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Multiple solutions for a Neumann system involving subquadratic nonlinearities

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ABSTRACT

In this paper, we consider the model semilinear Neumann system

$$\begin{cases} -\Delta u + a(x)u = \lambda c(x)F_u(u, v) & \text{in } \Omega, \\ -\Delta v + b(x)v = \lambda c(x)F_v(u, v) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{N_λ}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth open bounded domain, ν denotes the outward unit normal to $\partial\Omega$, $\lambda \geq 0$ is a parameter, $a, b, c \in L^\infty_+(\Omega) \setminus \{0\}$, and $F \in C^1(\mathbb{R}^2, \mathbb{R}) \setminus \{0\}$ is a nonnegative function which is subquadratic at infinity. Two nearby numbers are determined in explicit forms, $\underline{\lambda}$ and $\overline{\lambda}$ with $0 < \underline{\lambda} \leq \overline{\lambda}$, such that for every $0 \leq \lambda < \underline{\lambda}$, system ($N_\lambda$) has only the trivial pair of solution, while for every $\lambda > \overline{\lambda}$, system ($N_\lambda$) has at least two distinct nonzero pairs of solutions.

1. Introduction

Let us consider the quasilinear Neumann system

$$\begin{cases} -\Delta_p u + a(x)|u|^{p-2}u = \lambda c(x)F_u(u, v) & \text{in } \Omega, \\ -\Delta_q v + b(x)|v|^{q-2}v = \lambda c(x)F_v(u, v) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{$N_\lambda^{p,q}$}$$

where $p, q > 1$; $\Omega \subset \mathbb{R}^N$ is a smooth open bounded domain; ν denotes the outward unit normal to $\partial\Omega$; $a, b, c \in L^\infty(\Omega)$ are some functions; $\lambda \geq 0$ is a parameter; and F_u and F_v denote the partial derivatives of $F \in C^1(\mathbb{R}^2, \mathbb{R})$ with respect to the first and second variables, respectively.

Recently, problem ($N_\lambda^{p,q}$) has been considered by several authors. For instance, under suitable assumptions on a, b, c and F, El Manouni and Kbiri Alaoui [1] proved the existence of an interval $A \subset (0, \infty)$ such that ($N_\lambda^{p,q}$) has at least three solutions whenever $\lambda \in A$ and $p, q > N$. Lisei and Varga [2] also established the existence of at least three solutions for the system ($N_\lambda^{p,q}$) with nonhomogeneous and nonsmooth Neumann boundary conditions. Di Falco [3] proved the existence of infinitely many solutions for ($N_\lambda^{p,q}$) when the nonlinear function F has a suitable oscillatory behavior. Systems similar to ($N_\lambda^{p,q}$) with the Dirichlet boundary conditions were also considered by Afrouzi and Heidarkhani [4,5], Boccardo and de Figueiredo [6], Heidarkhani and Tian [7], and Li and Tang [8]; see also references therein.

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The aim of the present paper is to describe a new phenomenon for Neumann systems when the nonlinear term has a subquadratic growth. In order to avoid technicalities, instead of the quasilinear system $(N_\lambda^{p,q})$, we shall consider the semilinear problem

$$\begin{cases} -\Delta u + a(x)u = \lambda c(x)F_u(u,v) & \text{in } \Omega, \\ -\Delta v + b(x)v = \lambda c(x)F_v(u,v) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial v}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{N_λ}$$

We assume that the nonlinear term $F \in C^1(\mathbb{R}^2, \mathbb{R})$ satisfies the following properties:

($\mathbf{F}_+$) $F(s,t) \geq 0$ for every $(s,t) \in \mathbb{R}^2$, $F(0,0) = 0$, and $F \not\equiv 0$;
($\mathbf{F}_0$) $\lim_{(s,t)\to(0,0)} \frac{F_s(s,t)}{|s|+|t|} = \lim_{(s,t)\to(0,0)} \frac{F_t(s,t)}{|s|+|t|} = 0$;
($\mathbf{F}_\infty$) $\lim_{|s|+|t|\to\infty} \frac{F_s(s,t)}{|s|+|t|} = \lim_{|s|+|t|\to\infty} \frac{F_t(s,t)}{|s|+|t|} = 0$.

Example 1.1. A typical nonlinearity which fulfils hypotheses ($\mathbf{F}_+$), ($\mathbf{F}_0$) and ($\mathbf{F}_\infty$) is $F(s,t) = \ln(1 + s^2t^2)$.

We also introduce the set

$$\Pi_+(\Omega) = \{a \in L^\infty(\Omega) : \text{essinf}_\Omega a > 0\}.$$

For $a, b, c \in \Pi_+(\Omega)$ and for $F \in C^1(\mathbb{R}^2, R)$ which fulfils the hypotheses ($\mathbf{F}_+$), ($\mathbf{F}_0$) and ($\mathbf{F}_\infty$), we define the numbers

$$s_F = 2\|c\|_{L^1} \max_{(s,t)\neq(0,0)} \frac{F(s,t)}{\|a\|_{L^1}s^2 + \|b\|_{L^1}t^2}, \quad \text{and} \quad S_F = \max_{(s,t)\neq(0,0)} \frac{|sF_s(s,t) + tF_t(s,t)|}{\|c/a\|_{L^\infty}^{-1}s^2 + \|c/b\|_{L^\infty}^{-1}t^2}.$$

Note that these numbers are finite, positive and $S_F \geq s_F$, see Proposition 2.1 (here and in what follows, $\|\cdot\|_{L^p}$ denotes the usual norm of the Lebesgue space $L^p(\Omega)$, $p \in [1,\infty]$). Our main result reads as follows.

Theorem 1.1. *Let $F \in C^1(\mathbb{R}^2, \mathbb{R})$ be a function which satisfies ($\mathbf{F}_+$), ($\mathbf{F}_0$) and ($\mathbf{F}_\infty$), and $a, b, c \in \Pi_+(\Omega)$. Then, the following statements hold.*

(i) *For every $0 \leq \lambda < S_F^{-1}$, system (N_λ) has only the trivial pair of solution.*
(ii) *For every $\lambda > s_F^{-1}$, system (N_λ) has at least two distinct, nontrivial pairs of solutions $(u_\lambda^i, v_\lambda^i) \in H^1(\Omega)^2$, $i \in \{1,2\}$.*

Remark 1.1. (a) A natural question arises which is still open: how many solutions exist for (N_λ) when $\lambda \in [S_F^{-1}, s_F^{-1}]$? Numerical experiments show that s_F and S_F are usually not far from each other, although their origins are independent. For instance, if $a = b = c$, and F is from Example 1.1, we have $s_F \approx 0.8046$ and $S_F = 1$.

(b) Assumptions ($\mathbf{F}_+$), ($\mathbf{F}_0$) and ($\mathbf{F}_\infty$) imply that there exists $c > 0$ such that

$$0 \leq F(s,t) \leq c(s^2 + t^2) \quad \text{for all } (s,t) \in \mathbb{R}^2, \tag{1.1}$$

i.e., F has a subquadratic growth. Consequently, Theorem 1.1 completes the results of several papers, where F fulfils the Ambrosetti–Rabinowitz condition, i.e., there exist $\theta > 2$ and $r > 0$ such that

$$0 < \theta F(s,t) \leq sF_s(s,t) + tF_t(s,t) \quad \text{for all } |s|, |t| \geq r. \tag{1.2}$$

Indeed, (1.2) implies that for some $C_1, C_2 > 0$, one has $F(s,t) \geq C_1(|s|^\theta + |t|^\theta)$ for all $|s|, |t| > C_2$.

The next section contains some auxiliary notions and results, while in Section 3 we prove Theorem 1.1. First, a direct calculation proves (i), while a very recent three critical points result of Ricceri [9] provides the proof of (ii).

2. Preliminaries

A solution for (N_λ) is a pair $(u,v) \in H^1(\Omega)^2$ such that

$$\begin{cases} \displaystyle\int_\Omega (\nabla u \nabla \phi + a(x)u\phi)\mathrm{d}x = \lambda \int_\Omega c(x)F_u(u,v)\phi\mathrm{d}x & \text{for all } \phi \in H^1(\Omega), \\ \displaystyle\int_\Omega (\nabla v \nabla \psi + b(x)v\psi)\mathrm{d}x = \lambda \int_\Omega c(x)F_v(u,v)\psi\mathrm{d}x & \text{for all } \psi \in H^1(\Omega). \end{cases} \tag{2.1}$$

Let $a, b, c \in \Pi_+(\Omega)$. We associate to the system (N_λ) the energy functional $I_\lambda : H^1(\Omega)^2 \to \mathbb{R}$ defined by

$$I_\lambda(u,v) = \frac{1}{2}(\|u\|_a^2 + \|v\|_b^2) - \lambda\mathcal{F}(u,v),$$

where

$$\|u\|_a = \left(\int_\Omega |\nabla u|^2 + a(x)u^2\right)^{1/2}; \qquad \|v\|_b = \left(\int_\Omega |\nabla v|^2 + b(x)v^2\right)^{1/2},$$

and

$$\mathcal{F}(u,v) = \int_\Omega c(x)F(u,v).$$

It is clear that $\|\cdot\|_a$ and $\|\cdot\|_b$ are equivalent to the usual norm on $H^1(\Omega)$. Note that if $F \in C^1(\mathbb{R}^2, R)$ verifies the hypotheses $(\mathbf{F}_0)$ and $(\mathbf{F}_\infty)$ (see also relation (1.1)), the functional I_λ is well defined, of class C^1 on $H^1(\Omega)^2$ and its critical points are exactly the solutions for (N_λ). Since $F_s(0,0) = F_t(0,0) = 0$ from $(\mathbf{F}_0)$, $(0,0)$ is a solution of (N_λ) for every $\lambda \geq 0$.

In order to prove Theorem 1.1(ii), we must find critical points for I_λ. In order to do this, we recall the following Ricceri-type three critical point theorem. First, we need the following notion: if X is a Banach space, we denote by $\mathcal{W}_X$ the class of those functionals $E : X \to \mathbb{R}$ that possess the property that if $\{u_n\}$ is a sequence in X converging weakly to $u \in X$ and $\liminf_n E(u_n) \leq E(u)$ then $\{u_n\}$ has a subsequence strongly converging to u.

Theorem 2.1 *([9, Theorem 2]). Let X be a separable and reflexive real Banach space, let $E_1 : X \to \mathbb{R}$ be a coercive, sequentially weakly lower semicontinuous C^1 functional belonging to $\mathcal{W}_X$, bounded on each bounded subset of X and whose derivative admits a continuous inverse on X^*, and $E_2 : X \to \mathbb{R}$ a C^1 functional with a compact derivative. Assume that E_1 has a strict local minimum u_0 with $E_1(u_0) = E_2(u_0) = 0$. Setting the numbers*

$$\tau = \max\left\{0, \limsup_{\|u\|\to\infty} \frac{E_2(u)}{E_1(u)}, \limsup_{u\to u_0} \frac{E_2(u)}{E_1(u)}\right\}, \tag{2.2}$$

$$\chi = \sup_{E_1(u)>0} \frac{E_2(u)}{E_1(u)}, \tag{2.3}$$

assume that $\tau < \chi$.

Then, for each compact interval $[a,b] \subset (1/\chi, 1/\tau)$ (with the conventions $1/0 = \infty$ and $1/\infty = 0$) there exists $\kappa > 0$ with the following property: for every $\lambda \in [a,b]$ and every C^1 functional $E_3 : X \to \mathbb{R}$ with a compact derivative, there exists $\delta > 0$ such that for each $\theta \in [0,\delta]$, the equation

$$E_1'(u) - \lambda E_2'(u) - \theta E_3'(u) = 0$$

admits at least three solutions in X having norm less than κ.

We conclude this section with an observation which involves the constants s_F and S_F.

Proposition 2.1. *Let $F \in C^1(\mathbb{R}^2, \mathbb{R})$ be a function which satisfies $(\mathbf{F}_+)$, $(\mathbf{F}_0)$ and $(\mathbf{F}_\infty)$, and $a, b, c \in \Pi_+(\Omega)$. Then the numbers s_F and S_F are finite, positive and $S_F \geq s_F$.*

Proof. It follows from $(\mathbf{F}_0)$ and $(\mathbf{F}_\infty)$ and from the continuity of the functions $(s,t) \mapsto \frac{F_s(s,t)}{|s|+|t|}$, $(s,t) \mapsto \frac{F_t(s,t)}{|s|+|t|}$ away from $(0,0)$, that there exists $M > 0$ such that

$$|F_s(s,t)| \leq M(|s|+|t|) \quad \text{and} \quad |F_t(s,t)| \leq M(|s|+|t|) \quad \text{for all } (s,t) \in \mathbb{R}^2.$$

Consequently, a standard mean value theorem together with $(\mathbf{F}_+)$ implies that

$$0 \leq F(s,t) \leq 2M(s^2+t^2) \quad \text{for all } (s,t) \in \mathbb{R}^2. \tag{2.4}$$

We now prove that

$$\lim_{(s,t)\to(0,0)} \frac{F(s,t)}{s^2+t^2} = 0 \quad \text{and} \quad \lim_{|s|+|t|\to\infty} \frac{F(s,t)}{s^2+t^2} = 0. \tag{2.5}$$

From $(\mathbf{F}_0)$ and $(\mathbf{F}_\infty)$, for every $\varepsilon > 0$, there exists $\delta_\varepsilon \in (0,1)$ such that for every $(s,t) \in \mathbb{R}^2$ with $|s|+|t| \in (0,\delta_\varepsilon) \cup (\delta_\varepsilon^{-1}, \infty)$, one has

$$\frac{|F_s(s,t)|}{|s|+|t|} < \frac{\varepsilon}{4} \quad \text{and} \quad \frac{|F_t(s,t)|}{|s|+|t|} < \frac{\varepsilon}{4}. \tag{2.6}$$

From (2.6) and the mean value theorem, for every $(s,t) \in \mathbb{R}^2$ with $|s|+|t| \in (0,\delta_\varepsilon)$, we have

$$\begin{aligned} F(s,t) &= F(s,t) - F(0,t) + F(0,t) - F(0,0) \\ &\leq \frac{\varepsilon}{2}(s^2+t^2) \end{aligned}$$

which gives the first limit in (2.5). Now, for every $(s,t) \in \mathbb{R}^2$ with $|s|+|t| > \delta_\varepsilon^{-1}\max\{1, \sqrt{8M/\varepsilon}\}$, by using (2.4) and (2.6), we have

$$\begin{aligned} F(s,t) &= F(s,t) - F\left(\frac{\delta_\varepsilon^{-1}}{|s|+|t|}s, t\right) + F\left(\frac{\delta_\varepsilon^{-1}}{|s|+|t|}s, t\right) - F\left(\frac{\delta_\varepsilon^{-1}}{|s|+|t|}s, \frac{\delta_\varepsilon^{-1}}{|s|+|t|}t\right) + F\left(\frac{\delta_\varepsilon^{-1}}{|s|+|t|}s, \frac{\delta_\varepsilon^{-1}}{|s|+|t|}t\right) \\ &\le \frac{\varepsilon}{4}(|s|+|t|)^2 + 2M\delta_\varepsilon^{-2} \\ &\le \varepsilon(s^2+t^2), \end{aligned}$$

which leads us to the second limit in (2.5).

The facts above show that the numbers s_F and S_F are finite. Moreover, $s_F > 0$. We now prove that $S_F \ge s_F$. To do this, let $(s_0,t_0) \in \mathbb{R}^2 \setminus \{(0,0)\}$ be a maximum point of the function $(s,t) \mapsto \frac{F(s,t)}{\|a\|_{L^1}s^2+\|b\|_{L^1}t^2}$. In particular, its partial derivatives vanishes at (s_0,t_0), yielding

$$F_s(s_0,t_0)(\|a\|_{L^1}s_0^2 + \|b\|_{L^1}t_0^2) = 2\|a\|_{L^1}s_0F(s_0,t_0);$$
$$F_t(s_0,t_0)(\|a\|_{L^1}s_0^2 + \|b\|_{L^1}t_0^2) = 2\|b\|_{L^1}t_0F(s_0,t_0).$$

From the two relations above, we obtain

$$s_0F_s(s_0,t_0) + t_0F_t(s_0,t_0) = 2F(s_0,t_0).$$

On the other hand, since $a, b, c \in \Pi_+(\Omega)$, we have

$$\|c\|_{L^1} = \int_\Omega c(x)\mathrm{d}x = \int_\Omega \frac{c(x)}{a(x)}a(x)\mathrm{d}x \le \left\|\frac{c}{a}\right\|_{L^\infty}\int_\Omega a(x)\mathrm{d}x = \left\|\frac{c}{a}\right\|_{L^\infty}\|a\|_{L^1},$$

thus $\|c/a\|_{L^\infty}^{-1} \le \|a\|_{L^1}/\|c\|_{L^1}$ and in a similar way $\|c/b\|_{L^\infty}^{-1} \le \|b\|_{L^1}/\|c\|_{L^1}$. Combining these inequalities with the above argument, we conclude that $S_F \ge s_F$. □

3. Proof of Theorem 1.1

In this section we assume that the assumptions of Theorem 1.1 are fulfilled.

Proof of Theorem 1.1(i). Let $(u,v) \in H^1(\Omega)^2$ be a solution of (N_λ). Choosing $\phi = u$ and $\psi = v$ in (2.1), we obtain

$$\begin{aligned} \|u\|_a^2 + \|v\|_b^2 &= \int_\Omega (|\nabla u|^2 + a(x)u^2 + |\nabla v|^2 + b(x)v^2) \\ &= \lambda\int_\Omega c(x)(F_u(u,v)u + F_v(u,v)v) \\ &\le \lambda S_F\int_\Omega c(x)(\|c/a\|_{L^\infty}^{-1}u^2 + \|c/b\|_{L^\infty}^{-1}v^2) \\ &\le \lambda S_F\int_\Omega (a(x)u^2 + b(x)v^2) \\ &\le \lambda S_F(\|u\|_a^2 + \|v\|_b^2). \end{aligned}$$

Now, if $0 \le \lambda < S_F^{-1}$, we necessarily have $(u,v) = (0,0)$, which concludes the proof. □

Proof of Theorem 1.1(ii). In Theorem 2.1, we choose $X = H^1(\Omega)^2$ endowed with the norm $\|(u,v)\| = \sqrt{\|u\|_a^2 + \|v\|_b^2}$, and $E_1, E_2 : H^1(\Omega)^2 \to \mathbb{R}$ defined by

$$E_1(u,v) = \frac{1}{2}\|(u,v)\|^2 \quad \text{and} \quad E_2(u,v) = \mathcal{F}(u,v).$$

It is clear that both E_1 and E_2 are C^1 functionals and $I_\lambda = E_1 - \lambda E_2$. It is also a standard fact that E_1 is a coercive, sequentially weakly lower semicontinuous functional which belongs to $\mathcal{W}_{H^1(\Omega)^2}$, bounded on each bounded subset of $H^1(\Omega)^2$, and its derivative admits a continuous inverse on $(H^1(\Omega)^2)^*$. Moreover, E_2 has a compact derivative since $H^1(\Omega) \hookrightarrow L^p(\Omega)$ is a compact embedding for every $p \in (2, 2^*)$.

Now, we prove that the functional $(u,v) \mapsto \frac{E_2(u,v)}{E_1(u,v)}$ has similar properties as the function $(s,t) \mapsto \frac{F(s,t)}{s^2+t^2}$. More precisely, we shall prove that

$$\lim_{\|(u,v)\|\to 0}\frac{E_2(u)}{E_1(u)} = \lim_{\|(u,v)\|\to\infty}\frac{E_2(u)}{E_1(u)} = 0. \tag{3.1}$$

First, relation (2.5) implies that for every $\varepsilon > 0$ there exists $\delta_\varepsilon \in (0, 1)$ such that for every $(s, t) \in \mathbb{R}^2$ with $|s| + |t| \in (0, \delta_\varepsilon) \cup (\delta_\varepsilon^{-1}, \infty)$, one has

$$0 \le \frac{F(s,t)}{s^2+t^2} < \frac{\varepsilon}{4\max\{\|c/a\|_{L^\infty}, \|c/b\|_{L^\infty}\}}. \tag{3.2}$$

Fix $p \in (2, 2^*)$. Note that the continuous function $(s, t) \mapsto \frac{F(s,t)}{|s|^p+|t|^p}$ is bounded on the set $\{(s, t) \in \mathbb{R}^2 : |s| + |t| \in [\delta_\varepsilon, \delta_\varepsilon^{-1}]\}$. Therefore, for some $m_\varepsilon > 0$, we have in particular

$$0 \le F(s,t) \le \frac{\varepsilon}{4\max\{\|c/a\|_{L^\infty}, \|c/b\|_{L^\infty}\}}(s^2+t^2) + m_\varepsilon(|s|^p+|t|^p) \quad \text{for all } (s,t) \in \mathbb{R}^2.$$

Therefore, for each $(u, v) \in H^1(\Omega)^2$, we get

$$\begin{aligned} 0 \le E_2(u,v) &= \int_\Omega c(x)F(u,v) \\ &\le \int_\Omega c(x)\left[\frac{\varepsilon}{4\max\{\|c/a\|_{L^\infty}, \|c/b\|_{L^\infty}\}}(u^2+v^2) + m_\varepsilon(|u|^p+|v|^p)\right] \\ &\le \int_\Omega \left[\frac{\varepsilon}{4}(a(x)u^2+b(x)v^2) + m_\varepsilon c(x)(|u|^p+|v|^p)\right] \\ &\le \frac{\varepsilon}{4}\|(u,v)\|^2 + m_\varepsilon\|c\|_{L^\infty}S_p^p(\|u\|_a^p + \|v\|_b^p) \\ &\le \frac{\varepsilon}{4}\|(u,v)\|^2 + m_\varepsilon\|c\|_{L^\infty}S_p^p\|(u,v)\|^p, \end{aligned}$$

where $S_l > 0$ is the best constant in the inequality $\|u\|_{L^l} \le S_l \min\{\|u\|_a, \|u\|_b\}$ for every $u \in H^1(\Omega)$, $l \in (1, 2^*)$ (we used the fact that the function $\alpha \mapsto (s^\alpha + t^\alpha)^{\frac{1}{\alpha}}$ is decreasing, $s, t \ge 0$). Consequently, for every $(u, v) \ne (0, 0)$, we obtain

$$0 \le \frac{E_2(u,v)}{E_1(u,v)} \le \frac{\varepsilon}{2} + 2m_\varepsilon\|c\|_{L^\infty}S_p^p\|(u,v)\|^{p-2}.$$

Since $p > 2$ and $\varepsilon > 0$ is arbitrarily small when $(u, v) \to 0$, we obtain the first limit from (3.1).

Now, we fix $r \in (1, 2)$. The continuous function $(s, t) \mapsto \frac{F(s,t)}{|s|^r+|t|^r}$ is bounded on the set $\{(s, t) \in \mathbb{R}^2 : |s|+|t| \in [\delta_\varepsilon, \delta_\varepsilon^{-1}]\}$, where $\delta_\varepsilon \in (0, 1)$ is from (3.2). Combining this fact with (3.2), one can find a number $M_\varepsilon > 0$ such that

$$0 \le F(s,t) \le \frac{\varepsilon}{4\max\{\|c/a\|_{L^\infty}, \|c/b\|_{L^\infty}\}}(s^2+t^2) + M_\varepsilon(|s|^r+|t|^r) \quad \text{for all } (s,t) \in \mathbb{R}^2.$$

The Hölder inequality and a similar calculation as above show that

$$0 \le E_2(u,v) \le \frac{\varepsilon}{4}\|(u,v)\|^2 + 2^{1-\frac{r}{2}}M_\varepsilon\|c\|_{L^\infty}S_r^r\|(u,v)\|^r.$$

For every $(u, v) \ne (0, 0)$, we have

$$0 \le \frac{E_2(u,v)}{E_1(u,v)} \le \frac{\varepsilon}{2} + 2^{2-\frac{r}{2}}M_\varepsilon\|c\|_{L^\infty}S_r^r\|(u,v)\|^{r-2}.$$

Due to the arbitrariness of $\varepsilon > 0$ and $r \in (1, 2)$, by letting the limit $\|(u, v)\| \to \infty$, we obtain the second relation from (3.1).

Note that E_1 has a strict global minimum $(u_0, v_0) = (0, 0)$, and $E_1(0, 0) = E_2(0, 0) = 0$. The definition of the number τ in Theorem 2.1, see (2.2), and the limits in (3.1) imply that $\tau = 0$. Furthermore, since $H^1(\Omega)$ contains the constant functions on Ω, keeping the notation from (2.3), we obtain

$$\chi = \sup_{E_1(u,v)>0}\frac{E_2(u,v)}{E_1(u,v)} \ge 2\|c\|_{L^1}\max_{(s,t)\ne(0,0)}\frac{F(s,t)}{\|a\|_{L^1}s^2+\|b\|_{L^1}t^2} = s_F.$$

Therefore, applying Theorem 2.1 (with $E_3 \equiv 0$), we obtain, in particular, for every $\lambda \in (s_F^{-1}, \infty)$, the equation $I'_\lambda(u, v) \equiv E'_1(u, v) - \lambda E'_2(u, v) = 0$ admits at least three distinct pairs of solutions in $H^1(\Omega)^2$. Due to condition $(\mathbf{F}_0)$, system (N_λ) has the solution $(0, 0)$. Therefore, for every $\lambda > s_F^{-1}$, the system (N_λ) has at least two distinct, nontrivial pairs of solutions, which concludes the proof. □

Remark 3.1. The conclusion of Theorem 2.1 gives a much more precise information about the Neumann system (N_λ); namely, one can see that (N_λ) is stable with respect to small perturbations. To be more precise, let us consider the perturbed system

$$\begin{cases} -\Delta u + a(x)u = \lambda c(x)F_u(u,v) + \mu d(x)G_u(u,v) & \text{in } \Omega, \\ -\Delta v + b(x)v = \lambda c(x)F_v(u,v) + \mu d(x)G_v(u,v) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial v}{\partial n} = 0 & \text{on } \partial\Omega \end{cases} \qquad (N_{\lambda,\mu})$$

where $\mu \in \mathbb{R}$, $d \in L^\infty(\Omega)$, and $G \in C^1(\mathbb{R}^2, \mathbb{R})$ is a function such that for some $c > 0$ and $1 < p < 2^* - 1$,

$$\max\{|G_s(s,t)|, |G_t(s,t)|\} \le c(1 + |s|^p + |t|^p) \quad \text{for all } (s,t) \in \mathbb{R}^2.$$

One can prove in a standard manner that $E_3 : H^1(\Omega)^2 \to \mathbb{R}$ defined by

$$E_3(u,v) = \int_\Omega d(x)G(u,v)\mathrm{d}x,$$

is of class C^1 and it has a compact derivative. Thus, we may apply Theorem 2.1 in its generality to show that for small enough values of μ system $(N_{\lambda,\mu})$ still has three distinct pairs of solutions.

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