Research Article

Soraya Fareh, Kamel Akrout, Abdeljabbar Ghanmi, and Dušan D. Repovš*

Multiplicity results for fractional Schrödinger-Kirchhoff systems involving critical nonlinearities

https://doi.org/10.1515/anona-2022-0318
received July 7, 2022; accepted April 30, 2023

Abstract: In this article, we study certain critical Schrödinger-Kirchhoff-type systems involving the fractional p-Laplace operator on a bounded domain. More precisely, using the properties of the associated functional energy on the Nehari manifold sets and exploiting the analysis of the fibering map, we establish the multiplicity of solutions for such systems.

Keywords: variational method, Nehari manifold, elliptic equation, multiplicity of solutions

MSC 2020: 35P30, 35J35, 35J60

1 Introduction

In recent years, a lot of attention has been paid to problems involving fractional and nonlocal operators. These types of problems arise in applications in many fields, e.g., in materials science [9], phase transitions [5,39], water waves [16,17], minimal surfaces [13], and conservation laws [10]. For more applications of such problems in physical phenomena, probability, and finances, we refer interested readers to [12,14,47]. Due to their importance, there are many interesting works on the existence and multiplicity of solutions for fractional and nonlocal problems either on bounded domains or on the entire space, see [1,3,4,6,23,24,34,36–38].

In the last decade, many scholars have paid extensive attention to Kirchhoff-type elliptic equations with critical exponents, see [20,25,33], for the bounded domains and [26,28,29] for the entire space. In particular, in [22], the authors considered the following Kirchhoff problem:

$$\begin{cases} M\left(\iint\limits_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\mathrm{d}x\mathrm{d}y\right)(-\Delta)^s u = \lambda f(x,u) + |u|^{2_s^*-2}u \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \mathbb{R}^n\backslash\Omega, \end{cases} \tag{1.1}$$

*** Corresponding author: Dušan D. Repovš,** Department of Mathematics and Computer Science, Faculty of Education, University of Ljubljana, 1000 Ljubljana, Slovenia; Department of Mathematics, Faculty of Mathematics and Physics, University of Ljubljana, 1000 Ljubljana, Slovenia; Department of Mathematics, Institute of Mathematics, Physics and Mechanics, 1000 Ljubljana, Slovenia, e-mail: dusan.repovs@pef.uni-lj.si, dusan.repovs@fmf.uni-lj.si, dusan.repovs@guest.arnes.si
Soraya Fareh: LAMIS Laboratory, Echadid Cheikh Larbi Tebessi University, 12002 Tebessa, Algeria, e-mail: soraya.fareh@univ-tebessa.dz
Kamel Akrout: LAMIS Laboratory, Echadid Cheikh Larbi Tebessi University, 12002 Tebessa, Algeria, e-mail: kamel.akrout@univ-tebessa.dz
Abdeljabbar Ghanmi: LR10ES09 Modélisation matématique, analyse harmonique et téorie du potentiel, Faculté des Sciences, Université de Tunis El Manar, 2092 Tunis, Tunisie, e-mail: abdeljabbar.ghanmi@lamsin.rnu.tn
ORCID: Soraya Fareh 0000-0002-5531-7924; Kamel Akrout 0000-0002-3275-4013; Abdeljabbar Ghanmi 0000-0001-8121-0496; Dušan D. Repovš 0000-0002-6643-1271

where $t \geq 0$ and $M(t) = a + bt$ for some $a > 0$ and $b \geq 0$. Here, and in the rest of this article, Ω will denote a bounded domain in $\mathbb{R}^n$ with Lipschitz boundary $\partial\Omega$.

Under suitable conditions and by using the truncation technique method combined with the mountain pass theorem, the authors proved that for $\lambda > 0$ large enough, problem (1.1) has at least one nontrivial solution. Later, the fractional Kirchhoff-type problems were extensively studied by many authors using different methods, see [7,8,15,21,27,30–32,35,40,42–45]. In particular, by using the Nehari manifold method and the symmetric mountain pass theorem, Xiang et al. [43] investigated the multiplicity of solutions for some p-Kirchhoff system with Dirichlet boundary conditions.

Mingqi et al. [30] studied the following Schrödinger-Kirchhoff-type system:

$$\begin{cases} M([(u,v)]_{s,p}^p + \|u,v\|_{p,V}^p)(\mathcal{L}_p^s u + V(x)|u|^{p-2}u) = \lambda H_u(x,u,v) + \dfrac{\alpha}{p_s^*}|v|^\beta |u|^{\alpha-2}u & \text{in } \mathbb{R}^n, \\ M([(u,v)]_{s,p}^p + \|u,v\|_{p,V}^p)(\mathcal{L}_p^s v + V(x)|v|^{p-2}v) = \lambda H_v(x,u,v) + \dfrac{\beta}{p_s^*}|u|^\alpha |v|^{\beta-2}v & \text{in } \mathbb{R}^n, \end{cases} \tag{1.2}$$

where $\lambda > 0$, $\alpha + \beta = p_s^* := \frac{np}{n-sp}$, $V : \mathbb{R}^n \to [0,\infty)$ is a continuous function, the Kirchhoff function $M : (0,\infty) \to (0,\infty)$ is continuous, and H_u and H_v are Caratheodory functions. Under some suitable assumptions and by applying the mountain pass theorem with Ekeland's variational principle, the authors obtained the existence and asymptotic behavior of solutions for system (1.2).

By the same methods as in [30], Fiscella et al. [21] studied the existence of solutions for a critical Hardy-Schrödinger-Kirchhoff-type system involving the fractional p-Laplacian in $\mathbb{R}^n$. Using the three critical points theorem, Azroul et al. [8] established the existence of three weak solutions for a fractional p-Kirchhoff-type system on a bounded domain with homogeneous Dirichlet boundary conditions. Recently, Azroul et al. [7] have established the existence of three solutions for the (p,q)-Schrödinger-Kirchhoff-type system in $\mathbb{R}^n$ via the three critical points theorem.

Motivated by the above-mentioned articles, we consider in this article the following Schrödinger-Kirchhoff-type system involving the fractional p-Laplacian and critical nonlinearities:

$$\begin{cases} M_1\left(\|u\|_{V_1}^p\right)((-\Delta)_p^s u + V_1(x)|u|^{p-2}u) = a_1(x)|u|^{p_s^*-2}u + \lambda f(x,u,v) & \text{in } \Omega, \\ M_2\left(\|v\|_{V_2}^p\right)((-\Delta)_p^s v + V_2(x)|v|^{p-2}v) = a_2(x)|v|^{p_s^*-2}v + \lambda g(x,u,v) & \text{in } \Omega, \\ u, v > 0 \quad \text{in } \Omega, \\ u = v = 0 \quad \text{on } \mathbb{R}^n \backslash \Omega, \end{cases} \tag{1.3}$$

where $\|.\|_{V_1}$ and $\|.\|_{V_2}$ will be given later (see (1.6)), $n > ps$, $0 < s < 1 < q < p$, λ is a positive parameter, the weight functions a_1 and a_2 are positive and bounded on Ω, and $(-\Delta)_p^s$ is the fractional p-Laplace operator, defined as follows:

$$(-\Delta)_p^s u = 2\lim_{\varepsilon \to 0} \int\limits_{\mathbb{R}^N \backslash B_\varepsilon(x)} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{n+ps}} \mathrm{d}y, \quad \text{for all } x \in \mathbb{R}^n,$$

where $B_\varepsilon(x) = \{y \in \mathbb{R}^n : |x-y| < \varepsilon\}$. For more details about the fractional p-Laplacian operator and the basic properties of fractional Sobolev spaces, we refer the reader to [18].

Throughout this article, the index i will denote integers 1 or 2, and we shall assume that the potential function $V_i : \Omega \to (0,\infty)$ is continuous and that there exists $v_i > 0$ such that $\inf_\Omega V_i \geq v_i$. In addition, we shall assume that $M_i : (0,\infty) \to (0,\infty)$ is a continuous function satisfying the following conditions:

(H_1) $\lim_{t\to\infty} t^{1-\frac{p_s^*}{p}} M_i(t) = 0$.
(H_2) There exists $m_i > 0$ such that for all $t > 0$, we have $M_i(t) \geq m_i$.
(H_3) There exists $\theta_i \in \left[1, \frac{p_s^*}{p}\right[$ such that for all $t > 0$, we have $M_i(t)t \leq \theta_i \widehat{M}_i(t)$, where $\widehat{M}_i(t) = \int_0^t M_i(s)\mathrm{d}s$.

Moreover, we shall assume that $f, g \in C(\bar{\Omega} \times \mathbb{R} \times \mathbb{R}, [0,\infty[)$ are positively homogeneous functions of degree $(q-1)$, i.e., for all $t > 0$ and $(x,u,v) \in \Omega \times \mathbb{R} \times \mathbb{R}$, we have

$$\begin{cases} f(x, tu, tv) = t^{q-1}f(x, u, v), \\ g(x, tu, tv) = t^{q-1}g(x, u, v). \end{cases} \tag{1.4}$$

Finally, we shall also assume that there exists a function $H : \bar{\Omega} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfying

$$H_u(x, u, v) = f(x, u, v) \quad \text{and} \quad H_v(x, u, v) = g(x, u, v),$$

where H_u (respectively, H_v) denotes the partial derivative of H with respect to u (respectively, v). We note that the primitive function H belongs to $C^1(\bar{\Omega} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ and satisfies the following assumptions for all $t > 0$, $(x, u, v) \in \bar{\Omega} \times \mathbb{R} \times \mathbb{R}$, and some constant $\gamma > 0$:

$$\begin{cases} H(x, tu, tv) = t^q H(x, u, v), \\ qH(x, u, v) = uf(x, u, v) + vg(x, u, v), \\ |H(x, u, v)| \le \gamma(|u|^q + |v|^q). \end{cases} \tag{1.5}$$

Before stating our main result, let us introduce some notations. For $s \in (0, 1)$, we define the functional space

$$W^{s,p}(Q) = \left\{ w : \mathbb{R}^n \to \mathbb{R} \quad \text{measurable: } w \in L^p(\Omega) \quad \text{and} \quad \frac{w(x) - w(y)}{|x - y|^{\frac{n}{p}+s}} \in L^p(Q) \right\},$$

which is endowed with the norm

$$\|w\|_{W^{s,p}(Q)} = \left(\|w\|^p_{L^p(\Omega)} + \int_Q \frac{|w(x) - w(y)|^p}{|x - y|^{n+ps}} \mathrm{d}x\mathrm{d}y \right)^{\frac{1}{p}},$$

where $Q = \mathbb{R}^{2n} \setminus (\Omega^c \times \Omega^c)$ and $\Omega^c = \mathbb{R}^n \setminus \Omega$. From now on, we shall denote by $\|\cdot\|_q$ the norm on the Lebesgue space $L^q(\Omega)$. It is well known that $(W^{s,p}(Q), \|\cdot\|_{W^{s,p}(Q)})$ is a uniformly convex Banach space.

Next, $L^p(\Omega, V_i)$ denotes the Lebesgue space of real-valued functions, with $V_i(x)|w|^p \in L^1(\Omega)$, endowed with the following norm:

$$\|w\|_{p,V_i} = \left(\int_\Omega V_i(x)|w|^p \mathrm{d}x \right)^{\frac{1}{p}}.$$

Let us denote by $W^{s,p}_{V_i}(Q)$ the completion of $C_0^\infty(Q)$ with respect to the norm

$$\|w\|_{V_i} = \left(\|w\|^p_{p,V_i} + \int_Q \frac{|w(x) - w(y)|^p}{|x - y|^{n+ps}} \mathrm{d}x\mathrm{d}y \right)^{\frac{1}{p}}. \tag{1.6}$$

According to [18, (Theorem 6.7]), the embedding $W^{s,p}_{V_i}(Q) \hookrightarrow L^\nu(\Omega)$ is continuous for any $\nu \in [p, p_s^*]$. Namely, there exists a positive constant C_ν such that

$$\|w\|_\nu \le C_\nu \|w\|_{V_i} \quad \text{for all } w \in W^{s,p}_{V_i}(Q).$$

Moreover, by [46, Lemma 2.1], the embedding from $W^{s,p}_{V_i}(Q)$ into $L^\nu(\Omega)$, is compact for any $\nu \in [1, p_s^*)$.

Let $W = W^{s,p}_{V_1}(Q) \times W^{s,p}_{V_2}(Q)$ be equipped with the norm $\|(u, v)\| = (\|u\|^p_{V_1} + \|v\|^p_{V_2})^{\frac{1}{p}}$. Then, $(W, \|.\|)$ is a reflexive Banach space. The interested reader can refer to [2] for more details. Let S_{p,V_i} be the best Sobolev constants for the embeddings from $W^{s,p}_{V_i}(Q)$ into $L^{p_s^*}(\Omega)$, which is given as follows:

$$S_{p,V_i} = \inf_{u \in W^{s,p}_{V_i}(Q) \setminus \{0\}} \frac{\|w\|^p_{V_i}}{\|w\|^p_{p_s^*}}. \tag{1.7}$$

For simplicity, in the rest of this article, S will denote the following expression:

$$S = \min\left(S_{p,V_1}, S_{p,V_2}\right). \tag{1.8}$$

Next, we define the notion of solutions for problem (1.3).

Definition 1.1. We say that $(u, v) \in W$ is a weak solution of problem (1.3), if

$$\begin{aligned} &M_1(\|u\|^p)\left(\int_Q \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(z(x)-z(y))}{|x-y|^{n+ps}}\mathrm{d}x\mathrm{d}y + \int_\Omega V_1(x)|u|^{p-2}uz\mathrm{d}x\right) \\ &+ M_2(\|v\|^p)\left(\int_Q \frac{|v(x)-v(y)|^{p-2}(v(x)-v(y))(w(x)-w(y))}{|x-y|^{n+ps}}\mathrm{d}x\mathrm{d}y + \int_\Omega V_2(x)|v|^{p-2}vw\mathrm{d}x\right) \\ &= \int_\Omega \left(a_1(x)|u|^{p_s^*-2}uz + a_2(x)|v|^{p_s^*-2}vw\right)\mathrm{d}x + \lambda\int_\Omega (H_u(x,u,v)z + H_v(x,u,v)w)\mathrm{d}x, \end{aligned}$$

for all $(z, w) \in W$.

The following theorem is the main result of this article.

Theorem 1.1. *Assume that* $s \in (0, 1)$, $n > ps$, $1 < q < p < p_s^*$, *and that equations* (1.4) *and* (1.5) *hold. If* M *satisfies conditions* (H_1)–(H_3), *then there exists* $\lambda^* > 0$ *such that for all* $\lambda \in (0, \lambda^*)$, *system* (1.3) *has at least two nontrivial weak solutions.*

This article is organized as follows. In Section 2, we present some notations and preliminary results related to the Nehari manifold and fibering maps. In Section 3, we prove Theorem 1.1.

2 The Nehari manifold method and fibering maps analysis

This section collects some basic results on the Nehari manifold method and the fibering maps analysis, which will be used in the forthcoming section; we refer the interested reader to [11,12,19] for more details. We begin by considering the Euler-Lagrange functional $J_\lambda : W \to \mathbb{R}$, which is defined as follows:

$$J_\lambda(u, v) = \frac{1}{p}(\widehat{M_1}(A_1(u)) + \widehat{M_2}(A_2(v))) - \frac{1}{p_s^*}B(u, v) - \lambda C(u, v), \tag{2.1}$$

where

$$A_i(w) = \|w\|_{V_i}^p, \quad B(u, v) = \int_\Omega \left(a_1(x)|u|^{p_s^*} + a_2(x)|v|^{p_s^*}\right)\mathrm{d}x,\ C(u, v) = \int_\Omega H(x, u, v)\mathrm{d}x.$$

We can easily verify that $J_\lambda \in C^1(W, \mathbb{R})$; moreover, its derivative J_λ' from the space W into its dual space W' is given as follows:

$$\langle J_\lambda'(u, v), (u, v)\rangle = A_1(u)M_1(A_1(u)) + A_2(v)M_2(A_2(v)) - B(u, v) - \lambda qC(u, v). \tag{2.2}$$

From the last equation, we can see that the critical points of the functional J_λ are exactly the weak solutions for problem (1.3). Moreover, since the energy functional J_λ is not bounded from below on W, we shall show that J_λ is bounded from below on a suitable subset of W, which is known as the Nehari manifold and is defined as follows:

$$\mathcal{N}_\lambda = \{(u, v) \in W\backslash\{(0, 0)\}, \langle J_\lambda'(u, v), (u, v)\rangle_W = 0\}.$$

It is clear that $(u, v) \in \mathcal{N}_\lambda$ if and only if

$$A_1(u)M_1(A_1(u)) + A_2(v)M_2(A_2(v)) - B(u,v) - \lambda q C(u,v) = 0. \tag{2.3}$$

Hence, from (2.2), we see that the elements of $\mathcal{N}_\lambda$ correspond to nontrivial critical points, which are solutions of problem (1.3).

It is useful to understand $\mathcal{N}_\lambda$ in terms of the stationary points of the fibering maps $\varphi_{u,v} : (0,\infty) \to \mathbb{R}$, is defined as follows:

$$\varphi_{u,v}(t) = J_\lambda(tu,tv) = \frac{1}{p}(\widehat{M_1}(t^pA_1(u)) + \widehat{M_2}(t^pA_2(v))) - \frac{t^{p_s^*}}{p_s^*}B(u,v) - \lambda t^q C(u,v).$$

A simple calculation shows that for all $t > 0$, we have

$$\varphi'_{u,v}(t) = t^{p-1}(A_1(u)M_1(t^pA_1(u)) + A_2(v)M_2(t^pA_2(v))) - t^{p_s^*-1}B(u,v) - \lambda q t^{q-1}C(u,v),$$

and

$$\begin{aligned}\varphi''_{u,v}(t) = {} & (p-1)t^{p-2}(A_1(u)M_1(t^pA_1(u)) + A_2(v)M_2(t^pA_2(v))) + pt^{2p-2}((A_1(u))^2M_1'(t^pA_1(u)) \\ & + (A_2(v))^2M_2'(t^pA_2(v))) - (p_s^*-1)t^{p_s^*-2}B(u,v) - \lambda q(q-1)t^{q-2}C(u,v).\end{aligned}$$

It is easy to see that for all $t > 0$, we have

$$\varphi'_{u,v}(t) = \langle J'_\lambda(tu,tv),(u,v)\rangle_W = \frac{1}{t^2}\langle J'_\lambda(tu,tv),(tu,tv)\rangle_W.$$

So, $(tu,tv) \in \mathcal{N}_\lambda$ if and only if $\varphi'_{u,v}(t) = 0$. In the special case, when $t = 1$, we obtain $(u,v) \in \mathcal{N}_\lambda$, if and only if $\varphi'_{u,v}(1) = 0$. On the other hand, from (2.3), we obtain

$$\begin{aligned}\varphi''_{u,v}(1) = {} & (p-1)(A_1(u)M_1(A_1(u)) + A_2(v)M_2(A_2(v))) - (p_s^*-1)B(u,v) \\ & + p((A_1(u))^2M_1'(A_1(u)) + (A_2(v))^2M_2'(A_2(v))) - \lambda q(q-1)C(u,v) \\ = {} & p((A_1(u))^2M_1'(A_1(u)) + (A_2(v))^2M_2'(A_2(v))) - (p_s^*-p)B(u,v) - \lambda q(q-p)C(u,v)\end{aligned} \tag{2.4}$$

$$\begin{aligned}= {} & p((A_1(u))^2M_1'(A_1(u)) + (A_2(v))^2M_2'(A_2(v))) + \lambda q(p_s^*-q)C(u,v) \\ & - (p_s^*-p)(A_1(u)M_1(A_1(u)) + A_2(v)M_2(A_2(v)))\end{aligned} \tag{2.5}$$

$$\begin{aligned}= {} & p((A_1(u))^2M_1'(A_1(u)) + (A_2(v))^2M_2'(A_2(v))) - (p_s^*-q)B(u,v) \\ & + (p-q)(A_1(u)M_1(A_1(u)) + A_2(v)M_2(A_2(v))).\end{aligned} \tag{2.6}$$

Now, in order to obtain a multiplicity of solutions, we divide $\mathcal{N}_\lambda$ into three parts as follows:

$$\begin{aligned}\mathcal{N}_\lambda^+ &= \{(u,v) \in \mathcal{N}_\lambda : \varphi''_{u,v}(1) > 0\} = \{(u,v) \in W : \varphi'_{u,v}(1) = 0 \text{ and } \varphi''_{u,v}(1) > 0\},\\ \mathcal{N}_\lambda^- &= \{(u,v) \in \mathcal{N}_\lambda : \varphi''_{u,v}(1) < 0\} = \{(u,v) \in W : \varphi'_{u,v}(1) = 0 \text{ and } \varphi''_{u,v}(1) < 0\},\\ \mathcal{N}_\lambda^0 &= \{(u,v) \in \mathcal{N}_\lambda : \varphi''_{u,v}(1) = 0\} = \{(u,v) \in W : \varphi'_{u,v}(1) = 0 \text{ and } \varphi''_{u,v}(1) = 0\}.\end{aligned}$$

Lemma 2.1. *Suppose that (u_0,v_0) is a local minimizer for J_λ on $\mathcal{N}_\lambda$, with $(u_0,v_0) \notin \mathcal{N}_\lambda^0$. Then, (u_0,v_0) is a critical point of J_λ.*

Proof. If (u_0,v_0) is a local minimizer for J_λ on $\mathcal{N}_\lambda$, then (u_0,v_0) solves the following optimization problem:

$$\begin{cases}\min\limits_{(u,v)\in\mathcal{N}_\lambda} J_\lambda(u,v) = J_\lambda(u_0,v_0),\\ \beta(u_0,v_0) = 0,\end{cases}$$

where

$$\beta(u,v) = A_1(u)M_1(A_1(u)) + A_2(v)M_2(A_2(v)) - B(u,v) - \lambda q C(u,v).$$

By the Lagrangian multipliers theorem, there exists $\delta \in \mathbb{R}$, such that

$$J'_\lambda(u_0, v_0) = \delta\beta'(u_0, v_0). \tag{2.7}$$

Since $(u_0, v_0) \in \mathcal{N}_\lambda$, we obtain

$$\delta\langle\beta'(u_0, v_0), (u_0, v_0)\rangle_W = \langle J'_\lambda(u_0, v_0), (u_0, v_0)\rangle_W = 0. \tag{2.8}$$

Moreover, by (2.3) and the constraint $\beta(u_0, v_0) = 0$, we have

$$\begin{aligned}\langle\beta'(u_0, v_0), (u_0, v_0)\rangle_W &= p((A_1(u_0))^2M_1'(A_1(u_0)) + (A_2(v_0))^2M_2'(A_2(v_0))) - (p_s^* - p)B(u_0, v_0) - \lambda q(q-p)C(u_0, v_0)\\ &= \varphi''_{u_0,v_0}(1).\end{aligned}$$

Since $(u_0, v_0) \notin \mathcal{N}_\lambda^0$, we have $\varphi''_{u_0,v_0}(1) \neq 0$. Thus, by (2.8), we obtain $\delta = 0$. Consequently, by substituting δ in (2.7), we obtain $J'_\lambda(u_0, v_0) = 0$. This completes the proof of Lemma 2.1. □

In order to understand the Nehari manifold and fibering maps, let us define the function $\psi_{u,v} : (0, \infty) \to \mathbb{R}$ as follows:

$$\psi_{u,v}(t) = t^{p-q}(A_1(u)M_1(t^pA_1(u)) + A_2(v)M_2(t^pA_2(v))) - t^{p_s^*-q}B(u, v) - \lambda qC(u, v). \tag{2.9}$$

We note that $t^{q-1}\psi_{u,v}(t) = \varphi'_{u,v}(t)$. Thus, it is easy to see that $(tu, tv) \in \mathcal{N}_\lambda$ if and only if

$$\psi_{u,v}(t) = 0. \tag{2.10}$$

Moreover, by a direct computation, we obtain

$$\begin{aligned}\psi'_{u,v}(t) &= (p-q)t^{p-q-1}(A_1(u)M_1(t^pA_1(u)) + A_2(v)M_2(t^pA_2(v))) + pt^{2p-q-1}(A_1^2(u)M_1'(t^pA_1(u)) + A_2^2(v)M_2'(t^pA_2(v)))\\ &\quad - (p_s^* - q)t^{p_s^*-q-1}B(u, v).\end{aligned}$$

Therefore,

$$t^{q-1}\psi'_{u,v}(t) = \varphi''_{u,v}(t). \tag{2.11}$$

Hence, $(tu, tv) \in \mathcal{N}_\lambda^+$, (respectively, $(tu, tv) \in \mathcal{N}_\lambda^-$) if and only if $\psi_{u,v}(t) = 0$ and $\psi'_{u,v}(t) > 0$ (respectively, $\psi_{u,v}(t) = 0$, and $\psi'_{u,v}(t) < 0$). Put

$$m = \min(m_1, m_2), \quad \theta = \max(\theta_1, \theta_2), \tag{2.12}$$

and

$$\lambda_* = \frac{(mS)^{\frac{p_s^*-q}{p_s^*-p}}}{\gamma q|\Omega|^{\frac{p_s^*-q}{p_s^*}}}\left(\frac{p_s^*-p}{p_s^*-q}\right)\left(\frac{p-q}{(p_s^*-q)a}\right)^{\frac{p-q}{p_s^*-p}}. \tag{2.13}$$

Now we shall prove the following crucial result.

Lemma 2.2. *Assume that conditions* (H_1) *and* (H_2) *hold. Then, for all* $(u, v) \in \mathcal{N}_\lambda$, *there exist* $\lambda_* > 0$ *and unique* $t_1 > 0$ *and* $t_2 > 0$, *such that for each* $\lambda \in (0, \lambda_*)$, *we have* $(t_1u, t_1v) \in \mathcal{N}_\lambda^+$ *and* $(t_2u, t_2v) \in \mathcal{N}_\lambda^-$.

Proof. We begin by noting that by (2.9), we have

$$\psi_{u,v}(t) \to -\lambda qC(u, v), \ \text{ as } t \to 0^+, \ \text{ and } \psi_{u,v}(t) \to -\infty, \ \text{ as } t \to \infty.$$

Now, if we combine equations (1.5) and (1.7) with the Hölder inequality, we obtain

$$\begin{aligned}B(u, v) &\le \|a_1\|_\infty\|u\|_{p_s^*}^{p_s^*} + \|a_2\|_\infty\|v\|_{p_s^*}^{p_s^*} \le a\left(\|u\|_{p_s^*}^{p_s^*} + \|v\|_{p_s^*}^{p_s^*}\right)\\ &\le a(S_{p,V_1}^{-\frac{p_s^*}{p}}(A_1(u))^{\frac{p_s^*}{p}} + S_{p,V_2}^{-\frac{p_s^*}{p}}(A_2(u))^{\frac{p_s^*}{p}})\\ &\le S^{-\frac{p_s^*}{p}}a(A(u, v))^{\frac{p_s^*}{p}},\end{aligned} \tag{2.14}$$

and

$$C(u,v) \le \gamma(\|u\|_q^q + \|v\|_q^q) \le \gamma|\Omega|^{\frac{p_s^*-q}{p_s^*}}\left(\|u\|_{p_s^*}^q + \|v\|_{p_s^*}^q\right) \le \gamma S^{-\frac{q}{p}}|\Omega|^{\frac{p_s^*-q}{p_s^*}}(A(u,v))^{\frac{q}{p}}, \tag{2.15}$$

where $a = \max(\|a_1\|_\infty, \|a_2\|_\infty)$, $A(u,v) = \|(u,v)\|^p$, and S is given by equation (1.8).

On the other hand, by combining equations (2.14) and (2.15) with (H_2), we obtain

$$\begin{aligned}\psi_{u,v}(t) &\ge t^{p-q}(m_1A_1(u) + m_2A_2(v)) - t^{p_s^*-q}S^{-\frac{p_s^*}{p}}a(A(u,v))^{\frac{p_s^*}{p}} - \lambda q\gamma S^{-\frac{q}{p}}|\Omega|^{\frac{p_s^*-q}{p_s^*}}(A(u,v))^{\frac{q}{p}} \\ &\ge mt^{p-q}A(u,v) - t^{p_s^*-q}S^{-\frac{p_s^*}{p}}a(A(u,v))^{\frac{p_s^*}{p}} - \lambda q\gamma S^{-\frac{q}{p}}|\Omega|^{\frac{p_s^*-q}{p_s^*}}(A(u,v))^{\frac{q}{p}} \ge (A(u,v))^{\frac{q}{p}}F_{u,v}(t),\end{aligned} \tag{2.16}$$

where m is given by equation (2.12) and $F_{u,v}$ is defined for $t > 0$ by

$$F_{u,v}(t) = mt^{p-q}(A(u,v))^{\frac{p-q}{p}} - t^{p_s^*-q}S^{-\frac{p_s^*}{p}}a(A(u,v))^{\frac{p_s^*-q}{p}} - \lambda q\gamma S^{-\frac{q}{p}}|\Omega|^{\frac{p_s^*-q}{p_s^*}}.$$

Since $1 < q < p < p_s^*$, it is easy to see that $\lim_{t\to 0^+}F_{u,v}(t) < 0$ and $\lim_{t\to\infty}F_{u,v}(t) = -\infty$. So, by a simple calculation, we can prove that $F_{u,v}$ attains its unique global maximum at

$$t_{\max}(u,v) = \left(\frac{m}{S^{-\frac{p_s^*}{p}}a}\left(\frac{p-q}{p_s^*-q}\right)\right)^{\frac{1}{p_s^*-p}}(A(u,v))^{\frac{-1}{p}}. \tag{2.17}$$

Moreover,

$$F_{u,v}(t_{\max}) = q\gamma S^{-\frac{q}{p}}|\Omega|^{\frac{p_s^*-q}{p_s^*}}(\lambda_* - \lambda), \tag{2.18}$$

where λ_* is given by (2.13).

If we choose $\lambda < \lambda_*$, then we obtain from (2.16)

$$\psi_{u,v}(t_{\max}) \ge (A(u,v))^{\frac{q}{p}}F_{u,v}(t_{\max}) > 0. \tag{2.19}$$

Hence, by a variation of $\psi_{u,v}(t)$, there exist unique $t_1 < t_{\max}(u,v)$ and unique $t_2 > t_{\max}(u,v)$, such that $\psi'_{u,v}(t_1) > 0$ and $\psi'_{u,v}(t_2) < 0$. Moreover, $\psi_{u,v}(t_1) = 0 = \psi_{u,v}(t_2)$. Finally, it follows from (2.10) and (2.11) that $(t_1u, t_1v) \in \mathcal{N}_\lambda^+$ and $(t_2u, t_2v) \in \mathcal{N}_\lambda^-$. This completes the proof of Lemma 2.2. □

We can see from Lemma 2.2 that sets $\mathcal{N}_\lambda^+$ and $\mathcal{N}_\lambda^-$ are nonempty. In the following lemma, we shall provide a property related to $\mathcal{N}_\lambda^0$.

Lemma 2.3. *Assume that condition (H_2) holds. Then, for all $\lambda \in (0, \lambda_*)$, we have $\mathcal{N}_\lambda^0 = \varnothing$.*

Proof. We shall argue by contradiction. Assume that there exists $\lambda > 0$ in $(0, \lambda_*)$ such that $\mathcal{N}_\lambda^0 \ne \varnothing$. Let $(u_0, v_0) \in \mathcal{N}_\lambda^0$. Then, invoking (H_2), (2.5), and (2.15), we have

$$\begin{aligned}0 = \varphi_u''(1) &= p((A_1(u))^2M_1'(A_1(u)) + (A_2(v))^2M_2'(A_2(v))) - (p_s^* - p)(A_1(u)M_1(A_1(u)) + A_2(v)M_2(A_2(v))) \\ &\quad + \lambda q(p_s^* - q)C(u,v) \\ &\le p((A_1(u))^2M_1'(A_1(u)) + (A_2(v))^2M_2'(A_2(v))) - (p_s^* - p)(m_1A_1(u) + m_2A_2(v)) \\ &\quad + \lambda q(p_s^* - q)C(u,v) \\ &\le p((A(u))^2M'(A(u)) + (A(v))^2N'(A(v))) - (p_s^* - p)mA(u,v) \\ &\quad + \lambda q(p_s^* - q)\gamma S^{-\frac{q}{p}}|\Omega|^{\frac{p_s^*-q}{p_s^*}}(A(u,v))^{\frac{q}{p}}.\end{aligned} \tag{2.20}$$

On the other hand, by (H_2), (2.6), and (2.14), one has

$$\begin{aligned}0 = \varphi_u''(1) &= p((A_1(u))^2M_1'(A_1(u)) + (A_2(v))^2M_2'(A_2(v))) + (p - q)(A_1(u)M_1(A_1(u)) + A_2(v)M_2(A_2(v))) \\ &\quad - (p_s^* - q)B(u,v) \\ &\ge p((A_1(u))^2M_1'(A_1(u)) + (A_2(v))^2M_2'(A_2(v))) + (p - q)(m_1A_1(u) + m_2A_2(v)) - (p_s^* - q)B(u,v) \\ &\ge p((A_1(u))^2M_1'(A_1(u)) + (A_2(v))^2M_2'(A_2(v))) + (p - q)mA(u,v) - (p_s^* - q)S^{-\frac{p_s^*}{p}}a(A(u,v))^{\frac{p_s^*}{p}}.\end{aligned} \tag{2.21}$$

Combining (2.20) and (2.21), we obtain

$$\lambda \geq \frac{m(A(u,v))^{\frac{p-q}{p}} - S^{-\frac{p_s^*}{p}} a(A(u,v))^{\frac{p_s^*-q}{p}}}{q\gamma S^{-\frac{q}{p}} |\Omega|^{\frac{p_s^*-q}{p_s^*}}}. \tag{2.22}$$

Next, we define the function H on $(0,\infty)$ by

$$H(t) = \frac{mt^{\frac{p-q}{p}} - S^{-\frac{p_s^*}{p}} a t^{\frac{p_s^*-q}{p}}}{q\gamma S^{-\frac{q}{p}} |\Omega|^{\frac{p_s^*-q}{p_s^*}}}.$$

Since $1 < q < p < p_s^*$, it follows that $\lim_{t\to 0^+} H(t) = 0$ and $\lim_{t\to\infty} H(t) = -\infty$. A simple computation now shows that H attains its maximum at

$$\tilde{t} = \left(\left(\frac{p-q}{p_s^* - q} \right) \frac{mS^{\frac{p_s^*}{p}}}{a} \right)^{\frac{p}{p_s^*-p}},$$

and

$$\max_{t>0} H(t) = H(\tilde{t}) = \lambda_*. \tag{2.23}$$

Hence, it follows from (2.22) and (2.23), that $\lambda \geq \max_{t>0} H(t) = \lambda_*$, which contradicts $\lambda \in (0, \lambda_*)$. Therefore, we can conclude that that indeed $\mathcal{N}_\lambda^0 = \varnothing$, for $\lambda \in (0,\lambda_*)$. This completes the proof of Lemma 2.3. □

Lemma 2.4. *Assume that conditions* (H_2) *and* (H_3) *hold. Then,* J_λ *is coercive and bounded from below on* $\mathcal{N}_\lambda$.

Proof. Let $(u,v) \in \mathcal{N}_\lambda$. Then, by (2.3), we obtain

$$B(u,v) = A_1(u)M_1(A_1(u)) + A_2(v)M_2(A_2(v)) - \lambda q C(u,v).$$

Therefore,

$$J_\lambda(u,v) = \frac{1}{p}(\widehat{M}_1(A_1(u)) + \widehat{M}_2(A_2(v))) - \frac{1}{p_s^*}(A_1(u)M_1(A_1(u)) + A_2(v)M_2(A_2(v))) - \lambda\left(1 - \frac{q}{p_s^*}\right)C(u,v).$$

Moreover, by (H_2), (H_3), and (2.15), we have

$$\begin{aligned} J_\lambda(u,v) &\geq \frac{1}{\theta_1 p} A_1(u)M_1(A_1(u)) + \frac{1}{\theta_2 p} A_2(v)M_2(A_2(v)) - \frac{1}{p_s^*} A_1(u)M_1(A_1(u)) \\ &\quad - \frac{1}{p_s^*} A_2(v)M_2(A_2(v)) - \lambda\left(1 - \frac{q}{p_s^*}\right)C(u,v) \\ &\geq \left(\frac{1}{\theta p} - \frac{1}{p_s^*}\right)(A_1(u)M_1(A_1(u)) + A_2(v)M_2(A_2(v))) - \lambda\left(1 - \frac{q}{p_s^*}\right)C(u,v) \\ &\geq \left(\frac{1}{\theta p} - \frac{1}{p_s^*}\right)(m_1 A_1(u) + m_2 A_2(v)) - \lambda\left(1 - \frac{q}{p_s^*}\right)C(u,v) \\ &\geq m\left(\frac{1}{\theta p} - \frac{1}{p_s^*}\right)A(u,v) - \lambda\left(1 - \frac{q}{p_s^*}\right)\gamma S^{-\frac{q}{p}} |\Omega|^{\frac{p_s^*-q}{p_s^*}} (A(u,v))^{\frac{q}{p}}. \end{aligned}$$

Since $q < p$ and $\theta p < p_s^*$, it follows that J_λ is coercive and bounded from below on $\mathcal{N}_\lambda$. This completes the proof of Lemma 2.4. □

By Lemma (2.3), we can write $\mathcal{N}_\lambda = \mathcal{N}_\lambda^+ \cup \mathcal{N}_\lambda^-$, and by Lemma (2.4), we can define

$$\alpha_\lambda^- = \inf_{(u,v)\in\mathcal{N}_\lambda^-} J_\lambda(u,v) \quad \text{and} \quad \alpha_\lambda^+ = \inf_{(u,v)\in\mathcal{N}_\lambda^+} J_\lambda(u,v).$$

3 Proof of the main result

In this section, we shall prove the main result of this article (Theorem 1.1). First, we need to prove two propositions.

Proposition 3.1. *Assume that conditions (H_2) and (H_3) hold. Then, there exist $t_0 > 0$ and $(u_0, v_0) \in W\backslash\{0\}$, with $(u_0, v_0) > 0$ in $\mathbb{R}^n$, such that*

$$\frac{1}{p}(\widehat{M_1}(A_1(u_0)t_0^p) + \widehat{M_2}(A_2(v_0)t_0^p)) - \frac{t_0^{p_s^*}}{p_s^*}B(u_0, v_0) = \left(\frac{s}{n} - \frac{\theta - 1}{\theta p}\right)a^{\frac{-n}{sp_s^*}}\left(\frac{mS}{\theta}\right)^{\frac{n}{sp}}. \tag{3.1}$$

Proof. For any $(u, v) \in W\backslash\{0\}$, we define the function $\zeta_{u,v} : (0, \infty) \to \mathbb{R}$ as follows:

$$\zeta_{u,v}(t) = \frac{1}{p}(\widehat{M_1}(A_1(tu)) + \widehat{M_2}(A_2(tv))) - \frac{1}{p_s^*}B(t(u, v)) = \frac{1}{p}(\widehat{M_1}(t^pA_1(u)) + \widehat{M_2}(t^pA_2(v))) - \frac{t^{p_s^*}}{p_s^*}B(u, v).$$

By (H_3), it can be shown that $\lim_{t\to 0^+}\zeta_{u,v}(t) \geq 0$ and $\lim_{t\to\infty}\zeta_{u,v}(t) = -\infty$. It is clear that ζ is of class C^1. Moreover, invoking (H_2) and (H_3), we obtain

$$\begin{aligned}
\zeta_{u,v}(t) &\geq \frac{t^p}{\theta_1 p}A_1(u)M_1(t^pA_1(u)) + \frac{t^p}{\theta_2 p}A_2(v)M_2(t^pA_2(v)) - \frac{t^{p_s^*}}{p_s^*}B(u, v)\\
&\geq \frac{t^p}{\theta p}(A_1(u)M_1(t^pA_1(u)) + A_2(v)M_2(t^pA_2(v))) - \frac{t^{p_s^*}}{p_s^*}B(u, v)\\
&\geq \frac{t^p}{\theta p}(m_1A_1(u) + m_2A_2(v)) - \frac{t^{p_s^*}}{p_s^*}B(u, v)\\
&\geq \frac{m}{\theta p}t^pA(u, v) - \frac{t^{p_s^*}}{p_s^*}B(u, v) = \omega_{u,v}(t).
\end{aligned}$$

Since $\lim_{t\to 0}\omega_{u,v}(t) = 0$ and $\lim_{t\to\infty}\omega_{u,v}(t) = -\infty$, it follows that $\omega_{u,v}$ attains its global maximum at

$$t_* = \left(\frac{mA(u, v)}{\theta B(u, v)}\right)^{\frac{1}{p_s^*-p}}.$$

Moreover, from (2.14) and the fact that $p_s^* > \theta p$, we have

$$\begin{aligned}
\sup_{t>0}\omega_{u,v}(t) &= \omega_{u,v}(t_*)\\
&= \left(\frac{p_s^* - p}{pp_s^*}\right)\left(\frac{m}{\theta}\right)^{\frac{p_s^*}{p_s^*-p}}(A(u, v))^{\frac{p_s^*}{p_s^*-p}}(B(u, v))^{-\frac{p}{p_s^*-p}}\\
&= \left(\frac{p_s^* - p}{pp_s^*}\right)\left(\frac{m}{\theta}\right)^{\frac{p_s^*}{p_s^*-p}}\left((A(u, v))^{-\frac{p_s^*}{p}}B(u, v)\right)^{-\frac{p}{p_s^*-p}}\\
&= \frac{s}{n}\left(\frac{m}{\theta}\right)^{\frac{n}{sp}}(A(u, v))^{\frac{n}{sp}}(B(u, v))^{-\frac{n}{sp_s^*}}\\
&\geq \frac{s}{n}a^{\frac{-n}{sp_s^*}}\left(\frac{mS}{\theta}\right)^{\frac{n}{sp}}\\
&\geq \left(\frac{s}{n} - \frac{\theta - 1}{\theta p}\right)a^{\frac{-n}{sp_s^*}}\left(\frac{mS}{\theta}\right)^{\frac{n}{sp}}\\
&= \left(\frac{p_s^* - \theta p}{\theta pp_s^*}\right)a^{\frac{-n}{sp_s^*}}\left(\frac{mS}{\theta}\right)^{\frac{n}{sp}} > 0.
\end{aligned} \tag{3.2}$$

Therefore, using the variations of the functions $\zeta_{u,v}$ and $\omega_{u,v}$, we obtain

$$\sup_{t>0}\zeta_{u,v} \geq \sup_{t>0}\omega_{u,v} \geq \left(\frac{s}{n} - \frac{\theta-1}{\theta p}\right)a^{\frac{-n}{sp_s^*}}\left(\frac{mS}{\theta}\right)^{\frac{n}{sp}}.$$

Hence, there exists $t_0 > 0$ such that

$$\zeta_{u,v}(t_0) = \left(\frac{s}{n} - \frac{\theta-1}{\theta p}\right)a^{\frac{-n}{sp_s^*}}\left(\frac{mS}{\theta}\right)^{\frac{n}{sp}}.$$

This completes the proof of Proposition 3.1. □

Set now

$$L = (p-q)\left(\frac{m}{q}\left(\frac{s}{n} - \frac{\theta-1}{\theta p}\right)\right)^{-\frac{q}{p-q}}\left(\frac{p_s^*-q}{\theta p^2}\right)^{\frac{p}{p-q}}\left(\gamma S^{-\frac{q}{p}}|\Omega|^{\frac{p_s^*-q}{p_s^*}}\right)^{\frac{p}{p-q}}. \tag{3.3}$$

Proposition 3.2. *Assume that conditions* (H_2) *and* (H_3) *hold. If* $1 < q < p < p_s^*$, *then every Palais-Smale sequence* $\{(u_k, v_k)\} \subset W$ *for* J_λ *at level* c, *with*

$$c < \left(\frac{s}{n} - \frac{\theta-1}{\theta p}\right)a^{\frac{-n}{sp_s^*}}\left(\frac{mS}{\theta}\right)^{\frac{n}{sp}} - \lambda^{\frac{p}{p-q}}L, \tag{3.4}$$

possesses a convergent subsequence.

Proof. Let $\{(u_k, v_k)\}$ be a Palais-Smale sequence for J_λ at level c, i.e.,

$$J_\lambda(u_k, v_k) \to c, \quad \text{and } J'_\lambda(u_k, v_k) \to 0, \quad \text{as } k \to \infty.$$

By Lemma (2.4), we know that $\{(u_k, v_k)\}$ is bounded in W. So up to a subsequence, still denoted by $\{(u_k, v_k)\}$, there exists $(u_*, v_*) \in W$, $\mu > 0$, and $\eta > 0$, such that as k tends to infinity, we have

$$\begin{cases}(u_k, v_k) \rightharpoonup (u_*, v_*) \quad \text{weakly in } W,\\ \|u_k\|_{V_1} \to \mu,\ \|v_k\|_{V_2} \to \eta,\\ (u_k, v_k) \rightharpoonup (u_*, v_*) \quad \text{weakly in } L^{p_s^*}(\Omega) \times L^{p_s^*}(\Omega),\\ (u_k, v_k) \to (u_*, v_*) \quad \text{strongly in } L^q(\Omega) \times L^q(\Omega), \quad 1 \leq q < p_s^*,\\ (u_k, v_k) \to (u_*, v_*) \quad \text{a.e. in } \Omega,\end{cases} \tag{3.5}$$

Since $1 \leq q < p_s^*$, it follows from [41, Theorem IV-9] that there exist functions $l_1, l_2 \in L^q(\Omega)$ such that for a.e. $x \in \Omega$, we have $|u_k(x)| \leq l_1(x)$, $|v_k(x)| \leq l_2(x)$. Hence, by the dominated convergence theorem,

$$C(u_k, v_k) \longrightarrow C(u_*, v_*) \quad \text{as } k \to \infty. \tag{3.6}$$

On the other hand, by the Brezis-Lieb lemma [21, Lemma 1.32], for k large enough, we have

$$A_1(u_k) = A_1(u_k - u_*) + A_1(u_*) + o(1),$$
$$A_2(v_k) = A_2(v_k - v_*) + A_2(v_*) + o(1),$$

and

$$B(u_k, v_k) = B(u_k - u_*, v_k - v_*) + B(u_*, v_*) + o(1).$$

Consequently, by letting k tend to infinity, we obtain

$$
\begin{aligned}
o(1) &= \langle J'_\lambda(u_k, v_k), (u_k - u_*, v_k - v_*)\rangle_W \\
&= M_1(A_1(u_k))\left(\int_Q \frac{|u_k(x) - u_k(y)|^{p-1}((u_k - u_*)(x) - (u_k - u_*)(y))}{|x - y|^{n+ps}} \mathrm{d}x\mathrm{d}y \right. \\
&\quad \left. + \int_\Omega V_1(x)|u_k|^{p-1}(u_k - u_*)\mathrm{d}x\right) - \int_\Omega a_1(x)|u_k|^{p_s^*-1}(u_k - u_*)\mathrm{d}x \\
&\quad + M_2(A_2(v_k))\left(\int_Q \frac{|v_k(x) - v_k(y)|^{p-1}((v_k - v_*)(x) - (v_k - v_*)(y))}{|x - y|^{n+ps}} \mathrm{d}x\mathrm{d}y \right. \\
&\quad \left. + \int_\Omega V_2(x)|v_k|^{p-1}(v_k - v_*)\mathrm{d}x\right) - \int_\Omega a_2(x)|v_k|^{p_s^*-1}(v_k - v_*)\mathrm{d}x \\
&\quad - \lambda\int_\Omega (H_u(x, u_k, v_k)(u_k - u_*) + H_v(x, u_k, v_k)(v_k - v_*))\mathrm{d}x \\
&= M_1(\mu^p)(\mu^p - A_1(u_*)) + M_2(\eta^p)(\eta^p - A_2(v_*)) \\
&\quad - \int_\Omega \left(a_1(x)|u_k|^{p_s^*} + a_2(x)|v_k|^{p_s^*}\right)\mathrm{d}x + \int_\Omega \left(a_1(x)|u_*|^{p_s^*} + a_2(x)|v_*|^{p_s^*}\right)\mathrm{d}x \\
&\quad - \lambda\int_\Omega (H_u(x, u_k, v_k)(u_k - u_*) + H_v(x, u_k, v_k)(v_k - v_*))\mathrm{d}x + o(1) \\
&= M_1(\mu^p)A_1(u_k - u_*) + M_2(\eta^p)A_2(v_k - v_*) - B(u_k - u_*, v_k - v_*) \\
&\quad - \lambda\int_\Omega (H_u(x, u_k, v_k)(u_k - u_*) + H_v(x, u_k, v_k)(v_k - v_*))\mathrm{d}x + o(1).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&M_1(\mu^p)\lim_{k\to\infty} A_1(u_k - u_*) + M_2(\eta^p)\lim_{k\to\infty} A_2(v_k - v_*) \\
&= \lim_{k\to\infty} B(u_k - u_*, v_k - v_*) + \lim_{k\to\infty} \lambda\int_\Omega (H_u(x, u_k, v_k)(u_k - u_*) + H_v(x, u_k, v_k)(v_k - v_*))\mathrm{d}x
\end{aligned}
$$

By (1.5), (3.5), and the Holder inequality, it follows that

$$
\begin{aligned}
&\int_\Omega (H_u(x, u_k, v_k)(u_k - u_*) + H_v(x, u_k, v_k)(v_k - v_*))\mathrm{d}x \\
&\quad \le \gamma q\int_\Omega |u_k|^{q-1}(u_k - u_*)\mathrm{d}x + \gamma q\int_\Omega |v_k|^{q-1}(v_k - v_*)\mathrm{d}x \\
&\quad \le \gamma q\|u_k\|_q^{q-1}\|u_k - u_*\|_q + \gamma q\|v_k\|_q^{q-1}\|v_k - v_*\|_q \\
&\quad \le C_q\gamma q\|u_k\|_{V_1}^{q-1}\|u_k - u_*\|_q + C_q\gamma q\|v_k\|_{V_2}^{q-1}\|v_k - v_*\|_q
\end{aligned}
$$

for some positive constant C_q. So, we obtain

$$
\lim_{k\to\infty}\int_\Omega (H_u(x, u_k, v_k)(u_k - u_*) + H_v(x, u_k, v_k)(v_k - v_*))\mathrm{d}x = 0. \tag{3.7}
$$

Thus, from (3.7), we can deduce that

$$
\lim_{k\to\infty} B(u_k - u_*, v_k - v_*) = M_1(\mu^p)\lim_{k\to\infty} A_1(u_k - u_*) + M_2(\eta^p)\lim_{k\to\infty} A_2(v_k - v_*).
$$

For simplicity, set $b := \lim_{k\to\infty} B(u_k - u_*, v_k - v_*)$. Note that $b \geq 0$. Moreover, to prove that (u_k, v_k) converges strongly to (u_*, v_*), it suffices to prove that $b = 0$. Suppose to the contrary, that $b > 0$. Then, by (H_2), we obtain

$$A_1(u_k - u_*)M_1(\mu^p) + A_2(v_k - v_*)M_2(\eta^p) \geq m_1A_1(u_k - u_*) + m_2A_2(v_k - v_*) \geq mA(u_k - u_*, v_k - v_*). \quad (3.8)$$

Using (2.14), we obtain

$$A(u_k - u_*, v_k - v_*) \geq Sa^{-\frac{p}{p_s^*}}(B(u_k - u_*, v_k - v_*))^{\frac{p}{p_s^*}}. \quad (3.9)$$

So by combining (3.8) and (3.9), we obtain

$$A_1(u_k - u_*)M_1(A_1(u_*)) + A_2(v_k - v_*)M_2(A_2(v_*)) \geq mSa^{-\frac{p}{p_s^*}}(B(u_k - u_*, v_k - v_*))^{\frac{p}{p_s^*}}.$$

By letting k tend to infinity, we conclude that

$$b \geq a^{\frac{-n}{sp_s^*}}(mS)^{\frac{n}{sp}}. \quad (3.10)$$

On the other hand, by (H_3), (3.6), and (3.10), one has

$$\begin{aligned}
c &= \lim_{k\longrightarrow\infty} J_\lambda(u_k, v_k) = \lim_{k\longrightarrow\infty}\left(J_\lambda(u_k, v_k) - \frac{1}{p_s^*}\langle J_\lambda'(u_k, v_k), (u_k, v_k)\rangle_W\right) \\
&= \lim_{k\longrightarrow\infty}\left[\frac{1}{p}(\widehat{M}_1(A_1(u_k)) + \widehat{M}_2(A_2(v_k))) - \frac{1}{p_s^*}A_1(u_k)M_1(A_1(u_k))\right. \\
&\qquad \left. - \frac{1}{p_s^*}A_2(v_k)M_2(A_2(v_k)) - \lambda\left(\frac{p_s^* - q}{p_s^*}\right)C(u_k, v_k)\right] \\
&\geq \lim_{k\longrightarrow\infty}\left[\frac{1}{\theta_1 p}A_1(u_k)M_1(A_1(u_k)) + \frac{1}{\theta_2 p}A_2(v_k)M_2(A_2(v_k))\right. \\
&\qquad \left. - \frac{1}{p_s^*}(A_1(u_k)M_1(A_1(u_k)) + A_2(v_k)M_2(A_2(v_k))) - \lambda\left(\frac{p_s^* - q}{p_s^*}\right)C(u_k, v_k)\right] \\
&\geq \lim_{k\longrightarrow\infty}\left[\left(\frac{1}{\theta p} - \frac{1}{p_s^*}\right)(A_1(u_k)M_1(A_1(u_k)) + A_2(v_k)M_2(A_2(v_k))) - \lambda\left(\frac{p_s^* - q}{p_s^*}\right)C(u_k, v_k)\right] \\
&= \lim_{k\longrightarrow\infty}\left[\left(\frac{p_s^* - \theta p}{\theta p p_s^*}\right)A_1(u_k)M_1(\mu^p) + \left(\frac{p_s^* - \theta p}{\theta p p_s^*}\right)A_2(v_k)M_2(\eta^p) - \lambda\left(\frac{p_s^* - q}{p_s^*}\right)C(u_k, v_k)\right] \\
&= \lim_{k\longrightarrow\infty}\left[\left(\frac{p_s^* - \theta p}{\theta p p_s^*}\right)(A_1(u_k - u_*)M_1(\mu^p) + A_2(v_k - v_*)M_2(\eta^p))\right. \\
&\qquad \left. + \left(\frac{p_s^* - \theta p}{\theta p p_s^*}\right)(A_1(u_*)M_1(\mu^p) + A_2(v_*)M_2(\eta^p)) - \lambda\left(\frac{p_s^* - q}{p_s^*}\right)C(u_k, v_k)\right] \\
&= \left(\frac{s}{n} - \frac{\theta - 1}{\theta p}\right)b + \left(\frac{s}{n} - \frac{\theta - 1}{\theta p}\right)(A_1(u_*)M_1(\mu^p) + A_2(v_*)M_2(\eta^p)) - \lambda\left(\frac{p_s^* - q}{p_s^*}\right)C(u_*, v_*) \\
&\geq \left(\frac{s}{n} - \frac{\theta - 1}{\theta p}\right)b + \left(\frac{s}{n} - \frac{\theta - 1}{\theta p}\right)(m_1A_1(u_*) + m_2A_2(v_*)) - \lambda\left(\frac{p_s^* - q}{p_s^*}\right)C(u_*, v_*) \\
&\geq \left(\frac{s}{n} - \frac{\theta - 1}{\theta p}\right)a^{\frac{-n}{sp_s^*}}(mS)^{\frac{n}{sp}} + \left(\frac{s}{n} - \frac{\theta - 1}{\theta p}\right)mA(u_*, v_*) - \lambda\left(\frac{p_s^* - q}{p_s^*}\right)C(u_*, v_*).
\end{aligned}$$

Now, from (2.15), and using the fact that $\theta p < p_s^*$, we obtain

$$\begin{aligned} c &\geq \left(\frac{s}{n} - \frac{\theta - 1}{\theta p}\right) a^{\frac{-n}{sp_s^*}} (mS)^{\frac{n}{sp}} + \left(\frac{s}{n} - \frac{\theta - 1}{\theta p}\right) mA(u_*, v_*) - \lambda \gamma S^{-\frac{q}{p}} |\Omega|^{\frac{p_s^* - q}{p_s^*}} \left(\frac{p_s^* - q}{\theta p}\right) (A(u_*, v_*))^{\frac{q}{p}} \\ &= \left(\frac{s}{n} - \frac{\theta - 1}{\theta p}\right) a^{\frac{-n}{sp_s^*}} (mS)^{\frac{n}{sp}} + h(A(u_*, v_*)), \end{aligned} \tag{3.11}$$

where h is defined on $[0, \infty)$ by

$$h(\xi) = \left(\frac{s}{n} - \frac{\theta - 1}{\theta p}\right) m\xi - \lambda \gamma S^{-\frac{q}{p}} |\Omega|^{\frac{p_s^* - q}{p_s^*}} \left(\frac{p_s^* - q}{\theta p}\right) \xi^{\frac{q}{p}}.$$

A simple computation shows that h attains its minimum at

$$\xi_0 = \left(\lambda q \gamma S^{-\frac{q}{p}} |\Omega|^{\frac{p_s^* - q}{p_s^*}} \left(\frac{p^* - q}{mp}\right) \frac{1}{\frac{s}{n}\theta p - (\theta - 1)}\right)^{\frac{p}{p-q}},$$

and

$$\inf_{\xi > 0} h(\xi) = h(\xi_0) = -\lambda^{\frac{p}{p-q}} L, \tag{3.12}$$

where L is given by (3.3).

Therefore, from (3.11), (3.12), and by considering $\theta \geq 1$, we obtain

$$c \geq \left(\frac{s}{n} - \frac{\theta - 1}{\theta p}\right) a^{\frac{-n}{sp_s^*}} (mS)^{\frac{n}{sp}} - \lambda^{\frac{p}{p-q}} L \geq \left(\frac{s}{n} - \frac{\theta - 1}{\theta p}\right) a^{\frac{-n}{sp_s^*}} \left(\frac{mS}{\theta}\right)^{\frac{n}{sp}} - \lambda^{\frac{p}{p-q}} L.$$

This contradicts (3.4). Hence, $b = 0$. So, we deduce that $(u_k, v_k) \to (u_*, v_*)$ strongly in W. This completes the proof. □

Proposition 3.3. *Assume that conditions* (H_2) *and* (H_3) *hold. Then, there exist* $\lambda^* > 0$, $t_0 > 0$, *and* $(u_0, v_0) \in W$ *such that*

$$J_\lambda(t_0 u_0, t_0 v_0) \leq \left(\frac{s}{n} - \frac{\theta - 1}{\theta p}\right) a^{\frac{-n}{sp_s^*}} \left(\frac{mS}{\theta}\right)^{\frac{n}{sp}} - \lambda^{\frac{p}{p-q}} L, \tag{3.13}$$

provided that $\lambda \in (0, \lambda^*)$. *In particular,*

$$\alpha_\lambda^- < \left(\frac{s}{n} - \frac{\theta - 1}{\theta p}\right) a^{\frac{-n}{sp_s^*}} \left(\frac{mS}{\theta}\right)^{\frac{n}{sp}} - \lambda^{\frac{p}{p-q}} L. \tag{3.14}$$

Proof. We put

$$\lambda_{**} = \left(\frac{1}{L}\left(\frac{s}{n} - \frac{\theta - 1}{\theta p}\right) a^{\frac{-n}{sp_s^*}} \left(\frac{mS}{\theta}\right)^{\frac{n}{sp}}\right)^{\frac{p-q}{p}}.$$

Then, for any $0 < \lambda < \lambda_{**}$, we have

$$\left(\frac{s}{n} - \frac{\theta - 1}{\theta p}\right) a^{\frac{-n}{sp_s^*}} \left(\frac{mS}{\theta}\right)^{\frac{n}{sp}} - \lambda^{\frac{p}{p-q}} L > 0. \tag{3.15}$$

By (3.1), there exist $t_0 > 0$ and $(u_0, v_0) \in W \backslash \{0\}$ such that

$$\begin{aligned} J_\lambda(t_0 u_0, t_0 v_0) &= \frac{1}{p}(\widehat{M}_1(t_0^p A_1(u_0)) + \widehat{M}_2(t_0^p A_2(v_0))) - \frac{t_0^{p^*}}{p^*} B(u_0, v_0) - \lambda t_0^q C(u_0, v_0) \\ &= \left(\frac{s}{n} - \frac{\theta - 1}{\theta p}\right) a^{\frac{-n}{sp_s^*}} \left(\frac{mS}{\theta}\right)^{\frac{n}{sp}} - \lambda t_0^q C(u_0, v_0). \end{aligned} \tag{3.16}$$

Let

$$\lambda_{***} = \left(\frac{t_0^q C(u_0, v_0)}{L},\right)^{\frac{p-q}{q}}.$$

Then, for all $\lambda \in (0, \lambda_{***})$, we have

$$-\lambda t_0^q C(u_0, v_0) < -\lambda^{\frac{p}{p-q}} L. \tag{3.17}$$

Thus, from (3.16) and (3.17), we obtain

$$J_\lambda(t_0 u_0, t_0 v_0) < \left(\frac{s}{n} - \frac{\theta - 1}{\theta p}\right) a^{\frac{-n}{sp_s^*}} \left(\frac{mS}{\theta}\right)^{\frac{n}{sp}} - \lambda^{\frac{p}{p-q}} L.$$

Hence, (3.13) holds. Finally, if we put $\lambda^* = \min(\lambda_*, \lambda_{**}, \lambda_{***})$, then for all $0 < \lambda < \lambda^*$ and using the analysis of the fibering maps $\varphi_{u,v}(t) = J_\lambda(tu, tv)$, we obtain

$$\alpha_\lambda^- < \left(\frac{s}{n} - \frac{\theta - 1}{\theta p}\right) a^{\frac{-n}{sp_s^*}} \left(\frac{mS}{\theta}\right)^{\frac{n}{sp}} - \lambda^{\frac{p}{p-q}} L.$$

This completes the proof of Proposition 3.2. □

Now, we are in a position to prove the main result of this article.

Proof of Theorem 1.1. By Lemma 2.4, J_λ is bounded from below on $\mathcal{N}_\lambda$. Consequently, it is bounded from below on $\mathcal{N}_\lambda^+$ and $\mathcal{N}_\lambda^-$. So, we can find sequences $\{(u_k^+, v_k^+)\} \subset \mathcal{N}_\lambda^+$ and $\{(u_k^-, v_k^-)\} \subset \mathcal{N}_\lambda^-$, such that if k tends to infinity, then

$$J_\lambda(u_k^+, v_k^+) \longrightarrow \inf_{(u,v)\in\mathcal{N}_\lambda^+} J_\lambda(u, v) = \alpha_\lambda^+,$$

and

$$J_\lambda(u_k^-, v_k^-) \longrightarrow \inf_{(u,v)\in\mathcal{N}_\lambda^-} J_\lambda(u, v) = \alpha_\lambda^-.$$

By an analysis of fibering maps $\varphi_{u,v}$, we can conclude that $\alpha_\lambda^+ < 0$ and $\alpha_\lambda^- > 0$. Moreover, by Propositions 3.2 and 3.3, we have

$$J_\lambda(u_k^+, v_k^+) \longrightarrow J_\lambda(u_*^+, v_*^+) = \inf_{(u,v)\in\mathcal{N}_\lambda^+} J_\lambda(u, v) = \alpha_\lambda^+, J_\lambda'(u_k^+, \ v_k^+) \longrightarrow 0,$$

and

$$J_\lambda(u_k^-, v_k^-) \longrightarrow J_\lambda(u_*^-, v_*^-) = \inf_{(u,v)\in\mathcal{N}_\lambda^-} J_\lambda(u, v) = \alpha_\lambda^-, \ J_\lambda'(u_k^-, v_k^-) \longrightarrow 0.$$

Therefore, (u_*^+, v_*^+) (respectively, (u_*^-, v_*^-)) is a minimizer of J_λ on $\mathcal{N}_\lambda^+$ (respectively, on $\mathcal{N}_\lambda^-$). Hence, by Lemma 2.1, problem (1.3) has two solutions $(u_*^+, v_*^+) \in \mathcal{N}_\lambda^+$ and $(u_*^-, v_*^-) \in \mathcal{N}_\lambda^-$. Moreover, since $\mathcal{N}_\lambda^+ \cap \mathcal{N}_\lambda^- = \varnothing$, it follows that these two solutions are distinct. Finally, the fact that $\alpha_\lambda^+ < 0$ and $\alpha_\lambda^- > 0$ imply that (u_*^+, v_*^+) and (u_*^-, v_*^-) are nontrivial solutions for problem (1.3). This completes the proof of Theorem 1.1. □

Acknowledgments: We thank the referees for their comments and suggestions.

Funding information: D.D.R. was supported by the Slovenian Research Agency program P1-0292 and grants N1-0278, N1-0114, N1-0083, J1-4031, and J1-4001.

Conflict of interest: The authors declare that they have no conflict of interest. D.D.R., who is an Honorary Member of the Advisory Board, declares to have no involvement in the decision process.

Data availability statement: The authors declare that all data analyzed during this study are included in this published article.

References

[1] D. Abid, K. Akrout, and A. Ghanmi, *Existence results for sub-critical and critical p-fractional elliptic equations via Nehari manifold method*, J. Elliptic Parabol. Equ. **8** (2022), no. 1, 293–312.

[2] R. A. Adams and J. J. Fournier, *Sobolev Spaces*, 2nd ed., Pure and Applied Mathematics, vol. 140, Elsevier, Amsterdam, 2003.

[3] K. Akrout, M. Azouzi, and H. Yousfi, *Existence and multiplicity results for critical and sub-critical p-fractional elliptic equations via Nehari manifold method*, Novi Sad J. Math. **52** (2022), no. 2, 11–126.

[4] K. Akrout, N. E. Soltani, and S. Zediri, *Existence and multiplicity of solutions for singular fractional elliptic system via the Nehari manifold approach*, Novi Sad J. Math. **51** (2021), no. 1, 163–183.

[5] G. Alberti, G. Bouchitté, and P. Seppecher, *Phase transition with the line-tension effect*, Arch. Ration. Mech. Anal. **144** (1998), no. 1, 1–46.

[6] G. Autuori and P. Pucci, *Elliptic problems involving the fractional Laplacian in* $\mathbb{R}^n$, J. Differ. Equ. **255** (2013), no. 8, 2340–2362.

[7] E. Azroul, A. Benkirane, A. Boumazourh, and M. Srati, *Multiple solutions for a nonlocal fractional* (p,q)*-Schrödinger-Kirchhoff-type system*, Nonlinear Stud. **27** (2020), no. 4, 915–933.

[8] E. Azroul, A. Benkirane, A. Boumazourh, and M. Srati, *Three solutions for a nonlocal fractional p-Kirchhoff-type elliptic system*, Appl. Anal. **100** (2021), no. 9, 1871–1888.

[9] P. W. Bates, *On some nonlocal evolution equations arising in materials science*, Nonlinear Dyn. Evol. Equ. **48** (2006), 13–52.

[10] P. Biler, G. Karch, and W. A. Woyczyński, *Critical nonlinearity exponent and self-similar asymptotics for Lévy conservation laws*, In: Annales de laInstitut Henri Poincaré C, Analyse non linéaire, Elsevier, Masson 2001, 613–637.

[11] K. J. Brown and T. F. Wu, *A fibering map approach to a semilinear elliptic boundary value problem*, Electronic J. Differ. Equ. **2007** (2007), no. 69, 1–9.

[12] K. Brown and Y. Zhang, *The Nehari manifold for a semilinear elliptic equation with a sign-changing weight function*, J. Differ. Equ. **193** (2003), no. 2, 481–499.

[13] L. Caffarelli and L. Silvestre, *An extension problem related to the fractional Laplacian*, Commun. Partial Differ. Equ. **32** (2007), no. 8, 1245–1260.

[14] F. Charro, E. Colorado, and I. Peral, *Multiplicity of solutions to uniformly elliptic fully nonlinear equations with concave-convex hand side*, J. Differ. Equ. **246** (2009), no. 9, 4221–4248.

[15] W. Chen, *Existence of solutions for fractional p-Kirchhoff-type equations with a generalized Choquard nonlinearity*, Topol. Methods Nonlinear Anal. **54** (2019), no. 2, 773–791.

[16] W. Craig and D. P. Nichols, *Traveling two and three-dimensional capillary gravity water waves*, SIAM J. Math. Anal. **32** (2000), no. 2, 323–359.

[17] W. Craig, U. Schanz, and C. Sulem, *The modulational regime of three-dimensional water waves and the Davey-Stewartson system*, Ann. Inst. Henri Poincare (C) Anal. Non Lineaire, vol. 14, Elsevier, Masson, 1997, pp. 615–667.

[18] E. DiNezza, G. Palatucci, and E. Valdinoci, *Hitchhiker's guide to the fractional sobolev spaces*, Bull. Sci. Math. **136** (2012), no. 5, 521–573.

[19] P. Drábek and S. I. Pohozaev, *Positive solutions for the p-Laplacian: Application of the fibrering method*, Proc. Roy. Soc. Edinb. A Math. **127** (1997), no. 4, 703–726.

[20] G. M. Figueiredo, *Existence of a positive solution for a Kirchhoff problem type with critical growth via truncation argument*, J. Math. Anal. Appl. **401** (2013), no. 2, 706–713.

[21] A. Fiscella, P. Pucci, and B. Zhang, *p-fractional Hardy-Schödinger-Kirchhoff systems with critical nonlinearities*, Adv. Nonlinear Anal. **8** (2019), no. 1, 1111–1131.

[22] A. Fiscella and E. Valdinoci, *A critical Kirchhoff-type problem involving a nonlocal operator*, Nonlinear Anal. Theory Methods Appl. **94** (2014), 156–170.

[23] A. Ghanmi, *Multiplicity of nontrivial solutions of a class of fractional p-Laplacian problem*, Z. fur Anal. ihre Anwend. **34** (2015), no. 3, 309–319.

[24] A. Ghanmi and K. Saoudi, *The Nehari manifold for a singular elliptic equation involving the fractional Laplace operator*, Fract. Differ. Calc. **6** (2016), no. 2, 201–217.

[25] X. He and W. Zou, *Ground states for nonlinear Kirchhoff equations with critical growth*, Ann. Mat. Pura Appl. **193** (2014), no. 2, 473–500.

[26] Y. He, G. Li, and S. Peng, *Concentrating bound states for Kirchhoff-type problems in $\mathbb{R}^3$ involving critical Sobolev exponents*, Adv. Nonlinear Stud. **14** (2014), no. 2, 483–510.
[27] S. Liang, D. D. Repovš, and B. Zhang, *On the fractional Schrödinger-Kirchhoff equations with electromagnetic fields and critical nonlinearity*, Comput. Math. Appl. **75** (2018), no. 5, 1778–1794.
[28] S. Liang and S. Shi, *Soliton solutions to Kirchhoff-type problems involving the critical growth in $\mathbb{R}^n$*, Nonlinear Anal. Theory Methods Appl. **81** (2013), 31–41.
[29] J. Liu, J. F. Liao, and C. L. Tang, *Positive solutions for Kirchhoff-type equations with critical exponent in $\mathbb{R}^n$*, J. Math. Anal. Appl. **429** (2015), no. 2, 1153–1172.
[30] X. Mingqi, V. D. Rădulescu, and B. Zhang, *Combined effects for fractional Schrödinger-Kirchhoff systems with critical nonlinearities*, ESAIM- Control Optim. Calc. Var. **24** (2018), no. 3, 1249–1273.
[31] G. Molica Bisci, V. D. Rădulescu, and R. Servadei, *Variational Methods for Nonlocal Fractional Problems*, Vol. 162, Cambridge University Press, 2016, pp. 1–400, ISBN: 9781107111943, Foreword by J. Mawhin.
[32] G. Molica Bisci and D. D. Repovš, *On doubly nonlocal fractional elliptic equations*, Rend. Lincet-Mat. Appl. **26** (2015), no. 2, 161–176.
[33] A. Ourraoui, *On a p-Kirchhoff problem involving a critical nonlinearity*, C. R. Math. **352** (2014), no. 4, 295–298.
[34] N. S. Papageorgiou, V. D. Rădulescu, and D. D. Repovš, *Nonlinear Analysis - Theory and Methods*. Springer Monographs in Mathematics, Springer, Cham, 2019.
[35] P. Pucci, M. Xiang, and B. Zhang, *Multiple solutions for nonhomogeneous Schrödinger-Kirchhoff-type equations involving the fractional p-Laplacian in V*, Calc. Var. Partial Differ. Equ. **54** (2015), no. 3, 2785–2806.
[36] K. Saoudi, A. Ghanmi, and S. Horrigue, *Multiplicity of solutions for elliptic equations involving fractional operator and sign-changing nonlinearity*, J. Pseudo-Differ. Oper. Appl. **11** (2020), no. 4, 1743–1756.
[37] R. Servadei and E. Valdinoci, *The Brezis-Nirenberg result for the fractional Laplacian*, Trans. Amer. Math. Soc. **367** (2015), no. 1, 67–102.
[38] R. Servadei and E. Valdinoci, *Fractional Laplacian equations with critical Sobolev exponent*, Rev. Mat. Complut. **28** (2015), no. 3, 655–676.
[39] Y. Sire and E. Valdinoci, *Fractional Laplacian phase transitions and boundary reactions: a geometric inequality and a symmetry result*, J. Funct. Anal. **256** (2009), no. 6, 1842–1864.
[40] F. Wang and M. Xiang, *Multiplicity of solutions to a nonlocal Choquard equation involving fractional magnetic operators and critical exponent*, Electron. J. Differ. Equ. **2016** (2016), no. 306, 1–11.
[41] M. Willem, *Minimax Theorems*, vol. 24, Springer Science and Business Media, Berlin, Germany, 1997.
[42] M. Xiang and B. Zhang, *A critical fractional p-Kirchhoff-type problem involving discontinuous nonlinearity*, Discrete Contin. Dyn. Syst. S, **12** (2019), no. 2, 413–433.
[43] M. Xiang, B. Zhang, and V. R. Rădulescu, *Multiplicity of solutions for a class of quasilinear Kirchhoff system involving the fractional p-Laplacian*, Nonlinearity **29** (2016), no. 10, 3186–3205.
[44] M. Xiang, B. Zhang, and V. R. Rădulescu, *Superlinear Schrödinger-Kirchhoff-type problems involving the fractional p-Laplacian and critical exponent*, Adv. Nonlinear Anal. **9** (2020), no. 1, 690–709.
[45] M. Xiang, B. Zhang, and X. Zhang, *A nonhomogeneous fractional p-Kirchhoff-type problem involving critical exponent in $\mathbb{R}^n$*, Adv. Nonlinear Stud. **17** (2017), no. 3, 611–640.
[46] L. Yang and T. An, *Infinitely many solutions for fractional p-Kirchhoff equations*, Mediterr. J. Math. **15** (2018), 1–13.
[47] H. Yin, *Existence results for classes of quasilinear elliptic systems with sign-changing weight*, Int. J. Nonlinear Sci. **10** (2010), no. 1, 53–60.