(x, x_1) + d(x_1, y_1) + d(y_1, y).$$

Case 1: There exists i such that $x_1, y_1 \in f^{-1}([t_i, t_{i+1}])$. Then $[x, x_1] \cup [x_1, y_1] \cup [y_1, y]$ is a segment of diameter $d(x, y)$ that connects x and y in $\hat{X}$.

Case 2: $f(x_1) \in [t_i, t_{i+1}]$ and $f(y_1) \in [t_j, t_{j+1}]$, where $i \neq j$. Without loss of generality, we can assume that $i < j$.

Obviously, $d(x_1, y_1) \le d(x, y)$. Since $|t_j - t_{j-1}| \le d(x_1, y_1)$, we see that $|t_j - t_{j-1}| \le d(x, y)$. This implies that $t_j/2 \le d(x, y)$ or, equivalently, $t_j \le d(x, y)$.

Moreover,

$$d(y_1, f^{-1}([0, t_{j-1}]))) \le d(x_1, y_1) \le d(a, b).$$

For every $k = i, i+1, \ldots, j_1$, we choose $z_k \in f^{-1}(t_k)$. Then

$$d(y_1, z_{j-1}) \le d(y_1, f^{-1}([0, t_{j-1}]) + \operatorname{diam}(f^{-1}([0, t_{j-1}])) \le d(a, b) + 2t_{j-1} \le d(a, b) + t_j \le 3d(a, b).$$

We connect x and y by the segment

$$J = [x, x_1] \cup [x_1, z_1] \cup \bigcup\nolimits_{k=i}^{j-1} [z_k, z_{k+1}]) \cup [z_{j-1}, y_1] \cup [y_1, y].$$

Then

$$\operatorname{diam} J \le d(x, x_1) + d(x_1, z_{i+1}) + \left(\sum_{k=i+1}^{j-1} d(z_k, z_{k+1}) \right) + d(z_{j-1}, y_1) + d(y_1, y)$$

$$= d(x, y) + 2t_{i+1} + \sum_{k=i+1}^{j-1} 2t_{k+1} + 5d(x, y) + d(x, y)$$

$$\le 7d(x, y) + 2(t_{i+1} + \ldots + t_j) \le 7d(x, y) + 4t_j \le 15d(x, y).$$

Proposition 3.3 is proved.

We need a version of the fact proved in [3] for geodesic spaces.

Proposition 3.4. *Let $f: X \to Y$ be a coarse uniform map of an almost geodesic space X. Then f is asymptotically Lipschitz.*

Proof. Let C be a constant from the definition of almost geodesic space. Suppose that $x, y \in X$. Then there exists a short map $\alpha: [0, Cd(x,y)] \to X$ such that $\alpha(0) = x$ and $\alpha(Cd(x,y)) = y$. There are points $0 = t_0 < t_1 < \ldots < t_{k-1} < t_k = Cd(x,y)$, where $k \le [d(x,y)] + 1$, such that $|t_i - t_{i-1}| \le C$ for any $i = 1, \ldots, k$.

Since f is coarse uniform, there exists $R > 0$ such that $d(f(x'), f(y')) < R$ whenever $d(x', y') \le C$. Then

$$d(f(x), f(y)) \le \sum_{i=1}^{k} d(f(\alpha(t_i)), f(\alpha(t_{i-1}))) \le kR \le ([d(x,y)] + 1)R \le Rd(x,y) + 2R.$$

Proposition 3.4 is proved.

4. Asymptotic Extension Dimension

Let P be an object of category $\mathcal{A}$. For any object X of $\mathcal{A}$, the *Kuratowski notation* $X\tau P$ means the following: For any proper asymptotically Lipschitz map $f: A \to P$ defined on a closed subset A of X, there is a proper asymptotically Lipschitz extension of f onto X.

Denote by $\mathcal{L}$ the class of compact absolute Lipschitz neighborhood Euclidean extensors (ALNER). Following [4], we define a preordering relation $\le$ on $\mathcal{L}$. For $L_1, L_2 \in \mathcal{L}$, we have $L_1 \le L_2$ if and only if $X\tau\mathcal{O}L_1$ implies that $X\tau\mathcal{O}L_2$ for all proper metric spaces X. The indicated preordering relation leads to the following equivalence relation $\sim$ on $\mathcal{L}$: $L_1 \sim L_2$ if and only if $L_1 \le L_2$ and $L_2 \le L_1$. By $[L]$ we denote the equivalence class containing $L \in \mathcal{L}$. The class $[L]$ is called the *type of asymptotic extension dimension* for L. The indicated preordering relation induces a partial ordering relation for all types of asymptotic extension dimensions.

For a proper metric space X, we say that its *asymptotic extension dimension does not exceed* $[\mathcal{O}L]$ (or, briefly, as-ext-dim $X \le [\mathcal{O}L]$), whenever $X\tau\mathcal{O}L$.

If as-ext-dim $X \le [\mathcal{O}L]$, then the equality as-ext-dim $X = [\mathcal{O}L]$ means the following: If we also have as-ext-dim $X \le [\mathcal{O}L']$, then $[\mathcal{O}L] \le [\mathcal{O}L']$.

According to the results of extension of asymptotically Lipschitz functions ([3]; see also [6]), the element $[*]$ is maximal.

Theorem 4.1. *Let L be a compact metric ALNER. The following conditions are equivalent:*

(1) as-ext-dim $X \le [\mathcal{O}L]$;

(2) ext-dim $\nu X \le [L]$.

Proof. (1) $\Rightarrow$ (2). Assume that as-ext-dim $X \le [\mathcal{O}L]$. Let $\varphi: C \to L$ be a map defined on a closed subset C of νX. Since $L \in$ ANE, there exists an extension $\varphi': V \to L$ of φ over a closed neighborhood V of C in $\bar{X} = X \cup \nu X$. Then $\mathrm{Var}_R\varphi'(x) \to 0$ as $x \to \infty$ for any fixed $R > 0$. By Lemma 3.1, the function

$$f_n: V \cap X \to \mathbb{R}_+ \cup \{\infty\}, \qquad f_n(x) = \frac{1}{\mathrm{Var}_R\varphi'(x)},$$

is coarsely proper for every $n \in \mathbb{N}$. By Proposition 3.2, there is a coarsely proper function $f: V \cap X \to \mathbb{R}_+$ and a filtration $V \cap X = \bigcup_{n=1}^{\infty} A_n$ such that $f|A_n \le n$ and $f|(X \setminus A_n) \le f_n$. By Proposition 3.5 from [3], there is an asymptotically Lipschitz function $q\colon V \cap X \to \mathbb{R}_+$ with $q \le f$. We assume that q is (λ, s)-Lipschitz for some $\lambda, s > 0$. A map $g\colon V \cap X \to \mathcal{O}L$ is defined by the formula $g(x) = [\varphi'(x), q(x)]$.

We are now going to check that the map $g(x)$ is asymptotically Lipschitz. Let $x, y \in V \cap X$ and $n - 1 \le d(x, y) \le n$.

Suppose that $x, y \in (V \cap X) \setminus A_n$. Then $q(x) \le f_n(x)$ and $q(y) \le f_n(y)$. We have

$$d(g(x), g(y)) = |q(x) - q(y)| + \min\{q(x), q(y)\} d(\varphi'(x), \varphi'(y))$$

$$\le \lambda d(x, y) + s + \min\{q(x), q(y)\} \mathrm{Var}_n \varphi'(x) \le \lambda d(x, y) + s + 1.$$

If $x \in A_n$, then $q(x) \le n$ and we find

$$d(g(x), g(y)) \le \lambda d(x, y) + s + n d(\varphi'(x), \varphi'(y))$$

$$\le \lambda d(x, y) + s + n \mathrm{diam}\, L \le \lambda d(x, y) + s + (d(x, y) + 1) \mathrm{diam}\, L$$

$$\le (\lambda + \mathrm{diam}\, L) d(x, y) + (s + \mathrm{diam}\, L).$$

For $y \in A_n$, the arguments are similar.

Further, by assumption, there is an asymptotically Lipschitz extension $\bar{g}\colon X \to \mathcal{O}L$ of g. Consider a composition $\eta_L \nu \bar{g}\colon \nu X \to \mathcal{O}L$. Clearly, $\eta_L \nu \bar{g}|C = \varphi$. We conclude that ext-dim$\nu X \le [L]$.

(2) $\Rightarrow$ (1). Let $f\colon A \to \mathcal{O}L$ be an asymptotically Lipschitz map defined on a proper closed subset A of a proper metric space X. By Proposition 2.2, there is a proper asymptotically Lipschitz map $\tilde{f}\colon W \to \mathcal{O}L$ defined in a neighborhood W of A and constants λ and s such that

$$\alpha_L f(a) \le \lambda d(a, X \setminus W) + s$$

for all $a \in A$. Denote by $\varphi\colon \nu X \to L$ an extension of the composition $\eta_L \nu \tilde{f}$. Since L is an absolute neighborhood extensor, there exists an extension $\psi\colon V \to L$ of φ onto a closed neighborhood of νX in the Higson compactification $\bar{X}$. We extend ψ to a map $\hat{\psi}\colon (V \cap X)\hat{} \to L$ as follows: Let J be a segment attached to V with endpoints a and b. We require that $\hat{\psi}$ must linearly map J onto a geodesic segment in L with endpoints $\psi(a)$ and $\psi(b)$.

We now show that $\hat{\psi}$ is a slowly oscillating map. Since ψ is slowly oscillating, for any $\varepsilon > 0$ and $R > 0$, there exists $K > 0$ such that $\mathrm{Var}_R \psi(x) < \varepsilon$ whenever $d(x, x_0) > K$. Suppose that $\hat{\psi}$ is not slowly oscillating. Then there exist $R > 0$, $C > 0$, and sequences (x_1^i) and (x_2^i) in $(V \cap X)\hat{}$ such that $d(x_1^i, x_2^i) < R$, $x_1^i \to \infty$, $x_2^i \to \infty$ and $d(\hat{\psi}(x_1^i), \hat{\psi}(x_2^i)) > C$ for any i. We assume that $x_1^i \in [a_1^i, b_1^i]$ and $x_2^i \in [a_2^i, b_2^i]$ for every i, where $a_1^i, b_1^i, a_2^i, b_2^i \in X \cap V$. Without loss of generality we can assume that $a_1^i \to \infty$ and there exists $C_1 > 0$ such that $d(\hat{\psi}(x_1^i), \hat{\psi}(a_1^i)) > C_1$ for every i. If $d(a_1^i, b_1^i) < K$ for all i and some $K > 0$, then

$$d(\hat{\psi}(x_1^i), \hat{\psi}(a_1^i)) < d(\hat{\psi}(a_1^i), \hat{\psi}(b_1^i)) \to 0,$$

and we arrive at a contradiction. Hence, we can assume that $d(a_1^i, b_1^i) \to \infty$. Then

$$d(a_1^i, x_1^i)/d(a_1^i, b_1^i) < R/d(a_1^i, b_1^i) \to 0$$

and, therefore, by the definition of the map $\hat{\psi}$, we get

$$d(\hat{\psi}(x_1^i), \hat{\psi}(a_1^i))/d(\hat{\psi}(a_1^i), \hat{\psi}(b_1^i)) \to 0.$$

Thus, clearly, $d(\hat{\psi}(x_1^i), \hat{\psi}(a_1^i)) \to 0$, and we arrive at a contradiction.

Since the map $\tilde{f}$ is asymptotically Lipschitz, there exists $K > 0$ such that, for any $a \in W$, we have

$$\operatorname{diam}\,(\alpha_L \tilde{f}(N_1(a)) + \alpha_L \tilde{f}(a) \operatorname{diam}\,(\psi(N_1(a)) \leq K.$$

We define a function $r\colon (X \cap V\hat{)} \to \mathbb{R}_+ \cup \{\infty\}$ by the formula $r(x) = K/(\psi(N_1(x)))$. We have $f(a) \leq r(a)$ for any $a \in A$. The function r is asymptotically proper and, by Proposition 3.1, there exists a (λ', s')-Lipschitz function $\bar{f}\colon X \to \mathbb{R}_+$ for some λ', s' with $\bar{f} \leq r$ and $\bar{f}|A = \alpha_L f$.

We define a map $g\colon (X \cap V\hat{)} \to \mathcal{O}L$ by the formula $g(x) = (\psi(x), \bar{f}(x))$. Obviously, $g|A = f$. We are now going to show that g is a coarse uniform map.

Suppose that $x, y \in X$, $d(x, y) < 1$. Then

$$d(g(x), g(y)) \leq |\bar{f}(x) - \bar{f}(y)| + \min\{\bar{f}(x), \bar{f}(y)\} d(\psi(x), \psi(y)) \leq \lambda' + s' + K.$$

Note that, since $\bar{f}$ is proper, g is also proper. Since g is coarse uniform, by Proposition 3.4, g is asymptotically Lipschitz. Therefore, as-ext-dim $X \leq [\mathcal{O}L]$.

Theorem 4.1 is proved.

Corollary 4.1 (finite-sum theorem)**.** *Assume that X is a proper metric space and $X = X_1 \cup X_2$, where X_1 and X_2 are closed subsets of X with* as-ext-dim $X_i \leq [\mathcal{O}L]$, $i = 1, 2$, *for some $L \in \mathcal{L}$. Then* as-ext-dim $X \leq [\mathcal{O}L]$.

Proof. Since $\nu X = \nu X_1 \cup \nu X_2$, the result follows from Theorem 4.1 and the finite-sum theorem for the extension dimension (see [7]).

5. Remarks and Open Problems

Problem 5.1. *Is the following equality true: as-ext-*dim $\mathbb{R}^n = S^n$ *?*

Problem 5.2. *Let L_1, L_2 be finite polyhedra in Euclidean spaces endowed with the induced metric. Is the inequality $[L_1] \leq [L_2]$ introduced in [4] equivalent to the inequality $[L_1] \leq [L_2]$ in Sec. 4?*

We can define a counterpart of the asymptotic extension dimension by using warped cones instead of open cones. Following [8], we now briefly review this construction. Let $\mathcal{F}$ be a foliation on a compact smooth manifold V. Also let N be an arbitrary subbundle complementary to $T\mathcal{F}$ in TM. We choose Euclidean metrics g_N in N and $g_{\mathcal{F}}$ in $T\mathcal{F}$. The *foliated warped cone* $\mathcal{O}_{\mathcal{F}}$ is the manifold $V \times [0, \infty)/V \times \{0\}$ equipped with the metric induced for $t \geq 1$ by the Riemannian metric $g_R + g_{\mathcal{F}} + t^2 g_N$. Since we are interested in the asymptotic properties of warped cones, the metric structure of any bounded neighborhood of the apex of the cone is irrelevant.

Problem 5.3. *Is the obtained warped cone an absolute neighborhood extensor in the asymptotic category?*

The affirmative answer to this question would allow us to introduce the asymptotic extension dimension theory with values in warped cones.

Problem 5.4. *Is it possible to characterize the dimension of the sublinear corona (see [9]) in terms of the asymptotic extension dimension?*

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