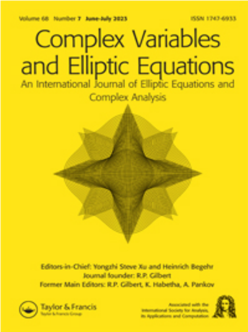

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Complex Variables and Elliptic Equations
An International Journal

ISSN: (Print) (Online) Journal homepage: https://www.tandfonline.com/loi/gcov20

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To cite this article: Zhongyi Zhang & Dušan D. Repovš (2023) On degenerate fractional Schrödinger–Kirchhoff–Poisson equations with upper critical nonlinearity and electromagnetic fields, Complex Variables and Elliptic Equations, 68:7, 1219-1238, DOI: 10.1080/17476933.2022.2040022

To link to this article: https://doi.org/10.1080/17476933.2022.2040022

Published online: 28 Feb 2022.

COMPLEX VARIABLES AND ELLIPTIC EQUATIONS
2023, VOL. 68, NO. 7, 1219–1238
https://doi.org/10.1080/17476933.2022.2040022

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On degenerate fractional Schrödinger–Kirchhoff–Poisson equations with upper critical nonlinearity and electromagnetic fields

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ABSTRACT

This paper intends to study the following degenerate fractional Schrödinger–Kirchhoff–Poisson equations with critical nonlinearity and electromagnetic fields in $\mathbb{R}^3$:

$$\begin{cases} \varepsilon^{2s}M([u]^2_{s,A})(-\Delta)^s_A u + V(x)u + \phi u \\ \quad = k(x)|u|^{r-2}u + \left(\mathcal{I}_\mu * |u|^{2^\sharp_s}\right)|u|^{2^\sharp_s-2}u, & x \in \mathbb{R}^3, \\ (-\Delta)^t\phi = u^2, & x \in \mathbb{R}^3, \end{cases}$$

where $\varepsilon > 0$ is a positive parameter, $3/4 < s < 1$, $0 < t < 1$, V is an electric potential satisfying suitable assumptions, and $0 < k_* \le k(x) \le k^*, \mathcal{I}_\mu(x) = |x|^{3-\mu}$ with $0 < \mu < 3, 2^\sharp_s = \frac{3+\mu}{3-2s}$ and $2 < r < 2^\sharp_s$. With the help of the concentration compactness principle and variational method, and together with some careful analytical skills, we prove the existence and multiplicity of solutions for the above problem as $\varepsilon \to 0$ in degenerate cases, that is the Kirchhoff term M can vanish at zero.

ARTICLE HISTORY
Received 28 October 2021
Accepted 3 February 2022

COMMUNICATED BY
T. Bartsch

KEYWORDS
Fractional Schrödinger–Kirchhoff–Poisson equations; degenerate cases; concentration compactness principle; upper critical nonlinearity; variational methods

AMS SUBJECT CLASSIFICATIONS
35A15; 35B99; 35J60; 47G20

1. Introduction

In this paper, we intend to study the following degenerate fractional Schrödinger–Kirchhoff–Poisson equations with upper critical nonlinearity and electromagnetic fields in $\mathbb{R}^3$:

$$\begin{cases} \varepsilon^{2s}M([u]^2_{s,A})(-\Delta)^s_A u + V(x)u + \phi u = k(x)|u|^{r-2}u + \left(\mathcal{I}_\mu * |u|^{2^\sharp_s}\right)|u|^{2^\sharp_s-2}u, & x \in \mathbb{R}^3, \\ (-\Delta)^t\phi = u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1}$$

where $\varepsilon > 0$ is a positive parameter, $3/4 < s < 1, 0 < t < 1, \mathcal{I}_\mu(x) = |x|^{3-\mu}$ with $0 < \mu < 3$, $0 < k_* \le k(x) \le k^*$, $2^\sharp_s = \frac{3+\mu}{3-2s}$ is the upper Sobolev critical exponent, $2 < r < 2^\sharp_s$, V is an electric potential, $(-\Delta)^s_A$ and A are called the magnetic operator and magnetic potential, respectively. This problem belong to fractional Schrödinger–Poisson system due to the

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potential ϕ satisfying a nonlinear fractional Poisson equation in problem (1). According to d'Avenia and Squassina [13], the fractional operator $(-\Delta)^s_A$ is defined by

$$(-\Delta)^s_A u(x) := 2 \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{u(x) - e^{i(x-y)\cdot A\left(\frac{x+y}{2}\right)} u(y)}{|x-y|^{N+2s}} \, dy, \quad x \in \mathbb{R}^N,$$

and magnetic potential A is given by

$$[u]^2_{s,A} := \iint_{\mathbb{R}^{2N}} \frac{|u(x) - e^{i(x-y)\cdot A\left(\frac{x+y}{2}\right)} u(y)|^2}{|x-y|^{N+2s}} \, dx \, dy.$$

Throughout the paper, the electric potential V and Kirchhoff function M satisfy the following assumptions:

- $(\mathcal{V})$ $V(x) \in C(\mathbb{R}^3, \mathbb{R})$, $V(0) = \min_{x \in \mathbb{R}^3} V(x) = 0$ and the set $V^b = \{x \in \mathbb{R}^3 : V(x) < b\}$ has finite Lebesgue measure, where $b > 0$ is a positive constant.
- $(\mathfrak{M})$ (M_1) There exists $\sigma \in (1, 2^\sharp_s/2)$ satisfying $\sigma \mathcal{M}(t) \geq M(t)t$ for all $t \geq 0$, where $\mathcal{M}(t) = \int_0^t M(s) \, ds$.
  (M_2) There exists $m_1 > 0$ such that $M(t) \geq m_1 t^{\sigma - 1}$ for all $t \in \mathbb{R}^+$ and $M(0) = 0$.

Remark 1.1: The typical function that satisfies conditions (M_1)-(M_2) is given by $M(t) = a + b\,t^{\sigma-1}$ for $t \in \mathbb{R}^+_0$, where $a \in \mathbb{R}^+_0$, $b \in \mathbb{R}^+_0$ and $a + b > 0$. In particular, when $M(t) \geq d > 0$ for some d and all $t \geq 0$, this case is said to be non-degenerate, while it is called degenerate if $M(0) = 0$ and $M(t) > 0$ for $t > 0$. However, in proving the compactness condition, the two cases of degenerate and non-degenerate are completely different, and it's more complicated in a degenerate case. In this paper, we deal with the critical fractional Schrödinger–Kirchhoff–Poisson equations with electromagnetic fields in degenerate cases.

Our motivation to study problem (1) mainly comes from the application of the fractional magnetic operator. We note that the equation with fractional magnetic operator often arises as a model for various physical phenomena, in particular in the study of the infinitesimal generators of Lévy stable diffusion processes [14]. Also, a lot of literature on nonlocal operators and their applications exists, and hence we refer interested readers to [15,18,19,33,42]. In order to further research this kind of equation by variational methods, many scholars have established the basic properties of fractional Sobolev spaces; readers are referred to [34,35].

First, we make a quick overview of the literature on the magnetic Schrödinger equation without Poisson term. For example, there are works on the magnetic Schrödinger equation

$$-(\nabla u - iA)^2 u + V(x)u = f(x, |u|)u, \tag{2}$$

where the magnetic operator in Equation (2) is given by

$$-(\nabla u - iA)^2 u = -\Delta u + 2iA(x) \cdot \nabla u + |A(x)|^2 u + iu \mathrm{div} A(x).$$

Squassina and Volzone [38] state that, up to correcting the operator by the factor $(1-s)$, it holds that $(-\Delta)^s_A u \to -(\nabla u - iA)^2 u$ as $s \to 1$. Thus, up to normalization, the nonlocal case can be seen as an approximation of the local one. Recently, many researchers have paid attention to the equations with fractional magnetic operator. In particular, Mingqi et al. [29] obtained some existence results of Schrödinger–Kirchhoff type equation involving the magnetic operator

$$M([u]_{s,A}^2)(-\Delta)_A^s u + V(x)u = f(x, |u|)u \quad \text{in } \mathbb{R}^N, \tag{3}$$

where f satisfies the subcritical growth condition. For the critical growth case, the authors in [10] first considered the following fractional Schrödinger equations:

$$\varepsilon^{2s}(-\Delta)_{A_\varepsilon}^s u + V(x)u = f(x, |u|)u + K(x)|u|^{2_\alpha^*-2}u \quad \text{in } \mathbb{R}^N. \tag{4}$$

They obtained the existence of ground state solution u_ε by using variational methods. For the non-degenerate case, Liang et al. [21] proved the existence and multiplicity of solutions for a class of Schrödinger–Kirchhoff type equation. Ambrosio [6] obtained the existence and concentration results for some fractional Schrödinger equations in $\mathbb{R}^N$ with magnetic fields. As for other results, we refer to [7,8,23,24] and references therein. We draw the attention of the reader to the degenerate case involving the magnetic operator in Liang et al. [22] and Mingqi et al. [29].

On the other hand, for the case $A \equiv 0$ in problem (1), some researchers began to use various methods to study this kind of problem. For example, using the perturbation approach, Zhang et al. in [43] obtained the existence results for the fractional Schrödinger–Poisson system with a general subcritical or critical nonlinearity. In [30], the authors proved that the number of positive solutions for a class of doubly singularly perturbed fractional Schrödinger–Poisson system via the Ljusternick–Schnirelmann category. Liu [27] was concerned with the existence of multi-bump solutions for the fractional Schrödinger–Poisson system by means of the Lyapunov–Schmidt reduction method. By using the non-Nehari manifold approach, Chen and Tang [12] proved the existence of ground state solutions for fractional Schrödinger–Poisson system. For more related results, we can cite the recent works [1,4,5,16,17,25,36,37,39,40,46] and the references therein.

Once we turn our attention to the Schrödinger–Kirchhoff–Poisson equations with electromagnetic fields, we immediately see that the literature is relatively scarce. In this case, we can cite the recent works [2,3,28]. Ambrosio [2] proved concentration results for a class of fractional Schrödinger–Poisson type equation with magnetic field and subcritical growth. For the critical growth case, Ambrosio [3] also obtained the multiplicity and concentration of nontrivial solutions to the fractional Schrödinger–Poisson equation with magnetic field. However, to the best of our knowledge, semiclassical solutions to the degenerate fractional Schrödinger–Kirchhoff–Poisson equations with critical nonlinearity and electromagnetic fields (1) has not been considered until now.

Inspired by the previously mentioned works, our main objective is to study the critical fractional Schrödinger–Kirchhoff–Poisson equations with electromagnetic fields in degenerate cases. For our prove, we use the concentration compactness principle and variational method. For this purpose, we will use some minimax arguments. Moreover, due to the appearance of the critical term and degenerate nature of the Kirchhoff coefficient, the Sobolev embedding does not possess the compactness. To this end, we need some technical estimations.

We are now in a position to state our existence result as follows.

Theorem 1.1: *Assume that $(\mathcal{V})$ and $(\mathfrak{M})$ hold. Then, for any $\kappa > 0$, there is $\mathcal{E}_\kappa > 0$ such that if $0 < \varepsilon < \mathcal{E}_\kappa$, then problem* (1) *has at least one solution u_ε. Moreover, $u_\varepsilon \to 0$ in E as $\varepsilon \to 0$.*

We also obtain the following existence results for problem (1).

Theorem 1.2: *Under the assumptions of Theorem* 1.1. *Then, for any $m \in \mathbb{N}$ and $\kappa > 0$, there is $\mathcal{E}_{m\kappa} > 0$ such that if $0 < \varepsilon < \mathcal{E}_{m\kappa}$, then problem* (1) *has at least m pairs of solutions $u_{\varepsilon,i}$, $u_{\varepsilon,-i}$, $i = 1, 2, \ldots, m$. Moreover, $u_{\varepsilon,i} \to 0$ in E as $\varepsilon \to 0$, $i = 1, 2, \ldots, m$.*

The main feature of our paper is establishing some results for degenerate fractional Schrödinger–Kirchhoff–Poisson equations (1) under the critical nonlinearity and electromagnetic fields. The lack of compactness can lead to a lot of difficulties, in order to overcome the challenge, we use the concentration-compactness principles for fractional Sobolev spaces from [20,32,44], and prove the $(PS)_c$ condition at special levels c. On the other hand, we need to develop new techniques to construct sufficiently small minimax levels.

The plan of this paper is the following: In Section 2, we give some basic definitions of fractional Sobolev space and their properties. In Section 3, we show some compactness lemmas of the functional associated to our problem. Section 4 deals with the existence and multiplicity results for problem (1).

2. Preliminaries

In this section, we have collected some known results for the convenience and later use.

For any $s \in (0, 1)$, fractional Sobolev space $H^s_A(\mathbb{R}^3, \mathbb{C})$ is defined by

$$H^s_A(\mathbb{R}^3, \mathbb{C}) = \left\{u \in L^2(\mathbb{R}^N, \mathbb{C}) : [u]_{s,A} < \infty\right\}$$

and $[u]_{s,A}$ denotes the so-called Gagliardo semi-norm, that is

$$[u]_{s,A} = \left(\iint_{\mathbb{R}^6} \frac{|u(x) - \mathrm{e}^{i(x-y)\cdot A\left(\frac{x+y}{2}\right)} u(y)|^2}{|x-y|^{N+2s}} \,\mathrm{d}x\,\mathrm{d}y\right)^{1/2}$$

and $H^s_A(\mathbb{R}^3, \mathbb{C})$ is endowed with the norm

$$\|u\|_{H^s_A(\mathbb{R}^3,\mathbb{C})} = \left([u]^2_{s,A} + \|u\|^2_{L^2}\right)^{\frac{1}{2}}.$$

Now, we give the following embedding theorem, see Lemma 3.5 in [13].

Proposition 2.1: *The space $H^s_A(\mathbb{R}^3, \mathbb{C})$ is continuously embedded in $L^\vartheta(\mathbb{R}^3, \mathbb{C})$ for all $\vartheta \in [2, 2^*_s]$. Furthermore, the space $H^s_A(\mathbb{R}^3, \mathbb{C})$ is continuously compactly embedded in $L^\vartheta(K, \mathbb{C})$ for all $\vartheta \in [2, 2^*_s]$ and any compact set $K \subset \mathbb{R}^3$.*

Next, we have the following diamagnetic inequality, its proof can be found in d'Avenia and Squassina [13].

Lemma 2.1: *Let $u \in H^s_A(\mathbb{R}^3)$, then $|u| \in H^s(\mathbb{R}^3)$. That is*

$$\| |u| \|_s \le \|u\|_{s,A}.$$

From Proposition 3.6 in [14], for all $u \in H^s(\mathbb{R}^3)$, we have

$$[u]_s = \|(-\Delta)^{\frac{s}{2}}\|_{L^2(\mathbb{R}^3)},$$

i.e.

$$\iint_{\mathbb{R}^6} \frac{|u(x)-u(y)|^2}{|x-y|^{3+2s}}\,\mathrm{d}x\,\mathrm{d}y = \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}u(x)|^2\,\mathrm{d}x.$$

Moreover,

$$\iint_{\mathbb{R}^6} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{3+2s}}\,\mathrm{d}x\,\mathrm{d}y = \int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}}u(x)\cdot(-\Delta)^{\frac{s}{2}}v(x)\,\mathrm{d}x.$$

For problem (1), we will use the Banach space E defined by

$$E = \left\{ u \in H^s_A(\mathbb{R}^3,\mathbb{C}) : \int_{\mathbb{R}^3} V(x)|u|^2\,\mathrm{d}x < \infty \right\}$$

with the norm

$$\|u\|_E := \left([u]^2_{s,A} + \int_{\mathbb{R}^3} V(x)|u|^2\,\mathrm{d}x \right)^{\frac{1}{2}}.$$

It follows from the assumption $(\mathcal{V})$ that the embedding $E \hookrightarrow H^s_A(\mathbb{R}^3,\mathbb{C})$ is continuous. Moreover, the norm $\|\cdot\|_E$ is equivalent to the norm $\|\cdot\|_\varepsilon$, where

$$\|u\|_\varepsilon := \left([u]^2_{s,A} + \varepsilon^{-2s}\int_{\mathbb{R}^3} V(x)|u|^2\,\mathrm{d}x \right)^{\frac{1}{2}}$$

for each $\varepsilon > 0$.

Obviously, for each $\theta \in [2, 2^*_s]$, there is $c_\theta > 0$ such that

$$|u|_\theta \le c_\theta \|u\|_E \le c_\theta \|u\|_\varepsilon, \tag{5}$$

where $0 < \varepsilon < 1$. Hereafter, we shortly denote by $\|\cdot\|_\nu$ the norm of Lebesgue space $L^\nu(\Omega)$ with $\nu \ge 1$.

Now, we give the following Hardy–Littlewood–Sobolev inequality, see Lieb and Loss [26, Theorem 4.3].

Lemma 2.2: *Assume that $p, \iota > 1$ and $0 < \mu < N$, $N \ge 3$ with $1/p + (N-\mu)/N + 1/\iota = 2$, $f \in L^p(\mathbb{R}^N)$ and $h \in L^\iota(\mathbb{R}^N)$. There exists a sharp constant $C(p,\iota,\mu,N)$ independent of f, h, such that*

$$\int\int_{\mathbb{R}^{2N}} \frac{f(x)h(y)}{|x-y|^{N-\mu}}\,\mathrm{d}x\,\mathrm{d}y \le C(p,\iota,\mu,N)\|f\|_{L^p}\|h\|_{L^\iota}. \tag{6}$$

Set $p = \iota = 2N/(N+\mu)$, then

$$C(p,\iota,\mu,N) = C(N,\mu) = \pi^{\frac{N-\mu}{2}} \frac{\Gamma\left(\frac{\mu}{2}\right)}{\Gamma\left(\frac{N+\mu}{2}\right)} \left\{ \frac{\Gamma\left(\frac{N}{2}\right)}{\Gamma(N)} \right\}^{\frac{\mu}{N}}.$$

If $u = v = |w|^q$, Lemma 2.2 implies that

$$\int_{\mathbb{R}^N} \left(\mathcal{I}_\mu * |w|^q\right) |w|^q \, dx$$

is well defined if $w \in L^{tq}(\mathbb{R}^N)$ for some $r > 1$ satisfying $2/r + (N-\mu)/N = 2$. Moreover, in the upper critical case,

$$\int_{\mathbb{R}^N} \left(\mathcal{I}_\mu * |u|^{2_s^\sharp}\right) |u|^{2_s^\sharp} \, dx \le C(N,\mu) \|u\|_{2_s^*}^{22_s^\sharp} \tag{7}$$

and the equality holds if and only if

$$u = C\left(\frac{l}{l^2 + |x-m|^2}\right)^{\frac{N-2}{2}}, \tag{8}$$

for some $x_0 \in \mathbb{R}^N$, where $C > 0$ and $l > 0$, see [26]. Let

$$S = \inf_{u \in D^s(\mathbb{R}^N)\setminus\{0\}} \left\{ \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} \, dx\, dy : \int_{\mathbb{R}^N} |u|^{2_s^*} \, dx = 1 \right\} \tag{9}$$

and

$$S_H = \inf_{u \in D^s(\mathbb{R}^N)\setminus\{0\}} \left\{ \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} \, dx\, dy : \int_{\mathbb{R}^N} \left(\mathcal{I}_\mu * |u|^{2_s^\sharp}\right) |u|^{2_s^\sharp} \, dx = 1 \right\}. \tag{10}$$

By (9) and (7), S_H is achieved if and only if u satisfies (8) and $S_H = S/C(N,\mu)^{\frac{1}{p^*}}$, see Mukherjee and Sreenadh [31].

3. Proof of $(PS)_c$

In this section, in order to overcome the lack of compactness caused by the upper critical exponents, we intend to employ the second concentration-compactness principle, see [20,32,44] for more details.

Now, let $s, t \in (0,1)$ such that $4s + 2t \ge 3$, we can see that

$$H^s(\mathbb{R}^3, \mathbb{R}) \hookrightarrow L^{\frac{12}{3+2t}}(\mathbb{R}^3, \mathbb{R}). \tag{11}$$

Then, by (11), we have

$$\int_{\mathbb{R}^3} u^2 v \, dx \le \|u\|^2_{L^{\frac{12}{3+2t}}(\mathbb{R}^3)} \|v\|_{2_t^*} \le C\|u\|^2_{H^s(\mathbb{R}^3,\mathbb{R})} \|v\|_{D^{t,2}(\mathbb{R}^3)}$$

for $u \in H^s(\mathbb{R}^3, \mathbb{R})$, where

$$\|v\|^2_{D^{t,2}(\mathbb{R}^3)} = \iint_{\mathbb{R}^6} \frac{|u(x)-u(y)|^2}{|x-y|^{3+2t}} \, dx\, dy.$$

The Lax–Milgram Theorem implies that there exists a unique $\psi^t_{|u|}$ such that $\psi^t_{|u|} \in D^{t,2}(\mathbb{R}^3, \mathbb{R})$ such that

$$(-\Delta)^t \psi^t_{|u|} = |u|^2 \quad \text{in } \mathbb{R}^3. \tag{12}$$

Therefore, we have

$$\psi^t_{|u|}(x) = \pi^{-\frac{3}{2}} 2^{-2t} \frac{\Gamma(3-2t)}{\Gamma(t)} \int_{\mathbb{R}^3} \frac{|u(y)|^2}{|x-y|^{3-2t}}\, \mathrm{d}y, \quad \forall\, x \in \mathbb{R}^3. \tag{13}$$

Furthermore, (13) is convergent at infinity since $|u|^2 \in L^{\frac{6}{3+2t}}(\mathbb{R}^3, \mathbb{R})$.

Next, we collect some properties of $\psi^t_{|u|}$, which will be used in this paper. The following proposition can be proved by using similar arguments as [2,3].

Proposition 3.1: *Assume that $4s + 2t \geq 3$ holds. Then for any $u \in E$, we have*

- (i) $\psi^t_{|u|} : H^s(\mathbb{R}^3, \mathbb{R}) \to D^{t,2}(\mathbb{R}^3, \mathbb{R})$ *is continuous and maps bounded sets into bounded sets;*
- (ii) *if $u_n \rightharpoonup u$ in E, then $\psi^t_{|u_n|} \rightharpoonup \psi^t_{|u|}$ in $D^{t,2}(\mathbb{R}^3, \mathbb{R})$;*
- (iii) $\psi^t_{|\alpha u|} = \alpha^2 \psi^t_{|u|}$ *for any $\alpha \in \mathbb{R}$ and $\psi^t_{|u(\cdot+y)|}(x) = \psi^t_{|u|}(x+y)$;*
- (iv) $\psi^t_{|u|} \geq 0$ *for all $u \in E$. Moreover*

$$\|\psi^t_{|u|}\|_{D^{t,2}(\mathbb{R}^3,\mathbb{R})} \leq C\|u\|^2_{L^{\frac{12}{3+2t}}(\mathbb{R}^3)} \leq C\|u\|^2_{\varepsilon}$$

and

$$\int_{\mathbb{R}^3} \psi^t_{|u|} |u|^2\, \mathrm{d}x \leq C\|u\|^4_{L^{\frac{12}{3+2t}}(\mathbb{R}^3)} \leq C\|u\|^4_{\varepsilon}.$$

Now, we will use the following equivalent form

$$M\left([u]^2_{s,A}\right)(-\Delta)^s_A u + \varepsilon^{-2s} V(x) u + \varepsilon^{-2s} \psi^t_{|u|} u = \varepsilon^{-2s} k(x)|u|^{r-2}u + \varepsilon^{-2s}|u|^{2^*_s-2}u, \tag{14}$$

for $x \in \mathbb{R}^3$. Now, the energy functional $J_\varepsilon : E \to \mathbb{R}$ associated to problem (1) is defined by

$$\begin{aligned} J_\varepsilon(u) := {} & \frac{1}{2}\mathcal{M}\left([u]^2_{s,A}\right) + \frac{\varepsilon^{-2s}}{2}\int_{\mathbb{R}^3} V(x)|u|^2\, \mathrm{d}x + \frac{\varepsilon^{-2s}}{4}\int_{\mathbb{R}^3} \psi^t_{|u|}|u|^2\, \mathrm{d}x \\ & - \frac{\varepsilon^{-2s}}{r}\int_{\mathbb{R}^3} k(x)|u|^r\, \mathrm{d}x - \frac{\varepsilon^{-2s}}{22^\sharp_s}\int_{\mathbb{R}^3}\left(\mathcal{I}_\mu * |u|^{2^\sharp_s}\right)|u|^{2^\sharp_s}\, \mathrm{d}x. \end{aligned} \tag{15}$$

Clearly, J_ε is of class $C^1(E, \mathbb{R})$ (see [41]). Moreover, the Fréchet derivative of J_ε is given by

$$\begin{aligned} & \langle J'_\varepsilon(u), v\rangle \\ & = M\left([u]^2_{s,A}\right)\mathscr{R}\iint_{\mathbb{R}^6} \frac{(u(x) - e^{i(x-y)\cdot A\left(\frac{x+y}{2}\right)}u(y))\overline{(v(x) - \mathrm{e}^{i(x-y)\cdot A\left(\frac{x+y}{2}\right)}v(y))}}{|x-y|^{3+2s}}\, \mathrm{d}x\, \mathrm{d}y \\ & \quad + \varepsilon^{-2s}\mathscr{R}\int_{\mathbb{R}^3} V(x)u\bar{v}\, \mathrm{d}x + \varepsilon^{-2s}\mathscr{R}\int_{\mathbb{R}^3} \psi^t_{|u|}u\bar{v}\, \mathrm{d}x - \varepsilon^{-2s}\mathscr{R}\int_{\mathbb{R}^3} k(x)|u|^{r-2}u\bar{v}\, \mathrm{d}x \\ & \quad - \varepsilon^{-2s}\mathscr{R}\int_{\mathbb{R}^3}\left(\mathcal{I}_\mu * |u|^{2^\sharp_s}\right)|u|^{2^\sharp_s-2}u\bar{v}\, \mathrm{d}x, \quad \forall\, u, v \in E. \end{aligned} \tag{16}$$

Lemma 3.1: *Let $(\mathcal{V})$ and $(\mathfrak{M})$ hold. Then for any $0 < \varepsilon < 1$, $(PS)_c$ sequence $\{u_n\}_n$ for J_ε is bounded in E and $c \geq 0$.*

Proof: Let sequence $\{u_n\}_n$ be a $(PS)_c$ sequence for J_ε, that is $J_\varepsilon(u_n) \to c$ and $J'_\varepsilon(u_n) \to 0$ in E'. It follows from $(\mathcal{V})$ and $(\mathfrak{M})$ that

$$\begin{aligned}
c + o(1)\|u_n\|_\varepsilon = J_\varepsilon(u_n) - \frac{1}{r}\langle J'_\varepsilon(u_n), u_n\rangle &= \frac{1}{2}\mathcal{M}\left([u_n]_{s,A}^2\right) - \frac{1}{p}M\left([u_n]_{s,A}^2\right)[u_n]_{s,A}^2 \\
&\quad + \left(\frac{1}{2} - \frac{1}{r}\right)\varepsilon^{-2s}\int_{\mathbb{R}^3} V(x)|u_n|^2\,\mathrm{d}x + \left(\frac{1}{4} - \frac{1}{p}\right)\varepsilon^{-2s}\int_{\mathbb{R}^3}\psi^t_{|u_n|}|u_n|^2\,\mathrm{d}x \\
&\quad + \left(\frac{1}{r} - \frac{1}{22_s^\sharp}\right)\varepsilon^{-2s}\int_{\mathbb{R}^3}\left(\mathcal{I}_\mu * |u_n|^{2_s^\sharp}\right)|u_n|^{2_s^\sharp}\,\mathrm{d}x \\
&\geq \left(\frac{1}{2\sigma} - \frac{1}{r}\right)M\left([u_n]_{s,A}^2\right)[u_n]_{s,A}^2 + \left(\frac{1}{2} - \frac{1}{r}\right)\varepsilon^{-2s}\int_{\mathbb{R}^3} V(x)|u_n|^2\,\mathrm{d}x \\
&\geq \left(\frac{1}{2\sigma} - \frac{1}{r}\right)m_1[u_n]_{s,A}^{2\sigma} + \left(\frac{1}{2} - \frac{1}{r}\right)\varepsilon^{-2s}\int_{\mathbb{R}^3} V(x)|u_n|^2\,\mathrm{d}x. \qquad (17)
\end{aligned}$$

This fact together with $2 < r < 2_s^*$ implies that $\{u_n\}_n$ is bounded in E. Moreover, we can obtain $c \geq 0$ by passing to the limit in (17). ■

Lemma 3.2: *Under the assumptions of Lemma* 3.1. *Then for any* $0 < \varepsilon < 1$, *the energy functional* J_ε *satisfies* $(PS)_c$ *condition, for all* $c \in (0, \alpha_0\varepsilon^\tau)$, *where*

$$\alpha_0 := \min\left\{\left(\frac{1}{r} - \frac{1}{22_s^\sharp}\right)\left(m_1 S_H^\sigma\right)^{\frac{2_s^\sharp}{2_s^\sharp-\sigma}}, \left(\frac{1}{2\sigma} - \frac{1}{r}\right)\left(m_1^{\frac{2_s^\sharp}{2\sigma}}\hat{C_\mu}^{-1}S^{\frac{2_s^\sharp}{2}}\right)^{\frac{2\sigma}{2_s^\sharp-2\sigma}}\right\} \qquad (18)$$

and

$$\tau := \max\left\{\frac{2s2_s^\sharp}{2_s^\sharp - \sigma}, \frac{4\sigma s}{2_s^\sharp - 2\sigma}\right\}. \qquad (19)$$

Proof: If $\inf_{n\in\mathbb{N}}\|u_n\|_\varepsilon = 0$, then there exists a subsequence of $\{u_n\}$ such that $u_n \to 0$ in E as $n \to \infty$. Thus, we assume that $\inf_{n\in\mathbb{N}}\|u_n\|_\varepsilon = d_1 > 0$ in the following sequel.

From Lemma 3.1, we know that $\{u_n\}_n$ is bounded in E. Thus, by diamagnetic inequality, $\{|u_n|\}_n$ is bounded in $H^s(\mathbb{R}^3)$. Furthermore, we have $u_n \to u$ a.e. in $\mathbb{R}^3$ and $u_n \rightharpoonup u$ in E. Let

$$|(-\Delta)^{\frac{s}{2}}u_n|^2 \rightharpoonup \omega, \quad |u_n|^{2_s^*} \rightharpoonup \xi$$

and

$$\left(\mathcal{I}_\mu * |u_n|^{2_s^\sharp}\right)|u_n|^{2_s^\sharp} \rightharpoonup \nu \quad \text{weakly in the sense of measures,}$$

where ω, ξ and ν are bounded nonnegative measures on $\mathbb{R}^3$. Then, by using the fractional version of concentration compactness principle in the fractional Sobolev space (see [20]), up to a subsequence, there exists a (at most countable) set of distinct points $\{x_i\}_{i\in I} \subset \mathbb{R}^3$ and a family of positive numbers $\{\nu_i\}_{i\in I}$ such that

$$\nu = \left(\mathcal{I}_\mu * |u|^{2_s^\sharp}\right)|u|^{2_s^\sharp} + \sum_{i\in I}\nu_i\delta_{x_i}, \quad \sum_{i\in I}\nu_i^{\frac{3}{3+\mu}} < \infty, \qquad (20)$$

$$\xi \geq |u|^{2_s^*} + C_\mu^{-\frac{3}{3+\mu}} \sum_{i\in I} \nu_i^{\frac{3}{3+\mu}} \delta_{x_i}, \quad \xi_i \geq C_\mu^{-\frac{3}{3+\mu}} \nu_i^{\frac{3}{3+\mu}} \tag{21}$$

and

$$\omega \geq |(-\Delta)^{\frac{s}{2}} u|^2 + S_H \sum_{i\in I} \nu_i^{\frac{1}{2_s^\sharp}} \delta_{x_i}, \quad \omega_i \geq S_H \nu_i^{\frac{1}{2_s^\sharp}}, \tag{22}$$

where δ_{x_i} is the Dirac-mass of mass 1 concentrated at $x \in \mathbb{R}^3$.

Now, let $i \in I$, we claim that either $\nu_i = 0$ or

$$\nu_i \geq \left(m_1 S_H^\sigma\right)^{\frac{2_s^\sharp}{2_s^\sharp-\sigma}} \varepsilon^{\frac{2s2_s^\sharp}{2_s^\sharp-\sigma}}. \tag{23}$$

In order to prove (23), we take $\phi \in C_0^\infty(\mathbb{R}^3)$ satisfying $0 \leq \phi \leq 1; \phi \equiv 1$ in $B(x_i, \epsilon), \phi(x) = 0$ in $\mathbb{R}^3 \setminus B(x_i, 2\epsilon)$. For any $\epsilon > 0$, define $\phi_\epsilon := \phi(\frac{x-x_i}{\epsilon})$, where $i \in I$. Clearly, $\{\phi_\epsilon u_n\}$ is bounded in E and $\langle J'_\varepsilon(u_n), u_n\phi_\epsilon)\rangle \to 0$ as $n \to \infty$. Hence

$$\begin{aligned} &M\left(\|u_n\|_{s,A}^2\right)\left(\iint_{\mathbb{R}^6} \frac{|u_n(x) - e^{i(x-y)\cdot A\left(\frac{x+y}{2}\right)} u_n(y)|^2 \phi_\epsilon(y)}{|x-y|^{3+2s}} \, dx\, dy + \varepsilon^{-2s} \int_{\mathbb{R}^3} V(x)|u_n|^2 \phi_\epsilon(x)\, dx\right) \\ &= -\mathscr{R}\left\{M\left(\|u_n\|_{s,A}^2\right) \iint_{\mathbb{R}^6} \frac{(u_n(x) - e^{i(x-y)\cdot A\left(\frac{x+y}{2}\right)} u_n(y))\overline{u_n(x)(\phi_\epsilon(x) - \phi_\epsilon(y))}}{|x-y|^{3+2s}} \, dx\, dy\right\} \\ &+ \varepsilon^{-2s} \int_{\mathbb{R}^3} \left(\mathcal{I}_\mu * |u_n|^{2_s^\sharp}\right) |u_n|^{2_s^\sharp} \phi_\epsilon \, dx + \varepsilon^{-2s} \int_{\mathbb{R}^3} k(x)|u_n|^r \phi_\epsilon \, dx + o_n(1). \end{aligned} \tag{24}$$

We deduce from (M_2) and diamagnetic inequality that

$$\begin{aligned} &M\left(\|u_n\|_{s,A}^2\right)\left(\iint_{\mathbb{R}^6} \frac{|u_n(x) - e^{i(x-y)\cdot A\left(\frac{x+y}{2}\right)} u_n(y)|^2 \phi_\epsilon(y)}{|x-y|^{3+2s}} \, dx\, dy + \int_{\mathbb{R}^3} V(x)|u_n|^2 \phi_\epsilon(x)\, dx\right) \\ &\geq m_1 \left(\iint_{\mathbb{R}^6} \frac{|u_n(x) - e^{i(x-y)\cdot A\left(\frac{x+y}{2}\right)} u_n(y)|^2 \phi_\epsilon(y)}{|x-y|^{3+2s}} \, dx\, dy + \int_{\mathbb{R}^3} V(x)|u_n|^2 \phi_\epsilon(x)\, dx\right)^\sigma \\ &\geq m_1 \left(\iint_{\mathbb{R}^6} \frac{\left||u_n(x)| - |u_n(y)|\right|^2 \phi_\epsilon(y)}{|x-y|^{3+2s}} \, dx\, dy\right)^\sigma. \end{aligned} \tag{25}$$

Note that

$$\lim_{\epsilon\to 0} \lim_{n\to\infty} \iint_{\mathbb{R}^6} \frac{\left||u_n(x)| - |u_n(y)|\right|^2 \phi_\epsilon(y)}{|x-y|^{3+2s}} \, dx\, dy = \lim_{\epsilon\to 0} \int_{\mathbb{R}^3} \phi_\epsilon d\omega = \omega_i \tag{26}$$

and

$$\lim_{\epsilon\to 0} \lim_{n\to\infty} \int_{\mathbb{R}^3} \left(\mathcal{I}_\mu * |u_n|^{2_s^\sharp}\right) |u_n|^{2_s^\sharp} \phi_\epsilon \, dx = \lim_{\epsilon\to 0} \int_{\mathbb{R}^3} \phi_\epsilon \, d\nu = \nu_i. \tag{27}$$

From the Hölder inequality, we have

$$\left|\mathscr{R}\left\{M\left(\|u_n\|_{s,A}^2\right) \iint_{\mathbb{R}^6} \frac{(u_n(x) - e^{i(x-y)\cdot A\left(\frac{x+y}{2}\right)} u_n(y))\overline{u_n(x)(\phi_\epsilon(x) - \phi_\epsilon(y))}}{|x-y|^{3+2s}} \, dx\, dy\right\}\right|$$

$$\leq C \iint_{\mathbb{R}^6} \frac{|u_n(x) - e^{i(x-y)\cdot A\left(\frac{x+y}{2}\right)} u_n(y)| \cdot |\phi_\epsilon(x) - \phi_\epsilon(y)| \cdot |u_n(x)|}{|x-y|^{3+2s}} \, dx \, dy$$

$$\leq C \left(\iint_{\mathbb{R}^6} \frac{|u_n(x)|^2 |\phi_\epsilon(x) - \phi_\epsilon(y)|^2}{|x-y|^{3+2s}} \, dx \, dy \right)^{1/2}. \tag{28}$$

As the proof of Lemma 3.4 in Zhang et al. [45], we get

$$\lim_{\epsilon\to 0} \lim_{n\to\infty} \iint_{\mathbb{R}^6} \frac{|u_n(x)|^2 |\phi_\epsilon(x) - \phi_\epsilon(y)|^2}{|x-y|^{3+2s}} \, dx \, dy = 0. \tag{29}$$

Since ϕ_ϵ has compact support, by the definition of $\phi_\epsilon(x)$, we have

$$\lim_{\epsilon\to 0} \lim_{n\to\infty} \int_{\mathbb{R}^3} k(x)|u_n|^r \phi_\epsilon \, dx = 0. \tag{30}$$

Combining (25)–(30), we get that

$$\varepsilon^{-2s} \nu_i \geq m_1 \omega_i^\sigma.$$

It follows from (22) that $\nu_i = 0$ or

$$\nu_i \geq \left(m_1 S_H^\sigma\right)^{\frac{2_s^\sharp}{2_s^\sharp - \sigma}} \varepsilon^{\frac{2s2_s^\sharp}{2_s^\sharp - \sigma}}.$$

Next, we prove that $\nu_i = 0$, for all $i \in I$ and $\nu_\infty = 0$.

Indeed, if not, then there exists a $i \in I$ such that (23) holds. Similar to (17), we deduce

$$\begin{aligned}
c &= \lim_{\epsilon\to 0} \lim_{n\to\infty} \left(J_\varepsilon(u_n) - \frac{1}{r}\langle J_\varepsilon'(u_n), u_n\rangle \right) \\
&\geq \left(\frac{1}{r} - \frac{1}{22_s^\sharp}\right) \varepsilon^{-2s} \int_{\mathbb{R}^3} \left(\mathcal{I}_\mu * |u|^{2_s^\sharp}\right) |u|^{2_s^\sharp} \, dx \\
&\geq \left(\frac{1}{r} - \frac{1}{22_s^\sharp}\right) \varepsilon^{-2s} \int_{\mathbb{R}^3} \left(\mathcal{I}_\mu * |u|^{2_s^\sharp}\right) |u|^{2_s^\sharp} \phi_\epsilon \, dx \\
&\geq \left(\frac{1}{p} - \frac{1}{22_s^\sharp}\right) \varepsilon^{-2s} \nu_i \geq \left(\frac{1}{r} - \frac{1}{22_s^\sharp}\right) \left(m_1 S_H^\sigma\right)^{\frac{2_s^\sharp}{2_s^\sharp - \sigma}} \varepsilon^{\frac{2s\sigma}{2_s^\sharp - \sigma}}.
\end{aligned} \tag{31}$$

For the concentration at infinity, letting $R > 0$, we take a cut off function $\phi_R \in C^\infty(\mathbb{R}^3)$ such that

$$\phi_R(x) = \begin{cases} 0 & |x| < R, \\ 1 & |x| > R+1. \end{cases}$$

Define

$$\omega_\infty = \lim_{R\to\infty} \limsup_{n\to\infty} \int_{\{x\in\mathbb{R}^3:|x|>R\}} |(-\Delta)^{\frac{s}{2}} u_n|^2 \, dx,$$

$$\xi_\infty = \lim_{R\to\infty} \limsup_{n\to\infty} \int_{\{x\in\mathbb{R}^3:|x|>R\}} |u_n|^{2_s^*}\,\mathrm{d}x$$

and

$$\nu_\infty = \lim_{R\to\infty} \limsup_{n\to\infty} \int_{\{x\in\mathbb{R}^3:|x|>R\}} \left(\mathcal{I}_\mu * |u_n|^{2_s^\sharp}\right)|u_n|^{2_s^\sharp}\,\mathrm{d}x.$$

By using the fractional version of concentration compactness principle (see [20]), for the energy at infinity, we have

$$\limsup_{n\to\infty} \int_{\mathbb{R}^3} \left(\mathcal{I}_\mu * |u_n|^{2_s^\sharp}\right)|u_n|^{2_s^\sharp}\,\mathrm{d}x = \int_{\mathbb{R}^3} \mathrm{d}\nu + \nu_\infty, \tag{32}$$

$$\limsup_{n\to\infty} \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u_n|^2\,\mathrm{d}x = \int_{\mathbb{R}^3} \mathrm{d}\omega + \omega_\infty, \tag{33}$$

$$\limsup_{n\to\infty} \int_{\mathbb{R}^3} |u_n|^{2_s^*}\,\mathrm{d}x = \int_{\mathbb{R}^3} \mathrm{d}\xi + \xi_\infty, \tag{34}$$

$$\xi_\infty \le \left(S^{-1}\omega_\infty\right)^{\frac{2_s^*}{2}}, \tag{35}$$

$$\nu_\infty \le C_\mu \left(\int_{\mathbb{R}^3} \mathrm{d}\xi + \xi_\infty\right)^{\frac{3+\mu}{6}} \xi_\infty^{\frac{3+\mu}{6}} \tag{36}$$

and

$$\nu_\infty \le S_H^{-2_s^\sharp} \left(\int_{\mathbb{R}^3} \mathrm{d}\omega + \omega_\infty\right)^{\frac{2_s^\sharp}{2}} \omega_\infty^{\frac{2_s^\sharp}{2}}. \tag{37}$$

By using the Hardy–Littlewood–Sobolev and Hölder's inequality, we get

$$\begin{aligned}
\nu_\infty &= \lim_{R\to\infty}\lim_{n\to\infty} \int_{\mathbb{R}^3} \left(\mathcal{I}_\mu * |u_n|^{2_s^\sharp}\right)|u_n|^{2_s^\sharp}\phi_R(y)\,\mathrm{d}x \\
&\le C_\mu \lim_{R\to\infty}\lim_{n\to\infty} |u_n|_{2_s^*}^{2_s^\sharp} \left(\int_{\mathbb{R}^3} |u_n(x)|^{2_s^*}\phi_R(y)\,\mathrm{d}x\right)^{\frac{2_s^\sharp}{2_s^*}} \\
&\le \hat{C}_\mu \xi_\infty^{\frac{2_s^\sharp}{2_s^*}}.
\end{aligned} \tag{38}$$

Since $\{\phi_R u_n\}$ is also bounded in E. Hence, $\langle J_\varepsilon'(u_n), u_n\phi_R)\rangle \to 0$ as $n\to\infty$, which yields that

$$\begin{aligned}
&M\left(\|u_n\|_{s,A}^2\right)\left(\iint_{\mathbb{R}^6} \frac{|u_n(x) - \mathrm{e}^{i(x-y)\cdot A\left(\frac{x+y}{2}\right)}u_n(y)|^2\phi_R(y)}{|x-y|^{3+2s}}\,\mathrm{d}x\,\mathrm{d}y + \varepsilon^{-2s}\int_{\mathbb{R}^3} V(x)|u_n|^2\phi_R(x)\,\mathrm{d}x\right) \\
&= -\mathscr{R}\left\{M\left(\|u_n\|_{s,A}^2\right)\iint_{\mathbb{R}^6} \frac{(u_n(x) - \mathrm{e}^{i(x-y)\cdot A\left(\frac{x+y}{2}\right)}u_n(y))\overline{u_n(x)(\phi_R(x)-\phi_R(y))}}{|x-y|^{3+2s}}\,\mathrm{d}x\,\mathrm{d}y\right\}
\end{aligned}$$

$$+\varepsilon^{-2s}\int_{\mathbb{R}^3}\left(\mathcal{I}_\mu * |u_n|^{2_s^\sharp}\right)|u_n|^{2_s^\sharp}\phi_R\,\mathrm{d}x+\varepsilon^{-2s}\int_{\mathbb{R}^3}k(x)|u_n|^r\phi_R\,\mathrm{d}x+o_n(1). \tag{39}$$

It's easy to get

$$\limsup_{R\to\infty}\limsup_{n\to\infty}\iint_{\mathbb{R}^6}\frac{||u_n(x)|-|u_n(y)||^2\phi_R(y)}{|x-y|^{3+2s}}\,\mathrm{d}x\,\mathrm{d}y=\omega_\infty$$

and

$$\left|\mathscr{R}\left\{M\left(\|u_n\|_{s,A}^2\right)\iint_{\mathbb{R}^6}\frac{(u_n(x)-\mathrm{e}^{i(x-y)\cdot A(\frac{x+y}{2})}u_n(y))\overline{u_n(x)(\phi_R(x)-\phi_R(y))}}{|x-y|^{3+2s}}\,\mathrm{d}x\,\mathrm{d}y\right\}\right|$$
$$\le C\left(\iint_{\mathbb{R}^6}\frac{|u_n(x)|^2|\phi_R(x)-\phi_R(y)|^2}{|x-y|^{3+2s}}\,\mathrm{d}x\,\mathrm{d}y\right)^{1/2}.$$

Furthermore

$$\limsup_{R\to\infty}\limsup_{n\to\infty}\iint_{\mathbb{R}^6}\frac{|u_n(x)|^2|\phi_R(x)-\phi_R(y)|^2}{|x-y|^{3+2s}}\,\mathrm{d}x\,\mathrm{d}y$$
$$=\limsup_{R\to\infty}\limsup_{n\to\infty}\iint_{\mathbb{R}^6}\frac{|u_n(x)|^2|(1-\phi_R(x))-(1-\phi_R(y))|^2}{|x-y|^{3+2s}}\,\mathrm{d}x\,\mathrm{d}y.$$

From the proof of Lemma 3.4 in Zhang et al. [45], we have

$$\limsup_{R\to\infty}\limsup_{n\to\infty}\iint_{\mathbb{R}^6}\frac{|u_n(x)|^2|(1-\phi_R(x))-(1-\phi_R(y))|^2}{|x-y|^{3+2s}}\,\mathrm{d}x\,\mathrm{d}y=0.$$

It follows from (M_2) that

$$M\left(\|u_n\|_{s,A}^2\right)\left(\iint_{\mathbb{R}^6}\frac{|u_n(x)-\mathrm{e}^{i(x-y)\cdot A\left(\frac{x+y}{2}\right)}u_n(y)|^2\phi_R(y)}{|x-y|^{3+2s}}\,\mathrm{d}x\,\mathrm{d}y+\int_{\mathbb{R}^3}|u_n|^2\phi_R(x)\,\mathrm{d}x\right)$$
$$\ge m_1\left(\iint_{\mathbb{R}^6}\frac{|u_n(x)-\mathrm{e}^{i(x-y)\cdot A\left(\frac{x+y}{2}\right)}u_n(y)|^2\phi_R(y)}{|x-y|^{3+2s}}\,\mathrm{d}x\,\mathrm{d}y+\int_{\mathbb{R}^3}|u_n|^2\phi_R(x)\,\mathrm{d}x\right)^\sigma$$
$$\ge m_1\left(\iint_{\mathbb{R}^6}\frac{\left||u_n(x)|-|u_n(y)|\right|^2\phi_R(y)}{|x-y|^{3+2s}}\,\mathrm{d}x\,\mathrm{d}y\right)^\sigma=m_1\omega_\infty^\sigma.$$

By the definition of ϕ_R, we have

$$\lim_{R\to\infty}\lim_{n\to\infty}\int_{\mathbb{R}^3}k(x)|u_n|^r\phi_R\,\mathrm{d}x\le k^*\lim_{R\to\infty}\lim_{n\to\infty}\int_{\mathbb{R}^3}|u_n|^r\phi_R\,\mathrm{d}x=0.$$

Therefore, by (39) together with (38), we can obtain that

$$\varepsilon^{-2s}\hat{C}_\mu\xi_\infty^{\frac{2_s^\sharp}{2_s^*}}\ge\varepsilon^{-2s}\nu_\infty\ge m_1\omega_\infty^\sigma.$$

It follows from (35) that $\omega_\infty = 0$ or

$$\omega_\infty \geq \left(m_1 \hat{C_\mu}^{-1} S^{\frac{2_s^\sharp}{2}}\right)^{\frac{2}{2_s^\sharp - 2\sigma}} \varepsilon^{\frac{4s}{2_s^\sharp - 2\sigma}}. \tag{40}$$

If (40) holds, then we have

$$\begin{aligned} c &= \lim_{R\to\infty} \lim_{n\to\infty} \left(J_\varepsilon(u_n) - \frac{1}{r}\langle J_\varepsilon'(u_n), u_n\rangle\right) \\ &\geq \left(\frac{1}{2\sigma} - \frac{1}{r}\right) m_1 \left(\iint_{\mathbb{R}^6} \frac{\left||u_n(x)| - |u_n(y)|\right|^2 \phi_R(y)}{|x-y|^{3+2s}} \, \mathrm{d}x\, \mathrm{d}y\right)^\sigma \\ &\geq \left(\frac{1}{2\sigma} - \frac{1}{r}\right) m_1 \omega_\infty^\sigma \geq \left(\frac{1}{2\sigma} - \frac{1}{r}\right) \left(m_1^{\frac{2_s^\sharp}{2\sigma}} \hat{C_\mu}^{-1} S^{\frac{2_s^\sharp}{2}}\right)^{\frac{2\sigma}{2_s^\sharp - 2\sigma}} \varepsilon^{\frac{4\sigma s}{2_s^\sharp - 2\sigma}}. \end{aligned} \tag{41}$$

By the selection of α_0 and τ, for any $c < \alpha_0 \varepsilon^\tau$, this gives a contradiction. Thus, $\omega_\infty = 0$. By (37), we know that

$$\nu_i = 0 \text{ for all } i \in I \quad \text{and} \quad \nu_\infty = 0.$$

Thus

$$\int_{\mathbb{R}^3} \left(\mathcal{I}_\mu * |u_n|^{2_s^\sharp}\right) |u_n|^{2_s^\sharp} \, \mathrm{d}x \to \int_{\mathbb{R}^3} \left(\mathcal{I}_\mu * |u|^{2_s^\sharp}\right) |u|^{2_s^\sharp} \, \mathrm{d}x \quad \text{as } n \to \infty. \tag{42}$$

By the Brézis–Lieb Lemma [11], we get

$$\int_{\mathbb{R}^3} \left(\mathcal{I}_\mu * |u_n - u|^{2_s^\sharp}\right) |u_n - u|^{2_s^\sharp} \, \mathrm{d}x \to 0 \quad \text{as } n \to \infty.$$

Finally, with the aid of the weak lower semicontinuity of the norm, condition (m_1) and the Brézis–Lieb Lemma [11], we can obtain that

$$\begin{aligned} o(1)\|u_n\| = \langle J_\varepsilon'(u_n), u_n\rangle &= M\left([u_n]_{s,A}^2\right)[u_n]_{s,A}^2 + \varepsilon^{-2s} \int_{\mathbb{R}^3} V(x)|u_n|^2 \, \mathrm{d}x \\ &\quad + \varepsilon^{-2s} \int_{\mathbb{R}^3} \psi_{|u_n|}^t |u_n|^2 \, \mathrm{d}x - \varepsilon^{-2s} \int_{\mathbb{R}^3} \left(\mathcal{I}_\mu * |u_n|^{2_s^\sharp}\right) |u_n|^{2_s^\sharp} \, \mathrm{d}x \\ &\quad - \varepsilon^{-2s} \int_{\mathbb{R}^3} k(x)|u_n|^r \, \mathrm{d}x \\ &\geq m_0 \left([u_n]_{s,A}^2 - [u]_{s,A}^2\right) + \varepsilon^{-2s} \int_{\mathbb{R}^3} V(x)(|u_n|^2 - |u|^2) \, \mathrm{d}x + M\left([u]_{s,A}^2\right)[u]_{s,A}^2 \\ &\quad + \varepsilon^{-2s} \int_{\mathbb{R}^3} V(x)|u|^2 \, \mathrm{d}x + \varepsilon^{-2s} \int_{\mathbb{R}^3} \psi_{|u|}^t |u|^2 \, \mathrm{d}x \\ &\quad - \varepsilon^{-2s} \int_{\mathbb{R}^3} \left(\mathcal{I}_\mu * |u|^{2_s^\sharp}\right) |u|^{2_s^\sharp} \, \mathrm{d}x - \varepsilon^{-2s} \int_{\mathbb{R}^3} k(x)|u|^r \, \mathrm{d}x \\ &\geq \min\{m_0, 1\}\|u_n - u\|_\varepsilon^2 + o(1)\|u\|_\varepsilon. \end{aligned}$$

Here we use the fact that $J_\varepsilon'(u) = 0$. This fact implies that $\{u_n\}_n$ strongly converges to u in E. Hence the proof is complete. ■

4. Proof of main results

In order to prove Theorems 1.1 and 1.2, we first prove that functional $J_\varepsilon(u)$ satisfies the following mountain pass geometry.

Lemma 4.1: *Let $(\mathcal{V})$ and $(\mathfrak{M})$ hold. Then for any $0 < \varepsilon < 1$,*

- (C_1) *there exist $\beta_\varepsilon, \rho_\varepsilon > 0$ such that $J_\varepsilon(u) > 0$ if $u \in B_{\rho_\varepsilon} \setminus \{0\}$ and $J_\varepsilon(u) \geq \beta_\varepsilon$ if $u \in \partial B_{\rho_\varepsilon}$, where $B_{\rho_\varepsilon} = \{u \in E : \|u\|_\varepsilon \leq \rho_\varepsilon\}$;*
- (C_2) *For any finite dimensional subspace $H \subset E$,*

$$J_\varepsilon(u) \to -\infty \quad \text{as} \quad u \in H,\ \|u\|_\varepsilon \to \infty.$$

Proof: From the Hardy–Littlewood–Sobolev inequality and the Sobolev embedding theorem, we have

$$\begin{aligned} J_\varepsilon(u) &\geq \min\left\{\frac{m_1}{2\sigma}, \frac{1}{2}\right\} \|u\|_\varepsilon^2 - \varepsilon^{-2s} k^* |u|_r^r - \varepsilon^{-2s} \int_{\mathbb{R}^3} \left(\mathcal{I}_\mu * |u|^{2_s^\sharp}\right) |u|^{2_s^\sharp}\, \mathrm{d}x \\ &\geq \min\left\{\frac{m_1}{2\sigma}, \frac{1}{2}\right\} \|u\|_\varepsilon^2 - \varepsilon^{-2s} C \|u\|_\varepsilon^r - \varepsilon^{-2s} C_\mu C \|u\|_\varepsilon^{22_s^\sharp}. \end{aligned}$$

Since $r > 2$ and $22_s^\sharp > 2$, we can obtain the conclusion (C_1) in Lemma 4.1.

On the other hand, from (M_2), we have that

$$\mathcal{M}(t) \leq \mathcal{M}(1) t^\sigma \quad \text{for all } t \geq 1. \tag{43}$$

Let $v_0 \in C_0^\infty(\mathbb{R}^3, \mathbb{C})$ with $\|v_0\|_\varepsilon = 1$. Thus, we have

$$\begin{aligned} J_\varepsilon(tv_0) \leq\ & \mathcal{M}(1) t^{2\sigma} + \frac{1}{2} t^2 + \frac{\varepsilon^{-2s}}{4} C t^4 - \varepsilon^{-2s} t^{22_s^\sharp} \int_{\mathbb{R}^3} \left(\mathcal{I}_\mu * |v_0|^{2_s^\sharp}\right) |v_0|^{2_s^\sharp}\, \mathrm{d}x \\ & - \varepsilon^{-2s} k_* t^r |v_0|_r^r. \end{aligned}$$

Note that all norms in a finite-dimensional space H are equivalent and $\max\{4, 2\sigma\} < 22_s^\sharp$, we can also obtain the conclusion (C_2) in Lemma 4.1. ■

What we need to point out is that $J_\varepsilon(u)$ does not satisfy $(PS)_c$ condition for any $c > 0$. Thus, we need to construct a special finite-dimensional subspace by which we construct sufficiently small minimax levels.

On the one hand, from Lemma 3.5 in [10], we know that

$$\inf\left\{\iint_{\mathbb{R}^6} \frac{|\phi(x) - \phi(y)|^2}{|x-y|^{3+2s}}\, \mathrm{d}x\, \mathrm{d}y : \phi \in C_0^\infty(\mathbb{R}^3), |\phi|_r = 1\right\} = 0.$$

Thus, for any $1 > \delta > 0$ one can choose $\phi_\delta \in C_0^\infty(\mathbb{R}^3)$ with $|\phi_\delta|_r = 1$ and $\operatorname{supp}\phi_\delta \subset B_{r_\delta}(0)$ so that

$$\iint_{\mathbb{R}^6} \frac{|\phi_\delta(x) - \phi_\delta(y)|^2}{|x-y|^{3+2s}}\, \mathrm{d}x\, \mathrm{d}y \leq C\delta^{\frac{6-(3-2s)r}{r}}.$$

Let

$$q_\delta(x) = e^{iA(0)x}\phi_\delta(x) \tag{44}$$

and

$$q_{\varepsilon,\delta}(x) = q_\delta(\varepsilon^{-\frac{\tau+2s}{3}}x), \tag{45}$$

where τ is defined by (19).

On the other hand, since $22_s^\sharp > \sigma$, thus, there exists a finite number $t_0 \in [0, +\infty)$ such that

$$\begin{aligned}\max_{t\geq 0}\mathcal{I}_\varepsilon(tq_\delta) \leq \frac{C_0}{2}t_0^{2\sigma}&\left(\iint_{\mathbb{R}^6}\frac{|q_\delta(x)-e^{i(x-y)\cdot A\left(\frac{\varepsilon x+\varepsilon y}{2}\right)}q_\delta(y)|^2}{|x-y|^{3+2s}}\,dx\,dy\right)^{2\sigma}\\&+\frac{t_0^2}{2}\int_{\mathbb{R}^3}V(\varepsilon x)\,|q_\delta|^2\,dx+\frac{t_0^4}{4}\int_{\mathbb{R}^3}\psi^{t_0}_{|q_\delta|}|q_\delta|^2\,dx-k_*\int_{\mathbb{R}^3}|q_\delta|^r\,dx\\&:= I_\varepsilon(t_0q_\delta),\end{aligned}$$

where

$$\begin{aligned}\mathcal{I}_\varepsilon(u) := \frac{C_0}{2}&\left(\iint_{\mathbb{R}^6}\frac{|u(x)-e^{i(x-y)\cdot A\left(\frac{\varepsilon x+\varepsilon y}{2}\right)}u(y)|^2}{|x-y|^{3+2s}}\,dx\,dy\right)^{2\sigma}+\frac{1}{2}\int_{\mathbb{R}^3}V(\varepsilon x)\,|u|^2\,dx\\&+\frac{1}{4}\int_{\mathbb{R}^3}\psi^t_{|u|}|u|^2\,dx-k_*\int_{\mathbb{R}^3}|u|^r\,dx-\frac{1}{22_s^\sharp}\int_{\mathbb{R}^3}\left(\mathcal{I}_\mu*|u|^{2_s^\sharp}\right)|u|^{2_s^\sharp}\,dx.\end{aligned}$$

Therefore, for any $t > 0$, we get

$$\begin{aligned}J_\varepsilon(tq_{\varepsilon,\delta}) \leq \frac{C_0}{2}t^{2\sigma}&\left(\iint_{\mathbb{R}^6}\frac{|q_{\varepsilon,\delta}(x)-e^{i(x-y)\cdot A(\frac{x+y}{2})}q_{\varepsilon,\delta}(y)|^2}{|x-y|^{3+2s}}\,dx\,dy\right)^{2\sigma}\\&+\frac{t^2}{2}\varepsilon^{-2s}\int_{\mathbb{R}^3}V(x)|q_{\varepsilon,\delta}|^2\,dx+\frac{t^4}{4}\varepsilon^{-2s}\int_{\mathbb{R}^3}|\psi^t_{|q_{\varepsilon,\delta}|}||q_{\varepsilon,\delta}|^2\,dx\\&-t^rk_*\varepsilon^{-2s}\int_{\mathbb{R}^3}|q_{\varepsilon,\delta}|^r\,dx\\&\leq \varepsilon^\tau\left[\frac{C_0}{2}t^{2\sigma}\left(\iint_{\mathbb{R}^6}\frac{|q_\delta(x)-e^{i(x-y)\cdot A\left(\frac{\varepsilon x+\varepsilon y}{2}\right)}q_\delta(y)|^2}{|x-y|^{3+2s}}\,dx\,dy\right)^{2\sigma}\right.\\&\left.+\frac{t^2}{2}\int_{\mathbb{R}^3}V(\varepsilon x)\,|q_\delta|^2\,dx+\frac{t^4}{4}\varepsilon^{2t}\int_{\mathbb{R}^3}|\psi^t_{|q_\delta|}||q_\delta|^2\,dx-t^rk_*\int_{\mathbb{R}^3}|q_\delta|^r\,dx\right]\\&\leq \varepsilon^\tau I_\varepsilon(t_0q_\delta).\end{aligned}$$

Let $\psi_\delta(x) = e^{iA(0)x}\phi_\delta(x)$, where $\phi_\varsigma(x)$ is as defined above. From Lemma 3.6 in [10], we have the following lemma.

Lemma 4.2: *For any* $\delta > 0$ *there exists* $\varepsilon_0 = \varepsilon_0(\delta) > 0$ *such that*

$$\iint_{\mathbb{R}^6} \frac{|q_\delta(x) - e^{i(x-y)\cdot A\left(\frac{\varepsilon x+\varepsilon y}{2}\right)} q_\delta(y)|^2}{|x-y|^{3+2s}} \, dx\, dy \le C\delta^{\frac{6-(3-2s)r}{r}} + \frac{1}{1-s}\delta^{2s} + \frac{4}{s}\delta^{2s}$$

for all $0 < \varepsilon < \varepsilon_0$ *and some constant* $C > 0$ *depending only on* $[\phi]_{s,0}$.

On the one hand, since $V(0) = 0$ and note that $\operatorname{supp} \phi_\delta \subset B_{r_\delta}(0)$, there is $\varepsilon^* > 0$ such that

$$V(\varepsilon x) \le \frac{\delta}{|\phi_\delta|_2^2} \quad \text{for all } |x| \le r_\delta \text{ and } 0 < \varepsilon < \varepsilon^*.$$

This fact together with Lemma 4.2 implies that

$$\max_{t\ge 0} J_\varepsilon(tq_\delta) \le \mathcal{N}(\delta), \tag{46}$$

where

$$\mathcal{N}(\delta) := \frac{C_0}{2} t_0^{2\sigma} \left(C\delta^{\frac{6-(3-2s)r}{r}} + \frac{1}{1-s}\delta^{2s} + \frac{4}{s}\delta^{2s} \right)^{2\sigma} + \frac{t_0^2}{2}\delta + C\left(C\delta^{\frac{6-(3-2s)r}{r}} + \frac{1}{1-s}\delta^{2s} + \frac{4}{s}\delta^{2s} + \delta \right)^2. \tag{47}$$

Thus we have the following result.

Lemma 4.3: *Let* $(\mathcal{V})$ *and* $(\mathfrak{M})$ *hold. Then for any* $\kappa > 0$ *there exists* $\mathcal{E}_\kappa > 0$ *such that for each* $0 < \varepsilon < \mathcal{E}_\kappa$, *there is* $\widehat{e_\varepsilon} \in E$ *with* $\|\widehat{e_\varepsilon}\| > \varrho_\varepsilon$, $J_\varepsilon(\widehat{e_\varepsilon}) \le 0$ *and*

$$\max_{t\in[0,1]} J_\varepsilon(t\widehat{e_\varepsilon}) \le \kappa\varepsilon^\tau. \tag{48}$$

Proof: Let $\delta > 0$ satisfy $\mathcal{N}(\delta) \le \kappa$. Set $\mathcal{E}_\kappa = \min\{\varepsilon_0, \varepsilon^*\}$ and $\widehat{t_\varepsilon} > 0$ be such that $\widehat{t_\varepsilon}\|q_{\varepsilon,\delta}\|_\varepsilon > \varrho_\varepsilon$ and $J_\varepsilon(tq_{\varepsilon,\delta}) \le 0$ for all $t \ge \widehat{t_\varepsilon}$. Choose $\widehat{e_\varepsilon} = \widehat{t_\varepsilon} q_{\varepsilon,\delta}$. Then by (46), we can obtain the conclusion of Lemma 4.3 holds. ■

In order to get the multiplicity results of problem (1), one can choose $m^* \in \mathbb{N}$ functions $\phi_\delta^i \in C_0^\infty(\mathbb{R}^3)$ such that $\operatorname{supp} \phi_\delta^i \cap \operatorname{supp} \phi_\delta^k = \emptyset$, $i \ne k$, $|\phi_\delta^i|_s = 1$ and

$$\iint_{\mathbb{R}^6} \frac{|\phi_\delta^i(x) - \phi_\delta^i(y)|^2}{|x-y|^{3+2s}} \, dx\, dy \le C\delta^{\frac{6-(3-2s)r}{r}}.$$

Let $r_\delta^{m^*} > 0$ be such that $\operatorname{supp} \phi_\delta^i \subset B_{r_\varsigma}^i(0)$ for $i = 1, 2, \ldots, m^*$. Set

$$q_\delta^i(x) = e^{iA(0)x} \phi_\delta^i(x) \tag{49}$$

and

$$q_{\varepsilon,\delta}^i(x) = q_\delta^i(\varepsilon^{-\frac{\tau+2s}{3}} x). \tag{50}$$

Denote

$$F_{\varepsilon\delta}^{m^*} = \text{span}\{q_{\varepsilon,\delta}^1, q_{\varepsilon,\delta}^2, \dots, q_{\varepsilon,\delta}^{m^*}\}.$$

Let $u = \sum_{i=1}^{m^*} c_i q_{\varepsilon,\delta}^i \in F_{\varepsilon\delta}^{m^*}$. Then there exists constant $C > 0$ such that

$$J_\varepsilon(u) \le C \sum_{i=1}^{m^*} J_\varepsilon(c_i q_{\varepsilon,\delta}^i).$$

As discussed above, we have

$$J_\varepsilon(c_i q_{\varepsilon,\delta}^i) \le \varepsilon^\tau I_\varepsilon(|c_i| q_\delta^i).$$

As before, we can obtain the following estimate:

$$\max_{u \in F_{\varepsilon\delta}^{m^*}} J_\varepsilon(u) \le C m^* \mathcal{N}(\delta) \varepsilon^\tau \tag{51}$$

for all δ small enough and some constant $C > 0$.

From (51), we have the following lemma.

Lemma 4.4: *Let* $(\mathcal{V})$ *and* $(\mathfrak{M})$ *hold. Then, for any* $m^* \in \mathbb{N}$ *and* $\kappa > 0$ *there exists* $\mathcal{E}_{m^*\kappa} > 0$ *such that for each* $0 < \varepsilon < \mathcal{E}_{m^*\kappa}$, *there exists an* m^**-dimensional subspace* $F_{\varepsilon m^*}$ *satisfying*

$$\max_{u \in F_{\varepsilon\delta}^{m^*}} J_\varepsilon(u) \le \kappa \varepsilon^\tau.$$

Now, we began to prove our main results.

Proof of Theorem 1.1. For any $0 < \kappa < \alpha_0$, we choose $\mathcal{E}_\kappa > 0$ satisfies $0 < \varepsilon < \mathcal{E}_\kappa$, and define the minimax value

$$c_\varepsilon := \inf_{\gamma \in \Upsilon_\varepsilon} \max_{t \in [0,1]} J_\varepsilon(t\widehat{e}_\varepsilon),$$

where

$$\Upsilon_\varepsilon := \{\gamma \in C([0,1], E) : \gamma(0) = 0 \text{ and } \gamma(1) = \widehat{e}_\varepsilon\}.$$

By Lemma 4.1, we have $\alpha_\varepsilon \le c_\varepsilon \le \kappa\varepsilon^\tau$. Lemma 3.2 implies that J_ε satisfies the $(PS)_{c_\varepsilon}$ condition. Thus, using the mountain pass theorem, there is $u_\varepsilon \in E$ such that $J'_\varepsilon(u_\varepsilon) = 0$ and $J_\varepsilon(u_\varepsilon) = c_\varepsilon$, that is u_ε is a nontrivial solution of problem (11).

On the other hand, by (12), we have

$$\begin{aligned} \kappa\varepsilon^\tau &\ge J_\varepsilon(u_\varepsilon) = J_\varepsilon(u_\varepsilon) - \frac{1}{r} J'_\varepsilon(u_\varepsilon) u_\varepsilon \\ &\ge \left(\frac{1}{2\sigma} - \frac{1}{r}\right) m_1 [u_\varepsilon]_{s,A}^{2\sigma} + \left(\frac{1}{2} - \frac{1}{r}\right) \varepsilon^{-2s} \int_{\mathbb{R}^3} V(x) |u_\varepsilon|^2 \, dx. \end{aligned}$$

This fact implies that $u_\varepsilon \to 0$ in E as $\varepsilon \to 0$. This completes the proof of Theorem 1.1.

Proof of Theorem 1.2. Denote the set of all symmetric (in the sense that $-Z = Z$) and closed subsets of E by Σ, for each $Z \in \Sigma$. Denote by gen(Z) the Krasnoselski genus, and define

$$j(Z) := \min_{\iota \in \Gamma_{m^*}} \operatorname{gen}(\iota(Z) \cap \partial B_{\varrho_\varepsilon}),$$

where Γ_{m^*} is the set of all odd homeomorphisms $\iota \in C(E, E)$ and ϱ_ε is given by Lemma 4.1. Then j is a version of Benci's pseudoindex [9]. Let

$$c_{\varepsilon i} := \inf_{j(Z) \geq i} \sup_{u \in Z} J_\varepsilon(u), \quad 1 \leq i \leq m^*.$$

Since $J_\varepsilon(u) \geq \alpha_\varepsilon$ for all $u \in \partial B^+_{\varrho_\varepsilon}$ and since $j(F_{\varepsilon m^*}) = \dim F_{\varepsilon m^*} = m^*$, we obtain

$$\alpha_\varepsilon \leq c_{\varepsilon 1} \leq \cdots \leq c_{\varepsilon m^*} \leq \sup_{u \in H_{\varepsilon m^*}} J_\varepsilon(u) \leq \kappa \varepsilon^\tau.$$

Thus, Lemma 3.2 implies that J_ε satisfies the $(PS)_{c_\varepsilon}$ condition for any $c < \alpha_0 \varepsilon^\tau$. By using the mountain pass theorem, we know that $c_{\varepsilon i}$ are critical valves and J_ε has at least m^* pairs of nontrivial critical points satisfying

$$\alpha_\varepsilon \leq J_\varepsilon(u_\varepsilon) \leq \kappa \varepsilon^\tau.$$

Hence, problem (1) has at least m^* pairs of solutions. Moreover, we also have $u_{\varepsilon,i} \to 0$ in E as $\varepsilon \to 0$, $i = 1, 2, \ldots, m$.

Disclosure statement

No potential conflict of interest was reported by the author(s).

Funding

Z. Zhang was supported by 2020 Nanjing Vocational College of Information Technology Doctoral special fund project 'Study on the Climate Ensemble Forecast Model Based on the East Asian Monsoon Region' (YB20200902). D.D. Repovš was supported by the Slovenian Research Agency (No. P1-0292, N1-0114, N1-0083).

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References

[1] Ambrosio V. An existence result for a fractional Kirchhoff–Schrödinger–Poisson system. Z Angew Math Phys. 2018;69:455.
[2] Ambrosio V. Multiplicity and concentration results for a fractional Schrödinger–Poisson type equation with magnetic field. Proc R Soc Edinburgh Sect A. 2020;150:655–694.
[3] Ambrosio V. Multiplicity and concentration results for fractional Schrödinger–Poisson equations with magnetic fields and critical growth. Potential Anal. 2020;52:565–600.
[4] Ambrosio V. Multiplicity and concentration results for a class of critical fractional Schrödinger–Poisson systems via penalization method. Commun Contemp Math. 2020;22: 1850078.
[5] Ambrosio V. Nonlinear fractional Schrödinger equations in $\mathbb{R}^N$. Cham: Birkhäuser/Springer; 2021. p. xvii+662. (Frontiers in Elliptic and Parabolic Problems).

[6] Ambrosio V. Existence and concentration results for some fractional Schrödinger equations in $\mathbb{R}^N$ with magnetic fields. Commun Partial Differ Equ. 2019;44:637–680.
[7] Ambrosio V. Multiplicity and concentration of solutions for a fractional Kirchhoff equation with magnetic field and critical growth. Ann Henri Poincaré. 2019;20:2717–2766.
[8] Ambrosio V. Multiple concentrating solutions for a fractional Kirchhoff equation with magnetic fields. Discrete Contin Dyn Syst. 2020;40:781–815.
[9] Benci V. On critical point theory for indefinite functionals in presence of symmetries. Trans Am Math Soc. 1982;274:533–572.
[10] Binlin Z, Squassina M, Zhang X. Fractional NLS equations with magnetic field, critical frequency and critical growth. Manuscr Math. 2018;155:115–140.
[11] Brézis H, Nirenberg L. Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents. Commun Pure Appl Math. 1983;36:437–477.
[12] Chen S, Peng J, Tang X. Ground state solutions for asymptotically periodic fractional Schrödinger–Poisson problems with asymptotically cubic or super-cubic nonlinearities. Math Method Appl Sci. 2017;40:4948–4961.
[13] d'Avenia P, Squassina M. Ground states for fractional magnetic operators. ESAIM: COCV. 2018;24:1–24.
[14] Di Nezza E, Palatucci G, Valdinoci E. Hitchhiker's guide to the fractional Sobolev spaces. Bull Sci Math. 2012;136:521–573.
[15] Düzgün F, Gamze F, Iannizzotto A. Three nontrivial solutions for nonlinear fractional Laplacian equations. Adv Nonlinear Anal. 2018;7:211–226.
[16] Fiscella A, Pucci P, Zhang B. p-fractional Hardy-Schrödinger–Kirchhoff systems with critical nonlinearities. Adv Nonlinear Anal. 2018;8:1111–1131.
[17] Giammetta AR. Fractional Schrödinger–Poisson–Slater system in one dimension. arXiv:1405.2796.
[18] Laskin N. Fractional quantum mechanics and Lévy path integrals. Phys Lett A. 2000;268: 298–305.
[19] Ledesma L, César E. Multiplicity result for non-homogeneous fractional Schrödinger–Kirchhoff-type equations in $\mathbb{R}^N$. Adv Nonlinear Anal. 2018;7:247–257.
[20] Li X, Ma S, Zhang G. Solutions to upper critical fractional Choquard equations with potential. Adv Differ Equ. 2020;25:135–160.
[21] Liang S, Repovš DD, Zhang B. On the fractional Schrödinger–Kirchhoff equations with electromagnetic fields and critical nonlinearity. Comput Math Appl. 2018;75:1778–1794.
[22] Liang S, Repovš DD, Zhang B. Fractional magnetic Schrödinger–Kirchhoff problems with convolution and critical nonlinearities. Math Models Methods Appl Sci. 2019;43:2473–2490.
[23] Liang S, Chung NT, Zhang B. Multi-bump solutions for fractional Schrödinger equation with electromagnetic fields and critical nonlinearity. Adv Differ Equ. 2020;25:423–456.
[24] Liang S, Zhang J. Infinitely many solutions for the p-fractional Kirchhoff equations with electromagnetic fields and critical nonlinearity. Nonlinear Anal Model Control. 2018;23:599–618.
[25] Liang S, Pucci P, Zhang B. Multiple solutions for critical Choquard Kirchhoff type equations. Adv Nonlinear Anal. 2021;10:400–419.
[26] Lieb EH, Loss M. Analysis volume 14 of graduate studies in mathematics. Providence, RI: American Mathematical Society; 2001.
[27] Liu W. Existence of multi-bump solutions for the fractional Schrödinger–Poisson system. J Math Phys. 2016;57(9):091502.
[28] Liu L, Chen H. The nontrivial solutions for fractional Schrödinger–Poisson equations with magnetic fields and critical or supercritical growth. Appl Math Lett. 2021;121:107358.
[29] Mingqi X, Pucci P, Squassina M, et al. Nonlocal Schrödinger–Kirchhoff equations with external magnetic field. Discrete Contin Dyn Syst. 2017;37:503–521.
[30] Murcia E, Siciliano G. Positive semiclassical states for a fractional Schrödinger–Poisson system. Differ Integr Equ. 2017;30(3-4):231–258.
[31] Mukherjee T, Sreenadh K. Fractional Choquard equation with critical nonlinearities. Nonlinear Differ Equ Appl. 2017;24(6): 423.

[32] Palatucci G, Pisante A. Improved Sobolev embeddings, profile decomposition, and concentration-compactness for fractional Sobolev spaces. Calc Var Partial Differ Equ. 2014;50:799–829.
[33] Mingqi X, Rădulescu V, Zhang B. Fractional Kirchhoff problems with critical Trudinger-Moser nonlinearity. Calc Var Partial Differ Equ. 2019;58:351.
[34] Molica Bisci G, Rădulescu V, Servadei R. Variational methods for nonlocal fractional equations. Cambridge: Cambridge University Press; 2016. (Encyclopedia of Mathematics and its Applications; 162).
[35] Papageorgiou N, Rădulescu V, Repovš D. Nonlinear analysis–theory and methods. Berlin, Heidelberg: Springer; 2019.
[36] Shen L. Existence result for fractional Schrödinger–Poisson systems involving a Bessel operator without Ambrosetti-Rabinowitz condition. Comput Math Appl. 2018;75:296–306.
[37] Song Y, Shi S. Existence and multiplicity solutions for the $p-$fractional Schrödinger–Kirchhoff equations with electromagnetic fields and critical nonlinearity. Complex Var Elliptic Equ. 2019;64:1163–1183.
[38] Squassina M, Volzone B. Bourgain-Brézis-Mironescu formula for magnetic operators. C R Math. 2016;354:825–831.
[39] Teng K. Existence of ground state solutions for the nonlinear fractional Schrödinger–Poisson system with critical Sobolev exponent. J Differ Equ. 2016;261:3061–3106.
[40] Wang L, Rădulescu VD, Zhang B. Infinitely many solutions for fractional Kirchhoff–Schrödinger–Poisson systems. J Math Phys. 2019;60(1):011506.
[41] Willem M. Minimax theorems, progress in nonlinear differential equations and their applications. Vol. 24. Boston, Basel, Berlin: Birkhäser; 1996.
[42] Xiang M, Zhang B, Rădulescu V. Superlinear Schrödinger–Kirchhoff type problems involving the fractional p-Laplacian and critical exponent. Adv Nonlinear Anal. 2019;9:690–709.
[43] Zhang J, Do Ó JM, Squassina M. Fractional Schrödinger–Poisson systems with a general subcritical or critical nonlinearity. Adv Nonlinear Stud. 2016;16:15–30.
[44] Zhang X, Zhang BL, Xiang MQ. Ground states for fractional Schrödinger equations involving a critical nonlinearity. Adv Nonlinear Anal. 2016;5:293–314.
[45] Zhang X, Zhang B, Repovš DD. Existence and symmetry of solutions for critical fractional Schrödinger equations with bounded potentials. Nonlinear Anal. 2016;142:48–68.
[46] Zhao F, Liu Z, Liang S. Multiple solutions for critical fourth-order elliptic equations of Kirchhoff type. Bull Malays Math Sci Soc. 2021;44:1057–1064.