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Coideals as remainders of groups distinguishing between combinatorial covering properties ☆

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ABSTRACT

In this paper we construct consistent examples of subgroups of 2^ω with Menger remainders which fail to have other stronger combinatorial covering properties. This answers several open questions asked by Bella, Tokgoz and Zdomskyy (Arch. Math. Logic **55** (2016), 767–784).

1. Introduction

All topological spaces will be assumed to be Tychonoff. For a space X and its compactification bX the complement $bX \setminus X$ is called a *remainder* of X.

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The study of the interplay between covering properties of Tychonoff spaces and those of their remainders takes its origin in the seminal work of Henriksen and Isbell [16]. Being highly homogeneous objects, topological groups restrict the properties of their remainders in a special, strong way, unlike general topological spaces. The study of this phenomenon in our context was started by Arhangel'skii [1,2] and further pursued in his joint works with Choban, van Mill, and others, see [3,4,6,5] and the references therein.

This motivated the study in [8] of topological groups whose remainders have combinatorial covering properties which lie between the σ-compactness and Lindelöf property. Recall from [26] that a space X is said to be *Menger* (or has the *Menger property*) if for any sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of open covers of X one can pick finite sets $\mathcal{V}_n \subset \mathcal{U}_n$ in such a way that $\{\bigcup \mathcal{V}_n : n \in \omega\}$ is a cover of X. A family $\{W_n : n \in \omega\}$ of subsets of X is called an ω*-cover* of X, if for every $F \in [X]^{<\omega}$, the set $\{n \in \omega : F \subset W_n\}$ is infinite. The property of *Scheepers* is defined in the same way as the Menger property, the only difference being that we additionally demand that $\{\bigcup \mathcal{V}_n : n \in \omega\}$ is a ω-cover of X. It is immediate that

$$\sigma\text{-compact} \Rightarrow \text{Scheepers} \Rightarrow \text{Menger} \Rightarrow \text{Lindelöf}.$$

Through a sequence of reductions it was proved in [8] that there exists a Scheepers ultrafilter if and only if there exists a topological group G such that $\beta G \setminus G$ is Scheepers and not σ-compact if and only if there exists a topological group G such that all finite powers of $\beta G \setminus G$ are Menger and are not σ-compact. Here, $\mathcal{P}(\omega)$ is as usually identified with the Cantor space 2^ω via characteristic functions, and subsets of $\mathcal{P}(\omega)$ are considered with the subspace topology. Thus the existence of a topological group G such that $\beta G \setminus G$ is Scheepers (resp. has all finite powers Menger) and not σ-compact is independent from ZFC: Such a group exists under $\mathfrak{d} = \mathfrak{c}$, and its existence yields P-points, see [8] and the references therein. Furthermore, it was proved in [8] that the existence of a topological group G such that $(\beta G \setminus G)^2$ is Menger but not σ-compact is independent from ZFC as well.

Since the same approach does not allow to solve the question whether consistently every Menger remainder of a topological group is σ-compact,[1] it was asked in [8] whether the Scheepers and Menger properties can be distinguished by remainders of topological groups at all, and whether a Menger remainder of a topological group can have a non-Menger square. In case of a negative answer outright in ZFC this would give the consistency of all Menger remainders of topological groups being σ-compact, simply by applying the aforementioned results. However, we provide here two alternative proofs that both of these questions have consistently the affirmative answer, by constructing counterexamples using different approaches, see Theorems 2.4, 3.1, and Corollary 2.5. Our topological groups are actually non-meager P-filters on ω, hence metrizable, 0-dimensional, totally bounded, and hereditarily Baire by [23]. Since the Menger property is preserved by finite products of metrizable spaces in the Miller model [29] and coincides with the Scheepers property under $\mathfrak{u} < \mathfrak{g}$ by [28] (this inequality holds in the Miller model by [10, Theorem 2] combined with the results of [11]), filters like in Theorems 2.4 and 3.1 cannot be obtained in ZFC. Theorem 3.2 is a variation of Theorem 3.1 motivated by the question whether the density one filter can be diagonalized by a (proper) poset adding no dominating reals.

As discussed above, Theorem 2.4 (stating that there exists a filter $\mathcal{F}$ such that among other properties, both $\mathcal{F}$ and $\mathcal{F}^+$ are Menger, and hence $\mathcal{F}$ cannot be meager by [28, Prop. 2]), cannot be proved in ZFC. In section 4 we investigate how far the assumptions on $\mathcal{F}$ can be weakened so that it is still impossible to get such a filter in ZFC. In this context let us recall that there are Menger non-meager filters in ZFC, see [25]. On the other hand, the existence of a filter $\mathcal{F}$ in ZFC such that $\mathcal{F}^+$ is Menger is unknown and constructing such a filter without additional set-theoretic assumptions (if it is possible at all) would be extremely difficult: If $\mathcal{F}^+$ is Menger then $\mathcal{F}$ is non-meager and P, see Corollary 2.3, and it is a famous open

[1] This question remains open.

problem to construct a non-meager P-filter in ZFC. In Theorem 4.3 we show that consistently there are no Menger filters $\mathcal{F}$ such that $\mathcal{F}^+$ is Menger as well.

Let us note that the properties of Menger, Scheepers, having Menger square, etc., are preserved by perfect maps in both directions. This implies that if one of the remainders of a space X has one of these covering properties, then all other also have it, see the beginning of [8, §3] for more detailed explanations.

All undefined topological notions can be found in [13]. For the definitions and basic properties of cardinal characteristics used in this paper we refer the reader to [9].

2. The main result

We need to recall some standard as well as introduce some ad-hoc notation and terminology. A family $\mathcal{H} \subset [\omega]^\omega$ is called a *semifilter* if for every $H \in \mathcal{H}$, $n \in \omega$, and $X \supset H \setminus n$ we have $X \in \mathcal{H}$. For a set $\mathcal{F} \subset \mathcal{P}(\omega)$ and $n \in \omega$ we denote by $\langle \mathcal{F} \rangle_n$ the set $\{\bigcap \mathcal{F}' : \mathcal{F}' \in [\mathcal{F}]^{\leq n}\}$, and $\langle \mathcal{F} \rangle$ stands for

$$\{X \subset \omega : \exists n \exists Y \in \langle \mathcal{F} \rangle_n (Y \setminus n \subset X)\}.$$

$\mathcal{F}$ is said to be *centered* if $\langle \mathcal{F} \rangle_n \subset [\omega]^\omega$ for all $n \in \omega$, i.e., if the intersection of any finite subfamily of $\mathcal{F}$ is infinite. In this case, $\langle \mathcal{F} \rangle$ is the smallest free filter on ω containing $\mathcal{F}$. Let us note that if $\mathcal{F}$ is compact then so is $\langle \mathcal{F} \rangle_n$ for any $n \in \omega$, being a continuous image of $\mathcal{F}^n$. Similarly, by $\langle \mathcal{F} \rangle_s$ we shall denote the smallest semifilter containing $\mathcal{F}$, i.e.,

$$\langle \mathcal{F} \rangle_s = \{X \subset \omega \;:\; \exists F \in \mathcal{F} \exists n \in \omega (F \setminus n \subset X)\}.$$

For a family F of subsets of $\mathcal{P}(\omega)$ (i.e., $\mathsf{F} \subset \mathcal{P}(\mathcal{P}(\omega))$) we define the $\langle\rangle_*$-*saturation* of F as the smallest subfamily $\mathsf{F}_1 \supset \mathsf{F}$ of $\mathcal{P}(\mathcal{P}(\omega))$ such that $\langle \bigcup \mathsf{F}' \rangle_n \in \mathsf{F}_1$ for any $\mathsf{F}' \in [\mathsf{F}_1]^{<\omega}$ and $n \in \omega$. It is easy to write the $\langle\rangle_*$-saturation of a family F in a precise way, thus proving that the $\langle\rangle_*$-saturation of an infinite F has the same cardinality as F, and it consists of compact subsets of $\langle \bigcup \mathsf{F} \rangle$ if each $\mathcal{F} \in \mathsf{F}$ is compact. We say that F is $\langle\rangle_*$-*saturated* if it coincides with its $\langle\rangle_*$-saturation.

Given families $\mathcal{F}, \mathcal{H}$ of subsets of ω, we denote by $\mathcal{F} \wedge \mathcal{H}$ the family $\{F \cap H : F \in \mathcal{F}, H \in \mathcal{H}\}$. Again, if $\mathcal{F}, \mathcal{H}$ are compact, then so is $\mathcal{F} \wedge \mathcal{H}$. For $\mathsf{F}, \mathsf{H} \subset \mathcal{P}(\omega)$ we define the $(\mathsf{F}, \wedge)$-*saturation* of H as the smallest subfamily $\mathsf{H}_1 \supset \mathsf{H}$ of $\mathcal{P}(\mathcal{P}(\omega))$ such that $\mathcal{F} \wedge \mathcal{H} \in \mathsf{H}_1$ for any $\mathcal{F} \in \mathsf{F}$ and $\mathcal{H} \in \mathsf{H}_1$. Again, it is straightforward to write the $(\mathsf{F}, \wedge)$-saturation of H in the precise way, thus proving that it is a subfamily of $\mathcal{P}(\langle \bigcup \mathsf{F} \rangle \wedge \bigcup \mathsf{H})$ of size at most $\max\{\omega, |\mathsf{F}|, |\mathsf{H}|\}$, and it consists of compact sets if so do F and H. We call H to be $(\mathsf{F}, \wedge)$-*saturated* if it coincides with its $(\mathsf{F}, \wedge)$-saturation.

The next easy auxiliary fact is very similar to [14, Prop. 7] and can be derived from the latter one in a rather straightforward way. However, for reader's convenience we give a direct proof.

Lemma 2.1. *Let F be a family of compact subsets of $[\omega]^\omega$ of size $|\mathsf{F}| < \min\{\mathfrak{d}, \mathfrak{r}\}$ and $\phi : \omega \to \omega$ a finite-to-one surjection. Then there exists $Z \subset \omega$ such that*

$$|\phi^{-1}[Z] \cap F| = |(\omega \setminus \phi^{-1}[Z]) \cap F| = \omega$$

for all $F \in \bigcup \mathsf{F}$.

Proof. Let us first assume that ϕ is monotone and consider the strictly increasing number sequence $\langle k_n : n \in \omega \rangle$ with $k_0 = 0$ such that $\phi^{-1}(n) = [k_n, k_{n+1})$ for all $n \in \omega$.

For every $\mathcal{F} \in \mathsf{F}$ let $h_{\mathcal{F}} : \omega \to \omega$ be a strictly increasing function such that $[h_{\mathcal{F}}(n), h_{\mathcal{F}}(n+1)) \cap F \neq \emptyset$ for all $F \in \mathcal{F}$ and $n \in \omega$. Since $|\mathsf{F}| < \mathfrak{d}$, there exists a strictly increasing $h \in \omega^\omega$ such that

$$I_{\mathcal{F}} := \{n \in \omega \ : \ (|[h(n), h(n+1)) \cap h_{\mathcal{F}}[\omega]| \geq 2)\}$$

is infinite for all $\mathcal{F} \in \mathsf{F}$. Without loss of generality we may assume that $h[\omega] \subset \{k_n : n \in \omega\}$, and hence for every (infinite) $I \subset \omega$ there exists (an infinite) $Z \subset \omega$ such that $\bigcup_{n\in I}[h(n), h(n+1)) = \phi^{-1}[Z]$.

Since $|\mathcal{F}| < \mathfrak{r}$ there exists $I \subset \omega$ such that $|I \cap I_{\mathcal{F}}| = |I_{\mathcal{F}} \setminus I| = \omega$ for all $\mathcal{F} \in \mathsf{F}$. We claim that

$$\big|\bigcup_{n\in I}[h(n), h(n+1)) \cap F\big| = \big|\bigcup_{n\in\omega\setminus I}[h(n), h(n+1)) \cap F\big| = \omega$$

for all $F \in \bigcup \mathsf{F}$. Indeed, let us find $\mathcal{F} \in \mathsf{F}$ containing F and pick $n \in I \cap I_{\mathcal{F}}$. Then there exists $m \in \omega$ such that $[h(n), h(n+1)) \supset [h_{\mathcal{F}}(m), h_{\mathcal{F}}(m+1))$, and hence

$$\emptyset \neq F \cap [h_{\mathcal{F}}(m), h_{\mathcal{F}}(m+1)) \subset F \cap [h(n), h(n+1)) \subset F \cap \bigcup_{n\in I}[h(n), h(n+1)) \cap F.$$

The case of $\omega \setminus I$ is analogous.

Now suppose that ϕ is arbitrary finite-to-one surjection from ω to ω. Fix a bijection $\theta : \omega \to \omega$ such that $\phi \circ \theta$ is a monotone surjection. It follows from the above that there exists $Z \subset \omega$ such that

$$\big|(\phi \circ \theta)^{-1}[Z] \cap \theta^{-1}[F]\big| = \big|(\omega \setminus (\phi \circ \theta)^{-1}[Z]) \cap \theta^{-1}[F]\big| = \omega$$

for all $F \in \bigcup \mathsf{F}$, i.e.,

$$\big|\theta^{-1}[\phi^{-1}[Z]] \cap \theta^{-1}[F]\big| = \big|(\omega \setminus \theta^{-1}[\phi^{-1}[Z]]) \cap \theta^{-1}[F]\big| = \omega$$

and thus also

$$|\phi^{-1}[Z] \cap F| = |(\omega \setminus \phi^{-1}[Z]) \cap F| = \omega$$

for all $F \in \bigcup \mathsf{F}$ because θ is a bijection. $\square$

A semifilter $\mathcal{F}$ is called a *P-semifilter* if for every sequence $\langle F_n : n \in \omega\rangle$ of elements of $\mathcal{F}$ there exists a sequence $\langle K_n : n \in \omega\rangle$ such that $K_n \in [F_n]^{<\omega}$ for all $n \in \omega$ and $\bigcup_{n\in\omega} K_n \in \mathcal{F}$. If $\mathcal{F}^+ = \{X \subset \omega : \forall F \in \mathcal{F}(X \cap F \neq \emptyset)\}$ is a P-semifilter, then we also say that $\mathcal{F}$ is a *P^+-semifilter*. Each semifilter $\mathcal{F}$ on ω gives rise to the semifilter $\mathcal{F}^{(<\omega)}$ on $[\omega]^{<\omega} \setminus \{\emptyset\}$ generated by the family $\{[F]^{<\omega} \setminus \{\emptyset\} \ : \ F \in \mathcal{F}\}$. If $\mathcal{F}$ is a filter we shall call $\mathcal{F}^+$ the *coideal* of $\mathcal{F}$.

Next, we put together several known facts about Menger semifilters established in [12] and [14] and get a potentially useful characterization.

Theorem 2.2. *Let $\mathcal{F}$ be a semifilter on ω. Then the following statements are equivalent:*

(1) *$\mathcal{F}$ is Menger;*
(2) *For every sequence $\langle \mathcal{K}_n : n \in \omega\rangle$ of compact subsets of $\mathcal{F}^+$ there exists an increasing sequence $\langle m_n : n \in \omega\rangle \in \omega^\omega$ with the property $\bigcup_{n\in\omega}(K_n \cap m_n) \in \mathcal{F}^+$ for any $\langle K_n : n \in \omega\rangle \in \prod_{n\in\omega} \mathcal{K}_n$;*
(3) *For every sequence $\langle \mathcal{K}_n : n \in \omega\rangle$ of compact subsets of $\mathcal{F}^+$ there exists an increasing sequence $\langle m_n : n \in \omega\rangle \in \omega^\omega$ with the property $\bigcup_{n\in\omega}\big(K_n \cap [m_{n-1}, m_n)\big) \in \mathcal{F}^+$ for*[2] *any $\langle K_n : n \in \omega\rangle \in \prod_{n\in\omega} \mathcal{K}_n$; and*
(4) *$\mathcal{F}^{(<\omega)}$ is a P^+-semifilter.*

[2] We set here $m_{-1} = 0$.

Proof. The equivalence (1) $\Leftrightarrow$ (4) was established in [12, Claim 2.4], and its proof works verbatim also for semifilters. The implication (1) $\to$ (3) was obtained in [12, Prop. 3.4], its proof again works for semifilters without any changes, while (3) $\to$ (2) is straightforward. Thus it remains to prove (2) $\to$ (4). Let $\langle E_n : n \in \omega\rangle$ be a sequence of elements of $(\mathcal{F}^{(<\omega)})^+$. For each n set $\mathcal{K}_n = \{X \subset \omega : \forall e \in E_n (X \cap e \neq \emptyset)\}$ and note that $\mathcal{K}_n \subset \mathcal{F}^+$: If $K \cap F = \emptyset$ for some $K \in \mathcal{K}_n$ and $F \in \mathcal{F}$, then there is no $e \in E_n \cap [F]^{<\omega}$, which contradicts our choice of E_n. Let $\langle m_n : n \in \omega\rangle$ be as in (2). We claim that $\bigcup_{n\in\omega} E_n \cap \mathcal{P}(m_n) \in (\mathcal{F}^{(<\omega)})^+$, which will imply (4). Indeed, otherwise there exists $F \in \mathcal{F}$ such that $e \setminus F \neq \emptyset$ for each $e \in E_n \cap \mathcal{P}(m_n)$ and $n \in \omega$. Thus

$$K_n := \bigcup\{e \setminus F : e \in E_n, e \subset m_n\} \cup (\omega \setminus m_n) \in \mathcal{K}_n$$

for all n. However,

$$\begin{aligned} F \cap \bigcup_{n\in\omega}(K_n \cap m_n) &= F \cap \bigcup_{n\in\omega}\bigcup\{e \setminus F : e \in E_n, e \subset m_n\} = \\ &= \bigcup_{n\in\omega}(F \cap \bigcup\{e \setminus F : e \in E_n, e \subset m_n\}) = \emptyset, \end{aligned}$$

and hence $\bigcup\{K_n \cap m_n : n \in \omega\} \notin \mathcal{F}^+$, which contradicts (2). $\square$

Corollary 2.3. *Each Menger semifilter is P^+.*

Theorem 2.2 will be crucial for the proof of the following fact, which is the main result of the paper.

Theorem 2.4. *($\mathfrak{r} = \mathfrak{d} = \mathfrak{c}$). There exists a filter $\mathcal{G}$ on ω such that*

(1) *$\mathcal{G}$ and $\mathcal{G}^+$ are Menger;*
(2) *For every finite-to-one surjection $\phi : \omega \to \omega$ there exists $X \subset \omega$ such that $\phi^{-1}[X], \phi^{-1}[\omega \setminus X] \in \mathcal{G}^+$.*

Proof. Let us fix the following enumerations:

- $\{\langle \mathcal{K}^\alpha_n : n \in \omega\rangle : \alpha < \mathfrak{c}\}$=: the family of all sequences of compact subsets of $[\omega]^\omega$;
- $\{\phi_\alpha : \alpha < \mathfrak{c}\}$=: the family of all finite-to-one surjections from $\omega \to \omega$.

By recursion over $\alpha < \mathfrak{c}$ we shall construct a sequence $\langle\langle \mathsf{F}_\alpha, \mathsf{H}_\alpha\rangle : \alpha < \mathfrak{c}\rangle$ such that

(a) $\mathsf{F}_\alpha, \mathsf{H}_\alpha$ are families of compact subsets of $[\omega]^\omega$ of size $< \mathfrak{c}$ with $\mathsf{F}_0 = \mathsf{H}_0 = \{\{\omega\}\}$;
(b) $\bigcup \mathsf{F}_\alpha$ is centered and F_α is $\langle\rangle_*$-saturated;
(c) $\mathsf{F}_\alpha \subset \mathsf{F}_{\alpha'} \cap \mathsf{H}_{\alpha'}$ and $\mathsf{H}_\alpha \subset \mathsf{H}_{\alpha'}$ for any $\alpha \leq \alpha'$;
(d) $\bigcup \mathsf{H}_\alpha \subset \langle \bigcup \mathsf{F}_\alpha\rangle^+$, and H_α is $(\mathsf{F}_\alpha, \wedge)$-saturated;
(e) If $\bigcup_{n\in\omega} \mathcal{K}^\alpha_n \subset \langle \bigcup \mathsf{F}_\alpha\rangle$, then there exists an increasing number sequence $\langle m^\alpha_n : n \in \omega\rangle$ such that $\mathsf{F}_{\alpha+1} \ni \mathcal{K}_\alpha$, where $\mathcal{K}_\alpha = \big\{\bigcup_{n\in\omega}(K_n \cap m^\alpha_n) : \langle K_n : n \in \omega\rangle \in \prod_{n\in\omega} \mathcal{K}^\alpha_n\big\}$;
(f) If $\bigcup_{n\in\omega} \mathcal{K}^\alpha_n \not\subset \langle \bigcup \mathsf{F}_\alpha\rangle$, then there exists $K_\alpha \in \bigcup_{n\in\omega} \mathcal{K}^\alpha_n$ such that $\{\omega \setminus K_\alpha\} \in \mathsf{H}_{\alpha+1}$;
(g) If $\bigcup_{n\in\omega} \mathcal{K}^\alpha_n \subset \langle \bigcup \mathsf{H}_\alpha\rangle_s$, then there exists an increasing number sequence $\langle l^\alpha_n : n \in \omega\rangle$ such that $\mathsf{H}_{\alpha+1} \ni \mathcal{L}_\alpha$, where $\mathcal{L}_\alpha = \big\{\bigcup_{n\in\omega}(K_n \cap l^\alpha_n) : \langle K_n : n \in \omega\rangle \in \prod_{n\in\omega} \mathcal{K}^\alpha_n\big\}$;
(h) If $\bigcup_{n\in\omega} \mathcal{K}^\alpha_n \not\subset \langle \bigcup \mathsf{H}_\alpha\rangle_s$, then there exists $L_\alpha \in \bigcup_{n\in\omega} \mathcal{K}^\alpha_n$ such that $\{\omega \setminus L_\alpha\} \in \mathsf{F}_{\alpha+1}$;
(j) There exists $Z_\alpha \subset \omega$ such that $\{\phi^{-1}_\alpha[Z_\alpha], \phi^{-1}_\alpha[\omega \setminus Z_\alpha]\} \in \mathsf{H}_{\alpha+1}$.

First, let us assume that we have constructed a sequence $\langle\langle \mathsf{F}_\alpha, \mathsf{H}_\alpha\rangle : \alpha < \mathfrak{c}\rangle$ satisfying (a)-(j). We claim that

$$\mathcal{G} := \Big\langle \bigcup\{\bigcup \mathsf{F}_\alpha : \alpha < \mathfrak{c}\}\Big\rangle$$

is as required. It follows that $\mathcal{G}^+$ equals

$$\mathcal{G}_1 := \Big\langle \bigcup\{\bigcup \mathsf{H}_\alpha : \alpha < \mathfrak{c}\}\Big\rangle_s.$$

Indeed, $\mathcal{G}_1 \subset \mathcal{G}^+$ follows directly from (d). To prove that $\mathcal{G}^+ \subset \mathcal{G}_1$ let us fix any $X \notin \mathcal{G}_1$. Let $\alpha < \mathfrak{c}$ be such that $\mathcal{K}_n^\alpha = \{X\}$ for every $n \in \omega$ and note that $\langle \mathcal{K}_n^\alpha : n \in \omega\rangle$ satisfies the premises of (h). Therefore there exists $L_\alpha \in \bigcup_{n\in\omega} \mathcal{K}_n^\alpha = \{X\}$ (i.e., $L_\alpha = X$) such that $\{\omega \setminus L_\alpha\} \in \mathsf{F}_{\alpha+1}$, and hence $\omega \setminus X \in \mathcal{G}$ which yields $X \notin \mathcal{G}^+$.

Next, we shall establish that both $\mathcal{G}$ and $\mathcal{G}^+$ are Menger. Let $\langle \mathcal{K}_n : n \in \omega\rangle$ be a sequence of compact subsets of $\mathcal{G}$ and α be such that $\mathcal{K}^\alpha = \mathcal{K}_n^\alpha$ for all $n \in \omega$. It follows that $\bigcup_{n\in\omega} \mathcal{K}_n^\alpha \subset \langle \bigcup \mathsf{F}_\alpha\rangle$. Indeed, otherwise by (f) there exists $K_\alpha \in \bigcup_{n\in\omega} \mathcal{K}_n^\alpha$ such that $\{\omega \setminus K_\alpha\} \in \mathsf{H}_{\alpha+1}$, and therefore $\omega \setminus K_\alpha \in \mathcal{G}^+$, which contradicts $K_\alpha \in \mathcal{G}$. Thus $\langle \mathcal{K}_n^\alpha : n \in \omega\rangle$ fulfills the premises of (e), which yields an increasing sequence $\langle m_n : n \in \omega\rangle \in \omega^\omega$ with the property $\bigcup_{n\in\omega}(K_n \cap m_n) \in \mathcal{G}$ for any $\langle K_n : n \in \omega\rangle \in \prod_{n\in\omega} \mathcal{K}_n^\alpha$. Applying Theorem 2.2, we conclude that $\mathcal{G}^+$ is Menger.

To see that also $\mathcal{G}$ is Menger let us consider a sequence $\langle \mathcal{K}_n : n \in \omega\rangle$ of compact subsets of $\mathcal{G}^+$ and find α such that $\mathcal{K}^\alpha = \mathcal{K}_n^\alpha$ for all $n \in \omega$. It follows that $\bigcup_{n\in\omega} \mathcal{K}_n^\alpha \subset \langle \bigcup \mathsf{H}_\alpha\rangle_s$. Indeed, otherwise by (h) there exists $L_\alpha \in \bigcup_{n\in\omega} \mathcal{K}_n^\alpha$ such that $\{\omega \setminus L_\alpha\} \in \mathsf{F}_{\alpha+1}$, and therefore $\omega \setminus L_\alpha \in \mathcal{G}$, which contradicts $L_\alpha \in \mathcal{G}^+$. Thus $\langle \mathcal{K}_n^\alpha : n \in \omega\rangle$ fulfills the premises of (g), which yields an increasing sequence $\langle l_n : n \in \omega\rangle \in \omega^\omega$ with the property $\bigcup_{n\in\omega}(K_n \cap l_n) \in \mathcal{G}^+$ for any $\langle K_n : n \in \omega\rangle \in \prod_{n\in\omega} \mathcal{K}_n^\alpha$. Applying Theorem 2.2, we conclude that $\mathcal{G}$ is Menger.

Finally, we show how to construct a sequence as above satisfying (a)-(j). Limit stages are straightforward, so suppose that we have already constructed $\langle\langle \mathsf{F}_\beta, \mathsf{H}_\beta\rangle : \beta \le \alpha\rangle$ satisfying (e)-(j) for all $\beta < \alpha$, $(a), (b)$ and (d) for all $\beta \le \alpha$, and (c) for all $\beta \le \beta' \le \alpha$. Two cases are possible.

1). $\bigcup_{n\in\omega} \mathcal{K}_n^\alpha \subset \langle \bigcup \mathsf{F}_\alpha\rangle$. Then for every $\mathcal{H} \in \mathsf{H}_\alpha$ we can find an increasing sequence $\langle m_n^{\mathcal{H}} : n \in \omega\rangle \in \omega^\omega$ such that

$$K \cap H \cap (m_n^{\mathcal{H}} \setminus n) \neq \emptyset$$

for every $K \in \langle \mathcal{K}_n^\alpha\rangle_n$, $H \in \mathcal{H}$, and $n \in \omega$. Such an $m_n^{\mathcal{H}}$ exists because of the compactness of the involved sets and $\mathcal{H} \subset \bigcup \mathsf{H}_\alpha \subset \langle \bigcup \mathsf{F}_\alpha\rangle^+$. Since $|\mathsf{H}_\alpha| < \mathfrak{d}$, there exists an increasing sequence $\langle m_n^\alpha : n \in \omega\rangle \in \omega^\omega$ such that $|\{n \in \omega : m_n^{\mathcal{H}} \le m_n^\alpha\}| = \omega$ for all $\mathcal{H} \in \mathsf{H}_\alpha$. Set

$$\mathcal{K}_\alpha = \Big\{ \bigcup_{n\in\omega} (K_n \cap m_n^\alpha) : \langle K_n : n \in \omega\rangle \in \prod_{n\in\omega} \mathcal{K}_n^\alpha \Big\},$$

and $\mathsf{F}_{\alpha+1}$ to be the $\langle\rangle_*$-saturation of $\mathsf{F}_\alpha \cup \{\mathcal{K}_\alpha\}$. Since $|\mathsf{F}_{\alpha+1}| < \mathfrak{c}$, Lemma 2.1 yields $Z_\alpha \subset \omega$ such that $\{\phi_\alpha^{-1}[Z_\alpha], \phi_\alpha^{-1}[\omega \setminus Z_\alpha]\} \in \langle \bigcup \mathsf{F}_{\alpha+1}\rangle^+$. We define $\mathsf{H}_{\alpha+1}$ to be the $(\mathsf{F}_{\alpha+1}, \wedge)$-saturation of

$$\mathsf{H}_\alpha \cup \big\{\{\phi_\alpha^{-1}[Z_\alpha], \omega \setminus \phi_\alpha^{-1}[Z_\alpha]\}\big\}.$$

We are left with the task of checking that conditions (a)-(j) are satisfied. Indeed, (a), (c), and (e)-(j) hold immediately by the construction, in case of (f) and (h) because of the premises being violated.

Proof of (b), (d) for $\alpha+1$: By (b) and (d) for α it suffices to prove that $\langle \mathcal{K}_\alpha\rangle_k \wedge \mathcal{H} \subset [\omega]^\omega$ for any $\mathcal{H} \in \mathsf{H}_\alpha$ and $k \in \omega$. Let us fix

$$\big\{\langle K_n^i : n\in\omega\rangle : i\le k\big\} \subset \prod_{n\in\omega}\mathcal{K}_n^\alpha$$

and $H\in\mathcal{H}$, and find $n\ge k$ such that $m_n^{\mathcal{H}}\le m_n^\alpha$. Then $\bigcap_{i\le k}K_n^i\in\langle\mathcal{K}_n^\alpha\rangle_n$ and therefore

$$\emptyset\neq\bigcap_{i\le k}K_n^i\cap H\cap(m_n^{\mathcal{H}}\setminus n)\subset\bigcap_{i\le k}K_n^i\cap H\cap(m_n^\alpha\setminus n)\subset$$
$$\subset\Big(\bigcap_{i\le k}\bigcup\{K_n^i\cap m_n^\alpha : n\in\omega\}\cap H\Big)\setminus n,$$

which proves (b) and (d) for $\alpha+1$ and thus completes this case.

2). $\bigcup_{n\in\omega}\mathcal{K}_n^\alpha\not\subset\langle\bigcup\mathsf{F}_\alpha\rangle$. Two subcases of 2) are possible.

2_0). $\bigcup_{n\in\omega}\mathcal{K}_n^\alpha\not\subset\langle\bigcup\mathsf{H}_\alpha\rangle_s$. It follows that there exists $L_\alpha\in\bigcup_{n\in\omega}\mathcal{K}_n^\alpha$ such that $\omega\setminus L_\alpha\in\langle\bigcup\mathsf{H}_\alpha\rangle_s^+\subset\langle\bigcup\mathsf{F}_\alpha\rangle^+$, which allows us to define $\mathsf{F}_{\alpha+1}$ as the $\langle\rangle_*$-saturation of $\mathsf{F}_\alpha\cup\big\{\{\omega\setminus L_\alpha\}\big\}$. Lemma 2.1 yields $Z_\alpha\subset\omega$ such that $\{\phi_\alpha^{-1}[Z_\alpha],\phi_\alpha^{-1}[\omega\setminus Z_\alpha]\}\subset\langle\bigcup\mathsf{F}_{\alpha+1}\rangle^+$. Let $\mathsf{H}_{\alpha+1}$ be the $(\mathsf{F}_{\alpha+1},\wedge)$-saturation of

$$\mathsf{H}_\alpha\cup\{\{\phi_\alpha^{-1}[Z_\alpha],\omega\setminus\phi_\alpha^{-1}[Z_\alpha]\}\}.$$

Conditions (a)-(c), and (e)-(j) hold for $\alpha+1$ immediately by the construction, in case of (e) and (g) because of the premises being violated, and for (f) we can simply take K_α to be L_α. Regarding (d) for $\alpha+1$, by (b) and (d) for α it suffices to prove that $F_0\cap F_1\cap(\omega\setminus L_\alpha)\cap X$ is infinite for any $F_0,F_1\in\bigcup\mathsf{F}_\alpha$ and $X\in\{\phi_\alpha^{-1}[Z_\alpha],\omega\setminus\phi_\alpha^{-1}[Z_\alpha]\}$, which again has been guaranteed in the course of the construction above.

2_1). $\bigcup_{n\in\omega}\mathcal{K}_n^\alpha\subset\langle\bigcup\mathsf{H}_\alpha\rangle_s$. In this case we set $\mathsf{F}_{\alpha+1}=\mathsf{F}_\alpha$.

For every $\mathcal{F}\in\mathsf{F}_{\alpha+1}$ we can find an increasing sequence $\langle l_n^{\mathcal{F}} : n\in\omega\rangle\in\omega^\omega$ such that

$$K\cap F\cap(l_n^{\mathcal{F}}\setminus n)\neq\emptyset$$

for every $K\in\mathcal{K}_n^\alpha$ and $F\in\mathcal{F}$. Such an $l_n^{\mathcal{F}}$ exists because of the compactness of the involved sets and $\mathcal{K}_n^\alpha\subset\langle\bigcup\mathsf{H}_\alpha\rangle_s\subset\langle\bigcup\mathsf{F}_\alpha\rangle^+$. Since $|\mathsf{F}_\alpha|<\mathfrak{d}$, there exists an increasing sequence $\langle l_n^\alpha : n\in\omega\rangle\in\omega^\omega$ such that $|\{n\in\omega : l_n^{\mathcal{F}}\le l_n^\alpha\}|=\omega$ for all $\mathcal{F}\in\mathsf{F}_{\alpha+1}$. Set

$$\mathcal{L}_\alpha=\Big\{\bigcup_{n\in\omega}(K_n\cap l_n^\alpha) : \langle K_n : n\in\omega\rangle\in\prod_{n\in\omega}\mathcal{K}_n^\alpha\Big\}.$$

Lemma 2.1 yields $Z_\alpha\subset\omega$ such that $\{\phi_\alpha^{-1}[Z_\alpha],\phi_\alpha^{-1}[\omega\setminus Z_\alpha]\}\subset\langle\bigcup\mathsf{F}_{\alpha+1}\rangle^+$. From $\bigcup_{n\in\omega}\mathcal{K}_n^\alpha\not\subset\langle\bigcup\mathsf{F}_\alpha\rangle$ it follows that there exists $K_\alpha\in\bigcup_{n\in\omega}\mathcal{K}_n^\alpha$ such that $\{\omega\setminus K_\alpha\}\in\langle\bigcup\mathsf{F}_\alpha\rangle^+$. Finally, we define $\mathsf{H}_{\alpha+1}$ to be the $(\mathsf{F}_{\alpha+1},\wedge)$-saturation of

$$\mathsf{H}_\alpha\cup\big\{\{\phi_\alpha^{-1}[Z_\alpha],\{\omega\setminus\phi_\alpha^{-1}[Z_\alpha],\omega\setminus K_\alpha\}\big\}\cup\{\mathcal{L}_\alpha\}.$$

It is straightforward to check that conditions (a)-(j) are satisfied, the only slightly non-trivial step being $\mathcal{L}_\alpha\subset\langle\bigcup\mathsf{F}_{\alpha+1}\rangle^+$, which can be proved analogously to (but more easily) "$\langle\mathcal{K}_\alpha\rangle_k\wedge\mathcal{H}\subset[\omega]^\omega$ for any $\mathcal{H}\in\mathsf{H}_\alpha$ and $k\in\omega$" in case 1).

This concludes our proof, since all possible cases have been considered. □

In the proof of the following corollary we shall use the classical result of Hurewicz [19] (see also [20, Theorem 4.3]) stating that $X\subset\mathcal{P}(\omega)$ is Menger if and only if $f[X]$ is not dominating for any continuous $f : X\to\omega^\omega$.

Corollary 2.5. *If a semifilter $\mathcal{H}$ satisfies condition* (2) *of Theorem 2.4, then $\mathcal{H}^2$ is not Menger and $\mathcal{H}$ is not Scheepers. In particular, there is no continuous surjection from $\mathcal{H}$ onto $\mathcal{H}^2$.*

Proof. Suppose that $\mathcal{H}^2$ is Menger, then so is $\mathcal{H}\times(\sim\mathcal{H})$, where $\sim\mathcal{H}=\{\omega\setminus H : H\in\mathcal{H}\}$, because $\sim\mathcal{H}$ is homeomorphic to $\mathcal{H}$. Therefore $\mathcal{X} := \mathcal{H}\times(\sim\mathcal{H})\cap\{\langle X, X\rangle : X\subset\omega\}$ is also Menger being a closed subspace of $\mathcal{H}\times(\sim\mathcal{H})$, and hence so is $\mathcal{H}\cap(\sim\mathcal{H})$ as the projection of $\mathcal{X}$ to the first (as well as the second) coordinate. Let us note that $\mathcal{H}\cap(\sim\mathcal{H})$ consists of infinite co-infinite subsets of ω, and hence the map $h:\mathcal{H}\cap(\sim\mathcal{H})\to\omega^\omega$, $h(X)=\{n\in X : (n+1)\notin X\}$, must have a non-dominating range.

On the other hand, given a strictly increasing $x\in\omega^\omega$ with $x(0)=0$, let us consider the monotone surjection $\phi:\omega\to\omega$ such that $\phi^{-1}(n)=[x(n),x(n+1))$ for all n. It follows that there exists $Z\subset\omega$ such that $\phi^{-1}[Z]\in\mathcal{H}\cap(\sim\mathcal{H})$, and it is easy to see that $h[\phi^{-1}[Z]]\subset x[\omega]$. Thus for every x as above there exists $X\in\mathcal{H}\cap(\sim\mathcal{H})$ with $h(X)$ contained in the range of x, which clearly yields a dominating continuous range of $\mathcal{H}\cap(\sim\mathcal{H})$ and thus leads to a contradiction.

Now, suppose that $\mathcal{H}$ is Scheepers and consider the clopen cover $\mathsf{O}=\{\mathcal{O}_k : k\in\omega\}$ of $\mathcal{H}$, where $\mathcal{O}_k=\{X\subset\omega : k\in X\}$. By [7, Theorem 2] (see the equivalence of items 1 and 4 there) there exists a disjoint sequence $\langle\mathsf{O}_n : n\in\omega\rangle$ of finite subsets of O such that for every finite $\mathcal{H}'\subset\mathcal{H}$ there exists n with $\mathcal{H}'\subset\bigcup\mathsf{O}_n$. Let $s_n\in[\omega]^{<\omega}$ be such that $\mathsf{O}_n=\{\mathcal{O}_k : k\in s_n\}$ and consider the finite-to-one surjection $\phi:\omega\to\omega$ such that $\phi^{-1}(n)=s_n$ for all $n\in\omega$. It follows from the above that for every finite $\mathcal{H}'\subset\mathcal{H}$ there exists $n\in\omega$ such that $H\cap s_n\neq\emptyset$ for all $H\in\mathcal{H}'$.

On the other hand, using Theorem 2.4(2) pick $Z\subset\omega$ be such that $\mathcal{H}' := \{\phi^{-1}[Z],\omega\setminus\phi^{-1}[Z]\}\subset\mathcal{H}$ and note that there is no $n\in\omega$ such that both sets

$$\phi^{-1}[Z]\cap s_n=\phi^{-1}[Z]\cap\phi^{-1}(n)\ \text{ and }\ (\omega\setminus\phi^{-1}[Z])\cap s_n=(\omega\setminus\phi^{-1}[Z])\cap\phi^{-1}(n)$$

are non-empty, a contradiction. □

Let us note that the proof of Corollary 2.5 could be actually extracted from that of [8, Lemma 3.1], but we have nonetheless presented it for reader's convenience. Since every Menger subspace of $\mathcal{P}(\omega)$ has Menger square and is Scheepers in the Miller model, Theorem 2.4 cannot be proved in ZFC, as follows from Corollary 2.5.

Since each filter has a structure of a topological group, Theorem 2.4 combined with Corollary 2.5 answers [8, Question 1.8] in the affirmative, the coideal $\mathcal{G}^+$ being the needed counterexample. Another motivation for Corollary 2.5 comes from [24] where it was proved that each filter on ω is homeomorphic to its square. According to [24, Prop. 8], this result fails for semifilters, and the counterexample is a Borel comeager semifilter. However, until now no semifilter $\mathcal{F}$ such that both $\mathcal{F}$ and $\mathcal{F}^+$ are non-meager, which in addition is not homeomorphic to its square, was known, and Theorem 2.4 combined with Corollary 2.5 gives a consistent example of coideal like that.

For curiosity we exclude below one more possibility for spaces related to filters $\mathcal{G}$ satisfying Theorem 2.4 to be homeomorphic.

Corollary 2.6. *If a filter $\mathcal{G}$ satisfies Theorem 2.4, then $\mathcal{G}\times\mathcal{G}^+$ is not homeomorphic to $\mathcal{G}$.*

Proof. $\mathcal{G}^2$ is Menger by [24] (see also [12, Claim 5.5] for a simpler proof), whereas $(\mathcal{G}\times\mathcal{G}^+)^2$ can be mapped continuously onto $(\mathcal{G}^+)^2$ and hence is not Menger. □

3. Examples by forcing

This section was inspired by [15, §4] as well as [22, §6]. More precisely, one can obtain a filter $\mathcal{G}$ such as in Theorem 2.4 by countably complete forcing, namely let $\mathbb{P}$ be the poset consisting of conditions $p = \langle \mathsf{F}, \mathsf{H} \rangle$ such that

(i) F is a countable collection of compact subsets of $[\omega]^\omega$ such that $\bigcup \mathsf{F}$ is centered;
(ii) H is a countable collection of compact subsets of $[\omega]^\omega$ such that $\mathsf{F} \subset \mathsf{H}$;
(iii) $\bigcup \mathsf{H} \subset \langle \bigcup \mathsf{F} \rangle^+$.

A condition $\langle \mathsf{F}_1, \mathsf{H}_1 \rangle$ is stronger than $\langle \mathsf{F}_0, \mathsf{H}_0 \rangle$ (and written $\langle \mathsf{F}_1, \mathsf{H}_1 \rangle \leq \langle \mathsf{F}_0, \mathsf{H}_0 \rangle$) if $\mathsf{F}_1 \supset \mathsf{F}_0$ and $\mathsf{H}_1 \supset \mathsf{H}_0$.

Theorem 3.1. *Let G be a $\mathbb{P}$-generic filter, $\mathcal{G} = \bigcup\{\bigcup \mathsf{F} : \exists \mathsf{H}(\langle \mathsf{F}, \mathsf{H} \rangle \in G)\}$, and $\mathcal{H} = \bigcup\{\bigcup \mathsf{H} : \exists \mathsf{F}(\langle \mathsf{F}, \mathsf{H} \rangle \in G)\}$. Then $\mathcal{G}$ is a filter, $\mathcal{H} = \mathcal{G}^+$, and both $\mathcal{G}$ and $\mathcal{H}$ are Menger. Moreover, for every surjection $\phi : \omega \to \omega$ there exists $X \subset \omega$ such that $\phi^{-1}[X], \phi^{-1}[\omega \setminus X] \in \mathcal{G}^+$.*

We leave the proof of Theorem 3.1 to the reader, as it is more or less a kind of a repetition of that of Theorem 2.4, with the only difference being that now everything we need would happen "generically", i.e., the set of suitable conditions is dense, while for Theorem 2.4 we had to "manually" guarantee all that by going over appropriate enumerations.

Instead, we shall address a similar poset tailored to analyze the density one filter $\mathcal{Z}^*$ on ω consisting of those $Z \subset \omega$ such that $\lim_{n\to\infty} \frac{|Z\cap n|}{n} = 1$. It is a well known open problem whether there exists a proper poset adding no dominating reals but adding an infinite subset of ω almost included into all ground model elements of $\mathcal{Z}^*$, see, e.g., [17, Question 2.12]. This motivated us to introduce the following poset.

Let $\mathbb{Q}$ be the set of conditions $\langle \mathsf{F}, \mathcal{H}, \varepsilon \rangle$ such that

(i) F is a countable collection of compact subsets of $[\omega]^\omega$ such that $\bigcup \mathsf{F}$ is centered;
(ii) $\mathcal{H}$ is a countable subset of $[\omega]^\omega$ and $\varepsilon : \mathcal{H} \to (0, 1]$; and
(iii) For every $F \in \langle \mathsf{F} \rangle$ and $H \in \mathcal{H}$ there exists $X \in [\omega]^\omega$ such that $\lim_{n\in X} \frac{|F\cap n|}{n} = 1$ and $\liminf_{n\in X} \frac{|F\cap H\cap n|}{n} \geq \varepsilon(H)$.

A condition $\langle \mathsf{F}_1, \mathcal{H}_1, \varepsilon_1 \rangle$ is stronger than $\langle \mathsf{F}_0, \mathcal{H}_0, \varepsilon_0 \rangle$ (and written $\langle \mathsf{F}_1, \mathcal{H}_1, \varepsilon_1 \rangle \leq \langle \mathsf{F}_0, \mathcal{H}_0, \varepsilon_0 \rangle$) if $\mathsf{F}_1 \supset \mathsf{F}_0$, $\mathcal{H}_1 \supset \mathcal{H}_0$, and $\varepsilon_0 = \varepsilon_1 \restriction \mathcal{H}_0$. Clearly, $\mathbb{Q}$ is countably closed.

Theorem 3.2. *Let G be a $\mathbb{Q}$-generic filter and $\mathcal{G} = \bigcup\{\bigcup \mathsf{F} : \exists \mathcal{H} \exists \varepsilon\, (\langle \mathsf{F}, \mathcal{H}, \varepsilon \rangle \in G)\}$. Then $\mathcal{G}$ is a filter, $\mathcal{Z}^* \subset \mathcal{G}$, and $\mathcal{G}^+$ is Menger.*

Proof. To see that $\mathcal{G}$ is a filter note that for any $\langle \mathsf{F}, \mathcal{H}, \varepsilon \rangle \in \mathbb{Q}$ and finite $\mathcal{F} \subset \langle \bigcup \mathsf{F} \rangle$ we have $\langle \mathsf{F} \cup \{\{\bigcap \mathcal{F}\}\}, \mathcal{H}, \varepsilon \rangle \in \mathbb{Q}$, and hence the set of all conditions in $\mathbb{Q}$ whose first component contains $\{\bigcap \mathcal{F}\}$ as an element, is dense below $\langle \mathsf{F}, \mathcal{H}, \varepsilon \rangle$.

The fact that $\mathcal{Z}^* \subset \mathcal{G}$, follows from the observation that for any $T \in \mathcal{Z}^*$ and $\langle \mathsf{F}, \mathcal{H}, \varepsilon \rangle \in \mathbb{Q}$, we have that $\langle \mathsf{F} \cup \{\{T\}\}, \mathcal{H}, \varepsilon \rangle \in \mathbb{Q}$. Indeed, if for some $F \in \langle \mathsf{F} \rangle$, $H \in \mathcal{H}$, and $X \in [\omega]^\omega$ we have $\lim_{n\in X} |\frac{|F\cap n|}{n}| = 1$ and $\liminf_{n\in X} |\frac{|F\cap H\cap n|}{n}| \geq \varepsilon(H)$, then also $\lim_{n\in X} |\frac{|(F\cap T)\cap n|}{n}| = 1$ and $\liminf_{n\in X} |\frac{|(F\cap T)\cap H\cap n|}{n}| \geq \varepsilon(H)$.

Thus we are left with the task of showing that for every sequence $\langle \mathcal{K}_n : n \in \omega \rangle$ of compact subspaces of $\mathcal{G}$ there exists an increasing sequence $\langle m_n : n \in \omega \rangle \in \omega^\omega$ such that for every $\langle K_n : n \in \omega \rangle \in \prod_{n\in\omega} \mathcal{K}_n$ we have $\bigcup_{n\in\omega}(K_n \cap m_n) \in \mathcal{G}$. Let $\langle \mathsf{F}_0, \mathcal{H}_0, \varepsilon_0 \rangle \in G$ be such that $\langle \mathsf{F}_0, \mathcal{H}_0, \varepsilon_0 \rangle \Vdash \bigcup_{n\in\omega} \mathcal{K}_n \subset \mathcal{G}$. We claim that there exists $\langle \mathsf{F}_1, \mathcal{H}_1, \varepsilon_1 \rangle \in G$ such that $\bigcup_{n\in\omega} \mathcal{K}_n \subset \langle \bigcup \mathsf{F}_1 \rangle$. Indeed, given any condition $\langle \mathsf{F}, \mathcal{H}, \varepsilon \rangle \leq \langle \mathsf{F}_0, \mathcal{H}_0, \varepsilon_0 \rangle$ two cases are possible.

1. There exists $K \in \langle \bigcup_{n\in\omega} \mathcal{K}_n \rangle$ such that $\omega \setminus K \in \mathcal{H}$, or there exists $\delta > 0$ such that for every $F \in \langle \mathsf{F} \rangle$ there exists $X \in [\omega]^\omega$ with properties $\lim_{n\in X} \frac{|F\cap n|}{n} = 1$ and $\liminf_{n\in X} \frac{|F\cap(\omega\setminus K)\cap n|}{n} \geq \delta$. In this case $\langle \mathsf{F}, \mathcal{H} \cup \{\omega \setminus K\}, \varepsilon' \rangle \Vdash K \notin \mathcal{G}$, where $\varepsilon' \restriction \mathcal{H} = \varepsilon$ and $\varepsilon'(\omega \setminus K) = \delta$ if $\omega \setminus K \notin \mathcal{H}$. The latter leads to a contradiction.

2. For every $K \in \langle \bigcup_{n\in\omega} \mathcal{K}_n \rangle$ we have $\omega \setminus K \notin \mathcal{H}$, and for every $\delta > 0$ there exists $F \in \langle \mathsf{F} \rangle$ such that for every $X \in [\omega]^\omega$ with $\lim_{n\in X} \frac{|F\cap n|}{n} = 1$ the following holds:

$$\forall^* n \in X \ \frac{|F \cap (\omega \setminus K) \cap n|}{n} < \delta. \tag{1}$$

Let us note that if for some $\delta > 0$ an element $F \in \langle \mathsf{F} \rangle$ is a witness for Equation (1), then any smaller $F' \in \langle \mathsf{F} \rangle$ is also one. Given any $F \in \langle \mathsf{F} \rangle$ and $K \in \langle \bigcup_{n\in\omega} \mathcal{K}_n \rangle$, let us construct a decreasing sequence $\langle F_i : i \geq 1 \rangle$ of elements of $\langle \mathsf{F} \rangle$ such that $F_0 = F$ and

$$\forall X \in [\omega]^\omega \left(\lim_{n\in X} \frac{|F_i \cap n|}{n} = 1 \Rightarrow \forall^* n \in X \ \frac{|F_i \cap (\omega \setminus K) \cap n|}{n} < \frac{1}{i} \right). \tag{2}$$

Equation (2) implies

$$\forall X \in [\omega]^\omega \left(\lim_{n\in X} \frac{|F_i \cap n|}{n} = 1 \Rightarrow \forall^* n \in X \ \frac{|F_i \cap K \cap n|}{n} \geq 1 - \frac{2}{i} \right). \tag{3}$$

Let us fix now any $H \in \mathcal{H}$ and for every i find $X_i \in [\omega]^\omega$ such that $\lim_{n\in X_i} \frac{|F_i\cap n|}{n} = 1$ and $\frac{|F_i\cap H\cap n|}{n} > \varepsilon(H) - \frac{1}{i}$ for all $n \in X_i$. This is possible by item (iii) of the definition of $\mathbb{Q}$. Removing finitely many elements of X_i, if necessary, by Equation (3) we may assume that $\frac{|F_i\cap K\cap n|}{n} > 1 - \frac{2}{i}$ for all $n \in X_i$. Now let $\langle n_i : i \in \omega \rangle \in \omega^\omega$ be an increasing sequence such that $n_i \in X_i$ and $X(F, H, K) = \{n_i : i \in \omega\}$. It follows that $\frac{|F_i\cap H\cap n_i|}{n_i} > \varepsilon(H) - \frac{1}{i}$ and $\frac{|F_i\cap K\cap n_i|}{n_i} > 2 - 1/i$ for all $i \in \omega$. Since $F_i \subset F$ for all i, we have that $\frac{|F\cap H\cap n_i|}{n_i} > \varepsilon(H) - \frac{1}{i}$ and $\frac{|F\cap K\cap n_i|}{n_i} > 1 - 2/i$ and therefore $\frac{F\cap K\cap H\cap n_i}{n_i} > \varepsilon(H) - \frac{3}{i}$ for all $i \in \omega$. Thus $\langle \mathsf{F} \cup \{\mathcal{K}_n : n \in \omega\}, \mathcal{H}, \varepsilon \rangle \in \mathbb{Q}$, where for each $F \in \langle \mathsf{F} \rangle$, $H \in \mathcal{H}$, and $K \in \langle \bigcup_{n\in\omega\mathcal{K}_n} \rangle$ the infinite set $X(F, H, K)$ is such as required in (iii) for $F \cap K$ and H.

Summarizing, we have proved that for any $\langle \mathsf{F}, \mathcal{H}, \varepsilon \rangle \leq \langle \mathsf{F}_0, \mathcal{H}_0, \varepsilon_0 \rangle$ we have $\langle \mathsf{F} \cup \{\mathcal{K}_n : n \in \omega\}, \mathcal{H}, \varepsilon \rangle \in \mathbb{Q}$, and thus there exists $\langle \mathsf{F}_1, \mathcal{H}_1, \varepsilon_1 \rangle \in G$ with $\bigcup_{n\in\omega} \mathcal{K}_n \subset \langle \bigcup \mathsf{F}_1 \rangle$.

Let us now fix $\langle \mathsf{F}, \mathcal{H}, \varepsilon \rangle \leq \langle \mathsf{F}_1, \mathcal{H}_1, \varepsilon_1 \rangle$ and enumerations $\langle H_i : i \in \omega \rangle$ of $\mathcal{H}$ as well as $\langle \mathcal{F}_i : i \in \omega \rangle$ of F. Set

$$\mathcal{L}_i = \{ \bigcap \mathcal{Y} \, : \, \mathcal{Y} \in [\bigcup_{j\leq i} \mathcal{F}_j \cup \bigcup_{j\leq i} \mathcal{K}_j]^{\leq i} \}$$

and by recursion over i construct an increasing number sequence $\langle m_i : i \in \omega \rangle$ such that for every $L \in \mathcal{L}_i$ and $j \leq i$ there exists $n_j \in [m_{i-1}, m_i)$ such that $\frac{|L\cap n_j|}{n_j} > 1 - \frac{1}{i}$ and $\frac{|L\cap H_j\cap n_j|}{n_j} > \varepsilon(H_j)(1 - \frac{1}{i})$. This is possible by (iii) and the compactness of $\mathcal{L}_i$. Letting

$$\mathcal{K} = \Big\{ \bigcup_{i\in\omega} (K_i \cap m_i) \, : \, \langle K_i : i \in \omega \rangle \in \prod_{i\in\omega} \mathcal{K}_i \Big\},$$

we claim that $\langle \mathsf{F} \cup \{\mathcal{K}\}, \mathcal{H}, \varepsilon \rangle \in \mathbb{Q}$. Indeed, let us fix $k \in \omega$ and a family $\{Y_s : s \leq k\} \subset \mathcal{K}$, where $Y_s = \bigcup_{i\in\omega} (K_{i,s} \cap m_i)$ for some $K_{i,s} \in \mathcal{K}_i$. Also, let us fix a family $\{R_s : s \leq k\} \subset \bigcup_{s\leq k} \mathcal{F}_s$ and set $R = \bigcap_{s\leq k} R_s$. Then for every $i \geq 2k + 1$ and $s \leq k$ we have that

$$\mathcal{Y} := \{R_s : s \leq k\} \cup \{K_{i,s} : s \leq k\} \in [\bigcup_{j\leq i} \mathcal{F}_j \cup \bigcup_{j\leq i} \mathcal{K}_j]^{\leq i},$$

and therefore for every and $j \le i$ there exists $n_j \in [m_{i-1}, m_i)$ such that $\frac{|\bigcap \mathcal{Y} \cap n_j|}{n_j} > 1 - \frac{1}{i}$ and $\frac{|\mathcal{Y} \cap H_j \cap n_j|}{n_j} > \varepsilon(H_j)(1 - \frac{1}{i})$. Since

$$\bigcap \mathcal{Y} \cap n_j \subset \bigcap \{R_s : s \le k\} \cap \bigcap \{Y_s : s \le k\} \cap n_j, \tag{4}$$

(because $K_{i,s} \cap m_i \subset Y_s \cap m_i$ for all $s \le k$), we conclude that (iii) is satisfied for $\langle \mathsf{F} \cup \{\mathcal{K}\}, \mathcal{H}, \varepsilon \rangle$, and hence it is a member of $\mathbb{Q}$. Thus for arbitrary $\langle \mathsf{F}, \mathcal{H}, \varepsilon \rangle \le \langle \mathsf{F}_1, \mathcal{H}_1, \varepsilon_1 \rangle$ we have $\langle \mathsf{F} \cup \{\mathcal{K}\}, \mathcal{H}, \varepsilon \rangle \in \mathbb{Q}$, and hence there exists $\langle \mathsf{F}_2, \mathcal{H}_2, \varepsilon_2 \rangle \in G$ with $\mathcal{K} \in \mathsf{F}_2$, which yields $\mathcal{K} \subset \mathcal{G}$. It remains to apply Theorem 2.2. □

It is well-known and easy to see that the filter $\mathcal{Z}^*$ cannot be extended to any P^+-filter, and hence $\mathcal{G}$ from Theorem 3.2 is not Menger by Corollary 2.3. In addition, the Menger co-ideal $\mathcal{G}^+$ does not contain any P-point, and hence also no Menger ultrafilter. We do not know whether such examples can be obtained in ZFC, see Section 5 for more questions related to $\mathcal{Z}$.

4. Impossibility results

Here we show that Theorem 2.4 cannot be proved without additional set-theoretic assumptions, see Theorem 4.3 below. Following [8] for a semifilter $\mathcal{F}$ we denote by $\mathbb{P}_{\mathcal{F}}$ the poset consisting of all partial maps p from $\omega \times \omega$ to 2 such that for every $n \in \omega$ the domain of $p_n : k \mapsto p(n,k)$ is an element of $\sim \mathcal{F} = \{\omega \setminus F : F \in \mathcal{F}\}$. If, moreover, we assume that additionally $\mathrm{dom}(p_n) \subset \mathrm{dom}(p_{n+1})$ for all n, the corresponding poset will be denoted by $\mathbb{P}^*_{\mathcal{F}}$. A condition q is stronger than p (in this case we write $q \le p$) if $p \subset q$. For filters $\mathcal{F}$ the poset $\mathbb{P}^*_{\mathcal{F}}$ is obviously dense in $\mathbb{P}_{\mathcal{F}}$, and the latter is proper and ω^ω-bounding if $\mathcal{F}$ is a non-meager P-filter [27, Fact VI.4.3, Lemma VI.4.4]. This result has the following topological counterpart proved in [8]. Recall that a poset $\mathbb{P}$ is called *ω^ω-bounding* if $(\omega^\omega)^V$ is dominating in $V^{\mathbb{P}}$.

Lemma 4.1. *If $\mathcal{F}^+$ is a Menger semifilter, then both $\mathbb{P}_{\mathcal{F}}$ and $\mathbb{P}^*_{\mathcal{F}}$ are proper and ω^ω-bounding.*

We shall need the following game of length ω on a topological space X: In the nth move player I chooses an open cover $\mathcal{U}_n$ of X, and player II responds by choosing a finite $\mathcal{V}_n \subset \mathcal{U}_n$. Player II wins the game if $\bigcup_{n\in\omega} \bigcup \mathcal{V}_n = X$. Otherwise, player I wins. We shall call this game *the Menger game* on X. It is well-known that X is Menger if and only if player I has no winning strategy in the Menger game on X, see [18] or [26, Theorem 13].

For a relation R on ω and $x, y \in \omega^\omega$ we denote by $[x\, R\, y]$ the set $\{n : x(n)\, R\, y(n)\}$. The next lemma improves [8, Lemma 4.3].

Lemma 4.2. *Suppose that $\mathcal{F}$ is a semifilter such that $\mathcal{F} \subset \mathcal{F}^+$ and $\mathcal{F}^+$ is Menger. Let x be $\mathbb{P}^*_{\mathcal{F}}$-generic, $\mathbb{Q} \in V[x]$ be an ω^ω-bounding poset, and H a $\mathbb{Q}$-generic over $V[x]$. Then in $V[x * H]$ there is no Menger semifilter $\mathcal{G}$ containing $\mathcal{F}$ such that $\mathcal{G} \subset \mathcal{G}^+$ and $\mathcal{G}^+$ is Menger.*

Proof. Suppose that a semifilter $\mathcal{G} \in V[x * H]$ as above exists. Throughout the proof we shall identify x with $\bigcup x : \omega \times \omega \to 2$. Suppose to the contrary, that such a $\mathcal{G}$ exists. Set $x_j(n) = x(j,n)$. Since $\mathcal{G} \subset \mathcal{G}^+$, $\mathcal{G}$ cannot contain two disjoint elements, and hence for every j there exists $\varepsilon_j \in 2$ such that $X_j := x_j^{-1}(\varepsilon_j) \in \mathcal{G}^+$. Without loss of generality we may assume that there exists an infinite $T \subset \omega$ such that $\varepsilon_j = 1$ for all $j \in T$. Since $\mathbb{P}^*_{\mathcal{F}} * \mathbb{Q}$ is ω^ω-bounding we can get an increasing sequence $\langle j_k : k \in \omega \rangle \in V$ such that for every $k \in \omega$ there exists $t_k \in [j_k, j_{k+1}) \cap T$.

Given $k \in \omega$, for every $s \in \omega^{j_{k+1}}$ set

$$\mathcal{U}_{k,s} = \{X \subset \omega \ : \ s[j_{k+1}] \subset X\}.$$

Then $\{\mathcal{U}_{k,s} : s \in \Sigma_{k,l}\}$, where $\Sigma_{k,l} = \prod_{j<j_{k+1}}(X_j \setminus l)$, is an open cover of $\mathcal{G}$ for every $k, l \in \omega$, because each $Y \in \mathcal{G}$ has infinite intersection with each X_j. We shall define a strategy Θ for player I in the Menger game on $\mathcal{G}$ as follows. $\Theta(\emptyset) = \{\mathcal{U}_{0,s} : s \in \Sigma_{0,0}\}$. If II replied with some finite subset $\mathcal{V}_0$ of $\Theta(\emptyset)$, I finds $h(0) \in \omega$ such that this reply is contained in $\{\mathcal{U}_{0,s} : s \in h(0)^{j_1} \cap \Sigma_{0,0}\}$ and his next move is then $\Theta\langle\mathcal{V}_0\rangle := \{\mathcal{U}_{1,s} : s \in \Sigma_{1,h(0)}\}$. And so on, i.e., after k many rounds player I finds $h(k) \in \omega$ above $h(k-1)$ (here $h(-1) := 0$) such that the reply of II is contained in $\{\mathcal{U}_{k,s} : s \in h(k)^{j_{k+1}} \cap \Sigma_{k,h(k-1)}\}$ and his next move is then

$$\Theta\langle\mathcal{V}_0, \dots, \mathcal{V}_k\rangle := \{\mathcal{U}_{k+1,s} : s \in \Sigma_{k+1,h(k)}\}.$$

Since $\mathcal{G}$ is Menger, Θ is not winning, and hence there exists an increasing sequence $h \in \omega^\omega \in V[x * H]$ such that

$$\mathcal{G} \subset \bigcup\Big\{\bigcup\{\mathcal{U}_{k,s} : s \in h(k)^{j_{k+1}} \cap \Sigma_{k,h(k-1)}\} \;:\; k \in \omega\Big\}. \tag{5}$$

Next, we shall define a strategy Υ for player I in the Menger game on $\mathcal{G}^+$ as follows. $\Upsilon(\emptyset) = \{\mathcal{U}_{0,s} : s \in S_{0,0}\}$, where $S_{k,l} = \prod_{j<j_{k+1}}\big(\omega \setminus (\mathrm{dom}(p_j) \cup l)\big)$ for all $k, l \in \omega$. If II replied with some finite subset $\mathcal{V}_0$ of $\Upsilon(\emptyset)$, I finds $l_0 \in \omega$ such that this reply is contained in $\{\mathcal{U}_{0,s} : s \in h(l_0)^{j_1}\}$ and his next move is then $\Upsilon(\mathcal{V}_0) := \{\mathcal{U}_{1,s} : s \in S_{1,h(l_0)}\}$. And so on, i.e., after k many rounds player I finds $l_k \in \omega$ above l_{k-1} such that the reply of II is contained in $\{\mathcal{U}_{k,s} : s \in h_1(l_k)^{j_{k+1}}\}$ and his next move is then

$$\Upsilon\langle\mathcal{V}_0, \dots, \mathcal{V}_k\rangle := \{\mathcal{U}_{k+1,s} : s \in S_{k+1,h(l_k)}\}.$$

Since Υ is not winning, there exists an increasing sequence $\langle l_k : k \in \omega\rangle$ (we set also $l_{-1} = 0$ for convenience) such that

$$\mathcal{G}^+ \subset \bigcup_{k\in\omega}\bigcup\{\mathcal{U}_{k,s} : s \in h(l_k)^{j_{k+1}} \cap S_{k,h(l_{k-1})}\}. \tag{6}$$

Let $h_1 \in V \cap \omega^\omega$ be an increasing function such that $h_1(k) \geq h(k)$ for all $k \in \omega$, and $\langle p, \dot{q}\rangle \in x * H$ a condition forcing all the above to happen. Let also $\dot{h}$, $\dot{t}_k$, $\dot{\mathcal{G}}$, $\dot{\Sigma}_{k,l}$, $\dot{x}$, $\dot{x}_j$, $\dot{X}_j$, $\dot{T}$,... be $\mathbb{P}^*_{\mathcal{F}} * \mathbb{Q}$-names of the objects in $V[x * H]$ considered above. Consider the condition $p^1 \in \mathbb{P}^*_{\mathcal{F}}$ below p defined as follows:

$$p^1_j = p_j \cup \{\langle n, 0\rangle : n \in h_1(l_k) \setminus \mathrm{dom}(p_j)\}$$

whenever $j \in [j_k, j_{k+1})$ and $k \in \omega$. Thus $\langle p^1, \dot{q}\rangle$ forces Equations (5) and (6). Thus $\langle p^1, \dot{q}\rangle$ forces the following: If for every $k \in \omega$ and $s \in \dot{h}(k)^{j_{k+1}} \cap \Sigma_{k,\dot{h}(k-1)}$ we pick $j_s \in j_{k+1}$, then

$$\big\{s(j_s) \;:\; s \in \dot{h}(k)^{j_{k+1}} \cap \Sigma_{k,\dot{h}(k-1)}, k \in \omega\big\} \in \dot{\mathcal{G}}^+; \text{ and} \tag{7}$$

if for every $k \in \omega$ and $s \in \dot{h}(\dot{l}_k)^{j_{k+1}} \cap \dot{S}_{k,\dot{h}(\dot{l}_{k-1})}$ we pick $j'_s \in j_{k+1}$, then

$$\big\{s(j'_s) \;:\; s \in \dot{h}(\dot{l}_k)^{j_{k+1}} \cap \dot{S}_{k,\dot{h}(\dot{l}_{k-1})}, k \in \omega\big\} \in \dot{\mathcal{G}}. \tag{8}$$

Given a $\mathbb{P}^*_{\mathcal{F}} * \mathbb{Q}$-generic filter $x^1 * H^1$ containing $\langle p^1, \dot{q}\rangle$, we shall work in $V[x^1 * H^1]$ in what follows. For abuse of notation we shall again use notations for the evaluations with respect to $x^1 * H^1$ of all names considered above, obtained simply by removing "$\dot{}$", i.e., $h := \dot{h}^{x^1 * H^1}$ etc. This way letting $j'_s := t_k$ for each $s \in h(l_k)^{j_{k+1}} \cap S_{k,h(l_{k-1})}$, the set B' defined in Equation (8) belongs to $\mathcal{G}$. For every $k \in \omega$ and $s \in h(k)^{j_{k+1}} \cap \Sigma_{k,h(k-1)}$ set $j_s := t_{m(k)}$, where $m(k) = \min\{m : l_m \geq k\}$. The set B defined in Equation (7) for this choice of j_s's belongs to $\mathcal{G}^+$, and hence $B \cap B' \neq \emptyset$. Thus there exist $k, k' \in \omega$, $s' \in h(l_{k'})^{j_{k'+1}} \cap S_{k',h(l_{k'-1})}$, and $s \in h(k)^{j_{k+1}} \cap \Sigma_{k,h(k-1)}$ such that

$$s(t_{m(k)}) = s'(t_{k'}). \tag{9}$$

Since $s(t_{m(k)}) \in h(k) \setminus h(k-1)$ and $s'(t_{k'}) \in h(l_{k'}) \setminus h(l_{k'-1})$, from Equation (9) we conclude that $h(k) \setminus h(k-1) \subset h(l_{k'}) \setminus h(l_{k'-1})$, and hence $k' = \min\{m : l_m \geq k\}$, which yields $k' = m(k)$. Recall that $s(t_{m(k)}) \in X_{t_{m(k)}}$, and therefore

$$x_{t_{m(k)}}\big(s(t_{m(k)})\big) = \varepsilon_{t_{m(k)}} = 1. \tag{10}$$

On the other hand,

$$x_{t_{m(k)}}\big(s(t_{m(k)})\big) = x_{t_{m(k)}}\big(s'(t_{m(k)})\big) = p^1_{t_{m(k)}}\big(s'(t_{m(k)})\big) \tag{11}$$

because $p^1_{t_{m(k)}} \subset x_{t_{m(k)}}$ and

$$s'(t_{m(k)}) \in h(l_{m(k)}) \subset h_1(l_{m(k)}) \subset \mathrm{dom}(p^1_{t_{m(k)}}),$$

the last inclusion following from $t_{m(k)} \in [j_{m(k)}, j_{m(k)+1})$ and the definition of p^1. Recall that $s'(t_{m(k)}) \notin \mathrm{dom}(p_{t_{m(k)}})$ by the definition of $S_{k',h(l_{k'-1})} = S_{m(k),h(l_{m(k)-1})}$, which implies that $p^1_{t_{m(k)}}\big(s'(t_{m(k)})\big) = 0$ by the definition of p^1. Thus $x_{t_{m(k)}}\big(s(t_{m(k)})\big) = x_{t_{m(k)}}\big(s'(t_{m(k)})\big) = 0$ by Equation (11), which is impossible by Equation (10). This contradiction completes our proof. □

The next theorem improves [8, Theorem 4.5] and is the main result of this section.

Theorem 4.3. *It is consistent that there is no semifilter $\mathcal{G}$ on ω such that $\mathcal{G} \subset \mathcal{G}^+$ and both $\mathcal{G}, \mathcal{G}^+$ are Menger.*

Proof. Let us assume that $V = L$ and[3] consider a function $B : \omega_2 \to H(\omega_2)$, the family of all sets whose transitive closure has size $< \omega_2$. Let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \beta < \alpha \leq \omega_2 \rangle$ be the following iteration with at most countable supports: If $B(\alpha)$ is a $\mathbb{P}_\alpha$-name for $\mathbb{P}^*_{\dot{\mathcal{F}}}$ for some semifilter $\dot{\mathcal{F}}$ such that $\Vdash_{\mathbb{P}_\alpha}$ "$\dot{\mathcal{F}} \subset \dot{\mathcal{F}}^+$ and $\dot{\mathcal{F}}, \dot{\mathcal{F}}^+$ are Menger", then $\dot{\mathbb{Q}}_\alpha = B(\alpha)$. Otherwise we let $\dot{\mathbb{Q}}_\alpha$ be a $\mathbb{P}_\alpha$-name for the trivial forcing. Then $\mathbb{P}_{\omega_2}$ is ω^ω-bounding forcing notion with ω_2-c.c. being a countable support iteration of length ω_2 of proper ω^ω-bounding posets of size ω_1 over a model of GCH. Now, using the suitable diamond in V for the choice of B together with a standard reflection argument, we can guarantee in addition that for any $\mathbb{P}_{\omega_2}$-generic filter G over V and semifilter $\mathcal{F} \in V[G]$ such that $\mathcal{F} \subset \mathcal{F}^+$ and $\mathcal{F}, \mathcal{F}^+$ are Menger, the following holds:

> The set $\big\{\alpha : \mathcal{F}_\alpha := (\mathcal{F} \cap V[G \cap \mathbb{P}_\alpha]) \in V[G \cap \mathbb{P}_\alpha],\ \mathcal{F}_\alpha \subset \mathcal{F}^+_\alpha,\ \mathcal{F}_\alpha, \mathcal{F}^+_\alpha$ are Menger in $V[G \cap \mathbb{P}_\alpha]$, and $\dot{\mathbb{Q}}^{G \cap \mathbb{P}_\alpha}_\alpha = \mathbb{P}^*_{\dot{\mathcal{F}}_\alpha}\big\}$ is stationary in ω_2.

Now, a direct application of Lemma 4.2 implies that $\mathcal{F}_\alpha$ cannot be enlarged to any semifilter $\mathcal{U} \subset \mathcal{U}^+$ in $V[G]$ such that $\mathcal{U}, \mathcal{U}^+$ are Menger, which contradicts the fact that $\mathcal{F}$ is such an enlargement. □

5. Open questions

Theorem 2.4 together with Corollaries 2.5 and 2.6 motivate the following

[3] It suffices to assume $2^\omega = \omega_1$, $2^{\omega_1} = \omega_2$, and $\diamond_{\{\delta \in \omega_2 : \mathrm{cf}(\delta) = \omega_1\}}$, see [27, Theorem 5.13].

Question 5.1.

- Is there a ZFC example of a non-meager filter $\mathcal{F}$ such that $\mathcal{F}^+$ is not homeomorphic to $(\mathcal{F}^+)^2$?
- Is $\mathcal{F} \times \mathcal{F}^+$ homeomorphic to $\mathcal{F}^+$ for every non-meager filter $\mathcal{F}$?

In light of Theorem 3.2 it is natural to ask the next question. Let us recall from [12] that for a filter $\mathcal{F}$ the Mathias forcing associated to it adds no dominating reals iff $\mathcal{F}$ is Menger, so the following question is especially interesting when $\mathcal{F}$ is not an ultrafilter.

Question 5.2. Let $\mathcal{F}$ be a filter such that $\mathcal{F}^+$ is Menger. Is there a (proper) poset adding no dominating reals and adding an infinite pseudointersection of $\mathcal{F}$? What about the Mathias forcing associated to $\mathcal{F}^+$?

Theorem 4.3 leaves open one possibility whose inconsistency we are unable to establish.

Question 5.3.

- Is there a ZFC example of a filter $\mathcal{F}$ such that $\mathcal{F}^+$ is Menger?
- Is there a ZFC example of a semifilter $\mathcal{F} \subset \mathcal{F}^+$ such that $\mathcal{F}^+$ is Menger?

The first item of the question above has been mentioned to us by Mikołaj Krupski in private communication and is related to his work [21]. By [8, Lemma 3.1] the negative answer to the second item of Question 5.3 implies that consistently every Menger remainder of a topological group is σ-compact.

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