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An elliptic problem of the Prandtl–Batchelor type with a singularity

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Abstract

We establish the existence of at least two solutions of the *Prandtl–Batchelor* like elliptic problem driven by a power nonlinearity and a singular term. The associated energy functional is nondifferentiable, and hence the usual variational techniques do not work. We shall use a novel approach in tackling the associated energy functional by a sequence of C^1 functionals and a *cutoff function*. Our main tools are fundamental elliptic regularity theory and the mountain pass theorem.

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Keywords: Elliptic free boundary problems; Mountain pass theorem; Singularity

1 Introduction

We consider the following class of sublinear elliptic *free boundary* problems:

$$\begin{cases} -\Delta u = \alpha\chi_{\{u>1\}}(x)f(x,(u-1)_+) + \beta u^{-\gamma} & \text{in } \Omega \setminus G(u), \\ |\nabla u^+|^2 - |\nabla u^-|^2 = 2 & \text{on } G(u), \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Here, $\Omega \subset \mathbb{R}^N$ is a bounded domain, $N \geq 2$, $0 < \gamma < 1$, the boundary $\partial\Omega$ has $C^{2,a}$ regularity, $G(u) = \partial\{u : u > 1\}$, $\alpha, \beta > 0$ are parameters, and χ is an indicator function. Furthermore, $\nabla u^{\pm}$ are the limits of ∇u from the sets $\{u : u > 1\}$ and $\{u : u \leq 1\}^\circ$ respectively, and $(u-1)_+ = \max\{u-1, 0\}$. The nonlinear term f is a locally Hölder continuous function $f : \Omega \times \mathbb{R} \to [0, \infty)$ that satisfies the following conditions for all $x \in \Omega$, $t > 0$:

(f_1) For some $c_0, c_1 > 0$, $|f(x,t)| \leq c_0 + c_1 t^{p-1}$, where $1 < p < 2$.

(f_2) $f(x,t) > 0$. (1.2)

We shall prove the existence of two distinct nontrivial solutions of (1.1) for a sufficiently large α.

The case when $f(x,t) = 1$, $\beta = 0$ is the well-known *Prandtl–Batchelor* problem, where the region $\{u : u > 1\}$ represents the vortex patch bounded by the vortex line $\{u : u = 1\}$

in a steady state fluid flow for $N = 2$ (cf. Batchelor [4, 5]). This case has been studied by several authors, e.g., Caflisch [8], Elcrat and Miller [10], Acker [1], and Jerison and Perera [14]. We drew our motivation for studying the present problem in this paper from perera [18]. The problem studied by Perera [18] is the case when $\beta = 0$ in problem (1.1).

The nonlinearity f includes the sublinear case of $f(x,t) = t^{p-1}$. Jerison and Perera [14] considered problem (1.1) with $\beta = 0$ for $2 < p < \infty$ if $N = 2$, and $2 < p \leq 2^* = \frac{2N}{N-2}$ if $N \geq 3$. This problem has its application in the study of plasma that is confined in a magnetic field. The region there $\{u : u > 1\}$ represents the plasma, and the boundary of the plasma is modeled by the free boundary (cf. Caffarelli and Friedman [6], Friedman and Liu [11], and Temam [19]).

Elliptic problems driven by a singular term have, of late, been of great interest. However, we shall discuss only the seminal work of Lazer and McKenna [16] from 1991 that opened a new door for the researchers in elliptic and parabolic PDEs. The problem considered in [16] was as follows:

$$\begin{cases} -\Delta u = p(x)u^{-\gamma} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where $p > 0$ is a $C^a(\bar{\Omega})$ function, $\gamma > 0$, Ω is a bounded domain with a smooth boundary $\partial\Omega$ of C^{2+a} regularity ($0 < a < 1$), and $N \geq 1$. The authors in [16] proved that problem (1.3) has a unique solution $u \in C^{2,a}(\Omega) \cap C(\bar{\Omega})$ such that $u > 0$ in Ω. Another noteworthy work addressing the singularity driven elliptic problem is due to Giacomoni et al. [12]. Jerison and Perera [14] obtained a mountain pass solution of this problem for the superlinear subcritical case. Yang and Perera [20] addressed the problem for the critical case. Recently, Choudhuri and Repovš [9] established the existence of a solution for a semilinear elliptic PDE with a free boundary condition on a stratified Lie group. Furthermore, those readers looking to expand their knowledge on the techniques and trends of the topics in analysis of elliptic PDEs may refer to Papageorgiou et al. [17].

We shall prove that a solution of problem (1.1) is Lipschitz continuous of class $H_0^1(\Omega) \cap C^2(\bar{\Omega} \setminus G(u))$ and is a classical solution on $\Omega \setminus G(u)$. This solution vanishes on $\partial\Omega$ continuously and satisfies the *free boundary* condition in the following sense:

$$\lim_{\epsilon^+ \to 0} \int_{\{u=1+\epsilon^+\}} \left(2 - |\nabla u|^2\right) \psi \cdot \hat{n}\, dS - \lim_{\epsilon^+ \to 0} \int_{\{u=1-\epsilon^+\}} |\nabla u|^2 \psi \cdot \hat{n}\, dS = 0 \tag{1.4}$$

for all$\psi \in C_0^1(\Omega, \mathbb{R}^N)$ that are supported a.e. on $\{u : u \neq 1\}$. Here $\hat{n}$ is the outward drawn normal to $\{u : 1 - \epsilon^- < u < 1 + \epsilon^+\}$ and dS is the surface element.

The novelty of this work, which separates it from the work of Perera [18], lies in the efficient handling of the singular term that disallows the associated energy functional to be C^1 at $u = 0$. This difficulty is the reason why one cannot directly apply the results from the variational set up. To handle this situation, we shall define a *cut-off* function.

Remark 1.1 *Note that* $\int_\Omega |\nabla u|^2\, dx$ *will be often denoted by* $\|u\|^2$, *where* $\|\cdot\|$ *is the norm of an element in the Sobolev space* $H_0^1(\Omega)$.

We begin by defining a *weak solution* of problem (1.1).

Definition 1.1 A function $u \in H_0^1(\Omega)$, $u > 0$ a.e. in Ω is said to be a weak solution of problem (1.1) if it satisfies the following:

$$\begin{aligned} 0 = & \int_\Omega \nabla u \cdot \nabla \varphi \, dx - \alpha \int_\Omega g\big(x, (u-1)_+\big)\varphi \, dx \\ & - \beta \int_\Omega u^{-\gamma} \varphi \, dx \quad \text{for all } \varphi \in H_0^1(\Omega). \end{aligned} \tag{1.5}$$

We define the associated energy functional to problem (1.1) as follows:

$$\begin{aligned} E(u) = & \frac{1}{2}\|u\|^2 + \int_\Omega \big(\chi_{\{u>1\}}(x) - \alpha G\big(x, (u-1)_+\big)\big)\, dx \\ & - \frac{\beta}{1-\gamma} \int_\Omega \big(u^+\big)^{1-\gamma} \, dx \quad \text{for all } u \in H_0^1(\Omega), \end{aligned} \tag{1.6}$$

where $F(x,t) = \int_0^t f(x,t)\, dt$, $t \geq 0$.

The functional E fails to be of C^1 class due to the term $\int_\Omega (u^+)^{1-\gamma} \, dx$. Moreover, it is nondifferentiable due to the term $\int_\Omega \chi_{\{u>1\}}(x)\, dx$. We shall first tackle the singular term by defining a cut-off function ϕ_β as follows:

$$\phi_\beta(u) = \begin{cases} u^{-\gamma} & \text{if } u > u_\beta, \\ u_\beta^{-\gamma} & \text{if } u \leq u_\beta. \end{cases}$$

Here u_β is a solution of the following problem:

$$\begin{aligned} & -\Delta u = \beta u^{-\gamma} \quad \text{in } \Omega, \\ & u > 0 \quad \text{in } \Omega, \\ & u = 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{1.7}$$

The existence of u_β can be guaranteed by Lazer and McKenna [16]. Moreover, a solution of problem (1.7) is a *subsolution* of (1.1) (refer to Lemma 6.1 in Sect. 6). Note that we call (1.7) a *singular problem*. We denote $\Phi_\beta(u) = \int_0^u \phi_\beta(t)\, dt$.

Furthermore, the functional E is nondifferentiable, and hence we approximate it by C^1 functionals. This technique is adopted from the work of Jerison and Pererra [14]. Working along similar lines, we now define a smooth function $h : \mathbb{R} \to [0,2]$ as follows:

$$h(t) = \begin{cases} 0 & \text{if } t \leq 0, \\ \text{a positive function} & \text{if } 0 < t < 1, \\ 0 & \text{if } t \geq 1, \end{cases}$$

and $\int_0^1 h(t)\, dt = 1$. We let $H(t) = \int_0^t h(t)\, dt$. Clearly, H is a smooth and nondecreasing function such that

$$H(t) = \begin{cases} 0 & \text{if } t \leq 0, \\ \text{a positive function} < 1 & \text{if } 0 < t < 1, \\ 1 & \text{if } t \geq 1. \end{cases}$$

We further define for $\delta > 0$

$$f_\delta(x,t) = H\left(\frac{t}{\delta}\right) f(x,t), \qquad F_\delta(x,t) = \int_0^t f_\delta(x,t)\,dt \quad \text{for all } t \geq 0. \tag{1.8}$$

Define

$$E_\delta(u) = \frac{1}{2}\|u\|^2 + \int_\Omega \left[H\left(\frac{u-1}{\delta}\right) - \alpha F_\delta\big(x,(u-1)_+\big) - \beta \Phi_\beta(u) \right] dx \quad \text{for all } u \in H_0^1(\Omega). \tag{1.9}$$

The functional E_δ is of C^1 class. The main result of this paper is the following theorem.

Theorem 1.1 *Let conditions* $(f_1) - (f_2)$ *hold. Then there exist* $\Lambda, \beta_* > 0$ *such that for all* $\alpha > \Lambda, 0 < \beta < \beta_*$ *problem* (1.1) *has two Lipschitz continuous solutions, say* $u_1, u_2 \in H_0^1(\Omega) \cap C^2(\bar{\Omega} \setminus G(u))$, *satisfying* (1.1) *classically in* $\bar{\Omega} \setminus G(u)$. *These solutions also satisfy the free boundary condition in the generalized sense and vanish continuously on* $\partial\Omega$. *Furthermore,*

1. $E(u_1) < -|\Omega| \leq -|\{u : u = 1\}| < E(u_2)$, *where* $|\cdot|$ *denotes the Lebesgue measure in* $\mathbb{R}^N$, *hence* u_1, u_2 *are nontrivial solutions.*
2. $0 < u_2 \leq u_1$ *and the regions* $\{u_1 : u_1 < 1\} \subset \{u_2 : u_2 < 1\}$ *are connected where* $\partial\Omega$ *is connected. The sets* $\{u_2 > 1\} \subset \{u_1 > 1\}$ *are nonempty.*
3. u_1 *is a minimizer of* E (*but* u_2 *is not*).

The paper is organized as follows. In Sect. 2 we introduce the key preliminary facts. In Sect. 3 we prove a convergence lemma. In Sect. 4 we prove a free boundary condition. In Sect. 5 we prove two auxiliary lemmas. In Sect. 6 we prove a result on positive Radon measure. Finally, in Sect. 7 we prove the main theorem.

2 Preliminaries

An important result that will be used to pass to the limit in the proof of Lemma 3.1 is the following theorem due to Caffarelli et al. [7, Theorem 5.1].

Lemma 2.1 *Let* u *be a Lipschitz continuous function on the unit ball* $B_1(0) \subset \mathbb{R}^N$ *satisfying the distributional inequalities*

$$\pm \Delta u \leq A\left(\frac{1}{\delta}\chi_{\{|u-1|<\delta\}}(x) H\big(|\nabla u|\big) + 1\right)$$

for constants $A > 0$, $0 < \delta \leq 1$, H *is a continuous function obeying* $H(t) = o(t^2)$ *as* $t \to \infty$. *Then there exists a constant* $C > 0$ *depending on* N, A *and* $\int_{B_1(0)} u^2\,dx$, *but not on* δ, *such that*

$$\sup_{x \in B_{\frac{1}{2}}(0)} \big|\nabla u(x)\big| \leq C.$$

The following are the *Palais–Smale* condition and the mountain pass theorem.

Definition 2.1 (cf. Kesavan [15, Definition 5.5.1]) Let V be a Banach space and $J : V \to \mathbb{R}$ be a C^1-functional. Then J is said to satisfy the Palais–Smale (PS) condition if the following holds: Whenever (u_n) is a sequence in V such that $(J(u_n))$ is bounded and $(J'(u_n)) \to 0$ strongly in V^* (the dual space), then (u_n) has a strongly convergent subsequence in V.

Lemma 2.2 (cf. Alt and Caffarelli [3, Theorem 2.1]) *Let J be a C^1-functional defined on a Banach space V. Assume that J satisfies the (PS)-condition and that there exists an open set $U \subset V$, $v_0 \in U$, and $v_1 \in X \setminus \bar{U}$ such that*

$$\inf_{v\in\partial U} J(v) > \max\{J(v_0), J(v_1)\}.$$

Then J has a critical point at the level

$$c = \inf_{\psi\in\Gamma} \max_{u\in\psi([0,1])} J(v) \geq \inf_{u\in\partial U} J(u),$$

where $\Gamma = \{\psi \in C([0,1]) : \psi(0) = v_0, \psi(1) = v_1\}$ is the class of paths in V joining v_0 and v_1.

Before we prove Lemma 3.1, we would like to give an a priori estimate of the parameter β.

3 Convergence lemma

We denote the first eigenvalue of $(-\Delta)$ by α_1 and the first eigenvector by φ_1 (for an existence of α_1, φ_1, refer to Kesavan [15]). Fix α to, say, α_0 and let β be any positive real number. On testing problem (1.1) with φ_1, the following weak formulation has to hold if u is a weak solution of problem (1.1). Thus

$$\alpha_1 \int_\Omega u\varphi_1\,dx = \int_\Omega \nabla u \cdot \nabla\varphi\,dx = \alpha \int_\Omega f\big(x,(u-1)_+\big)\varphi_1\,dx + \beta \int_\Omega \big(u^+\big)^{-\gamma}\varphi\,dx. \tag{3.1}$$

So there exists $\beta_* > 0$, which depends on the chosen fixed α, such that $\beta_* t^{-\gamma} + \alpha f(x,(t-1)_+) > \alpha_1 t$ for all $t > 0$. This is a contradiction to (3.1). Therefore, $0 < \beta < \beta_*$.

Lemma 3.1 *Let conditions $(f_1) - (f_2)$ hold, $\delta_j \to 0$ $(\delta_j > 0)$ as $j \to \infty$, and let u_j be a critical point of E_{δ_j}. If (u_j) is bounded in $H_0^1(\Omega) \cap L^\infty(\Omega)$, then there exists a Lipschitz continuous function u on $\bar{\Omega}$ such that $u \in H_0^1(\Omega) \cap C^2(\bar{\Omega} \setminus G(u))$ and a subsequence such that*

(i) *$u_j \to u$ uniformly over $\bar{\Omega}$,*
(ii) *$u_j \to u$ locally in $C^1(\bar{\Omega} \setminus \{u = 1\})$,*
(iii) *$u_j \to u$ strongly in $H_0^1(\Omega)$,*
(iv) *$E(u) \leq \liminf E_{\delta_j}(u_j) \leq \limsup E_{\delta_j}(u_j) \leq E(u) + |\{u : u = 1\}|$, i.e., u is a nontrivial function if $\liminf E_{\delta_j}(u_j) < 0$ or $\limsup E_{\delta_j}(u_j) > 0$.*

Furthermore, u satisfies

$$-\Delta u = \alpha\chi_{\{u>1\}}(x) g\big(x,(u-1)_+\big) + \beta u^{-\gamma}$$

classically in $\Omega \setminus G(u)$, the free boundary condition is satisfied in the generalized sense and u vanishes continuously on $\partial\Omega$. If u is nontrivial, then $u > 0$ in Ω, the region $\{u : u < 1\}$ is connected, and the region $\{u : u > 1\}$ is nonempty.

Proof of Lemma 3.1 Let $0 < \delta_j < 1$. Consider the following problem:

$$\begin{cases} -\Delta u_j = -\frac{1}{\delta_j}h(\frac{u_j-1}{\delta_j}) + \alpha f_{\delta_j}(x,(u_j-1)_+) + \beta\phi_\beta(u_j) & \text{in } \Omega, \\ u_j > 0 & \text{in } \Omega, \\ u_j = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.2}$$

The nature of the problem being a sublinear one and driven by a singularity allows us to conclude by an iterative technique that the sequence (u_j) is bounded in $L^\infty(\Omega)$. Therefore, there exists C_0 such that $0 \leq f_{\delta_j}(x,(u_j-1)_+) \leq C_0$. Let φ_0 be a solution of the following problem:

$$\begin{cases} -\Delta\varphi_0 = \alpha C_0 + \beta u_\beta^{-\gamma} & \text{in } \Omega, \\ \varphi_0 = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.3}$$

Now, since $h \geq 0$, we have that $-\Delta u_j \leq \alpha C_0 + \beta u_\beta^{-\gamma} = -\Delta\varphi_0$ in Ω. Therefore by the maximum principle,

$$0 \leq u_j(x) \leq \varphi_0(x) \quad \text{for all } x \in \Omega. \tag{3.4}$$

From the argument used in the proof of Lemma 6.1, together with $\beta_* > 0$ and large $\Lambda > 0$, we conclude that $u_j > u_\beta$ in Ω for all $\beta \in (0,\beta_*)$. Since $\{u_j : u_j \geq 1\} \subset \{\varphi_0 : \varphi_0 \geq 1\}$, hence φ_0 gives a uniform lower bound, say d_0, on the distance from the set $\{u_j : u_j \geq 1\}$ to $\partial\Omega$. Furthermore, u_j is a positive function satisfying the singular problem in a d_0-neighborhood of $\partial\Omega$. Thus (u_j) is bounded with respect to the $C^{2,a}$ norm. Therefore, it has a convergent subsequence in the C^2-norm in a $\frac{d_0}{2}$ neighborhood of the boundary $\partial\Omega$. Obviously, $0 \leq h \leq 2\chi_{(-1,1)}$ and hence

$$\begin{aligned} \pm\Delta u_j &= \pm\frac{1}{\delta_j}h\left(\frac{u_j-1}{\delta_j}\right) \mp \alpha f_{\delta_j}\big(x,(u_j-1)_+\big) + \beta u_j^{-\gamma} \\ &\leq \frac{2}{\delta_j}\chi_{\{|u_j-1|<\delta_j\}}(x) + \alpha C_0 + \beta u_j^{-\gamma} \\ &\leq \frac{2}{\delta_j}\chi_{\{|u_j-1|<\delta_j\}}(x) + \alpha C_0 + \beta u_\beta^{-\gamma}. \end{aligned} \tag{3.5}$$

By Lazer and McKenna [16], for any subset K of Ω that is relatively compact in it, i.e., $K \Subset \Omega$, we have that $u_\beta \geq C_K$ for some $C_K > 0$. Therefore

$$\pm\Delta u_j \leq \frac{2}{\delta_j}\chi_{\{|u_j-1|<\delta_j\}}(x) + \alpha C_0 + \beta C_K^{-\gamma}. \tag{3.6}$$

Since (u_j) is bounded in $L^2(\Omega)$ and by Lemma 2.1, it follows that there exists $A > 0$ such that

$$\sup_{x\in B_{\frac{r}{2}}(x_0)} \left|\nabla u_j(x)\right| \leq \frac{A}{r} \tag{3.7}$$

for suitable $r > 0$ such that $B_r(0) \subset \Omega$. Therefore, (u_j) is uniformly Lipschitz continuous on the compact subsets of Ω such that its distance from the boundary $\partial\Omega$ is at least $\frac{d_0}{2}$ units.

Thus, by the *Ascoli–Arzela* theorem applied to (u_j), we have a subsequence, still denoted the same, such that it converges uniformly to a Lipschitz continuous function u in Ω with zero boundary values and with strong convergence in C^2 on a $\frac{d_0}{2}$-neighborhood of $\partial\Omega$. By the *Eberlein–Šmulian* theorem, we can conclude that $u_j \rightharpoonup u$ in $H_0^1(\Omega)$.

We now prove that u satisfies the following equation:

$$-\Delta u = \alpha\chi_{\{u>1\}}(x)f\big(x,(u-1)_+\big) + \beta u^{-\gamma}$$

in the set $\{u \neq 1\}$. This will include the cases (i) $0 < u_\beta < 1 < u$, (ii) $1 < u_\beta < u$, (iii) $0 < u_\beta < u < 1$. The cases (i)–(iii) do not pose any real mathematical obstacle. Let $\varphi \in C_0^\infty(\{u > 1\})$. Then $u \geq 1 + 2\delta$ on the support of φ for some $\delta > 0$. Using the convergence of u_j to u uniformly on Ω, we have $|u_j - u| < \delta$ for any sufficiently large $j, \delta_j < \delta$. So $u_j \geq 1 + \delta_j$ on the support of φ. Testing (3.1) with φ yields

$$\int_\Omega \nabla u_j \cdot \nabla\varphi\, dx = \alpha\int_\Omega f(x, u_j - 1)\varphi\, dx + \beta\int_\Omega u_j^{-\gamma}\varphi\, dx. \tag{3.8}$$

On passing to the limit $j \to \infty$, we get

$$\int_\Omega \nabla u \cdot \nabla\varphi\, dx = \alpha\int_\Omega f(x, u - 1)\varphi\, dx + \beta\int_\Omega u^{-\gamma}\varphi\, dx. \tag{3.9}$$

To arrive at (3.9), we have used the weak convergence of u_j to u in $H_0^1(\Omega)$ and the uniform convergence of the same in Ω. Hence u is a weak solution of $-\Delta u = \alpha f(x, u-1) + \beta u^{-\gamma}$ in $\{u > 1\}$. Since f, u are continuous and Lipschitz continuous respectively, we conclude by the Schauder estimates that it is also a classical solution of $-\Delta u = \alpha f(x, u-1) + \beta u^{-\gamma}$ in $\{u : u > 1\}$. Similarly, on choosing $\varphi \in C_0^\infty(\{u : u < 1\})$, one can find a $\delta > 0$ such that $u \leq 1 - 2\delta$. Therefore, $u_j < 1 - \delta$. Using the arguments as in (3.8) and (3.9), we find that u satisfies $-\Delta u = \beta u^{-\gamma}$ in the set $\{u : u < 1\}$.

Let us now see what is the nature of u in the set $\{u : u \leq 1\}^\circ$. On testing (3.1) with any nonnegative function, passing to the the limit $j \to \infty$, and using the fact that $h \geq 0, H \leq 1$, we can show that u satisfies

$$-\Delta u \leq \alpha f\big(x,(u-1)_+\big) + \beta u^{-\gamma} \quad \text{in } \Omega \tag{3.10}$$

in the distributional sense. Also, we see that u satisfies $-\Delta u = \beta u^{-\gamma}$ in the set $\{u : u < 1\}$. Furthermore, $\mu = \Delta u + \beta u^{-\gamma}$ is a positive Radon measure supported on $\Omega \cap \partial\{u : u < 1\}$ (refer to Lemma 6.2 in Sect. 6). From (3.10), the positivity of the Radon measure μ and the usage of Section 9.4 in Gilbarg and Trudinger [13], we conclude that $u \in W_{\mathrm{loc}}^{2,p}(\{u : u \leq 1\}^\circ)$, $1 < p < \infty$. Thus μ is supported on $\Omega \cap \partial\{u : u < 1\} \cap \partial\{u : u > 1\}$ and u satisfies $-\Delta u = \beta u^{-\gamma}$ in the set $\{u : u \leq 1\}^\circ$.

To prove (ii), we show that $u_j \to u$ locally in $C^1(\Omega \setminus \{u : u = 1\})$. Note that we have already proved that $u_j \to u$ in the C^2 norm in a neighborhood of $\partial\Omega$ of $\bar{\Omega}$. Suppose that $M \subset\subset \{u : u > 1\}$. In this set M we have $u \geq 1 + 2\delta$ for some $\delta > 0$. Thus, for sufficiently large j with $\delta_j < \delta$, we have $|u_j - u| < \delta$ in Ω, and hence $u_j \geq 1 + \delta_j$ in M. From (3.2) we derive that

$$-\Delta u = \alpha f(x, u-1) + \beta u^{-\gamma} \quad \text{in } M.$$

Clearly, $f(x,u_j-1)\to f(x,u-1)$ in $L^p(\Omega)$ for $1<p<\infty$ because f is a locally Hölder continuous function and $u_j\to u$ uniformly in Ω. Our analysis says something stronger. Since $-\Delta u=\alpha f(x,u-1)$ in M, we have that $u_j\to u$ in $W^{2,p}(M)$. By the embedding $W^{2,p}(M)\hookrightarrow C^1(M)$ for $p>2$, we have that $u_j\to u$ in $C^1(M)$. This shows that $u_j\to u$ in $C^1(\{u>1\})$. Working along similar lines we can also show that $u_j\to u$ in $C^1(\{u:u<1\})$.

We shall now prove (iii). Since $u_j\rightharpoonup u$ in $H_0^1(\Omega)$, we know that by the weak lower semi-continuity of the norm $\|\cdot\|$,

$$\|u\|\leq\liminf\|u_j\|.$$

It suffices to prove that $\limsup\|u_j\|\leq\|u\|$. To achieve this, we multiply (3.2) with u_j-1 and then integrate by parts. We shall also use the fact that $th(\frac{t}{\delta_j})\geq 0$ for any t. This gives

$$\begin{aligned}\int_\Omega|\nabla u_j|^2\,dx\leq{}&\alpha\int_\Omega f\big(x,(u_j-1)_+\big)(u_j-1)_+\,dx\\&-\int_{\partial\Omega}\frac{\partial u_j}{\partial\hat{n}}\,dS+\beta\int_\Omega u_j^{-\gamma}(u_j-1)_+\,dx\\\to{}&\alpha\int_\Omega f\big(x,(u-1)_+\big)(u-1)_+\,dx\\&-\int_{\partial\Omega}\frac{\partial u}{\partial\hat{n}}\,dS+\beta\int_\Omega u^{-\gamma}(u-1)_+\,dx\end{aligned}\tag{3.11}$$

as $j\to\infty$. Here, $\hat{n}$ is the outward drawn normal to $\partial\Omega$. We saw earlier that u is a weak solution to $-\Delta u=\alpha f(x,u-1)+\beta u^{-\gamma}$ in $\{u:u>1\}$. Let $0<\delta<1$. We test this equation with the function $\varphi=(u-1-\delta)_+$ and get

$$\int_{\{u>1+\delta\}}|\nabla u|^2\,dx=\alpha\int_\Omega f\big(x,(u-1)_+\big)(u-1-\delta)\,dx+\beta\int_\Omega u^{-\gamma}(u-1-\delta)_+\,dx.\tag{3.12}$$

Integrating $(u-1-\delta)_-\Delta u=\beta u^{-\gamma}(u-1-\delta)_-$ over Ω yields

$$\int_{u<1-\delta}|\nabla u|^2\,dx=-(1-\delta)\int_{\partial\Omega}\frac{\partial u}{\partial\hat{n}}\,dS+\beta\int_\Omega u^{-\gamma}(u-1-\delta)_-\,dx.\tag{3.13}$$

On adding (3.12) and (3.13) and passing to the limit $\delta\to 0$, we get

$$\begin{aligned}\int_\Omega|\nabla u|^2\,dx={}&\alpha\int_\Omega f\big(x,(u-1)_+\big)(u-1)_+\,dx\\&-\int_{\partial\Omega}\frac{\partial u}{\partial\hat{n}}\,dS+\beta\int_\Omega u^{-\gamma}(u-1)_+\,dx.\end{aligned}\tag{3.14}$$

Note that we have used $\int_{\{u:u=1\}}|\nabla u|^2\,dx=0$. Invoking (3.14) and (3.11), we get

$$\limsup\int_\Omega|\nabla u_j|^2\,dx\leq\int_\Omega|\nabla u|^2\,dx.\tag{3.15}$$

This proves (iii).

We shall now prove (iv). Consider

$$E_{\delta_j}(u_j) = \int_\Omega \left(\frac{1}{2}|\nabla u_j|^2 + H\left(\frac{u_j-1}{\delta_j}\right)\chi_{\{u\neq 1\}} - \alpha F_{\delta_j}\big(x,(u_j-1)_+\big) - \beta u_j^{-\gamma}(u_j-1)_+\right) dx + \int_{\{u=1\}} H\left(\frac{u_j-1}{\delta_j}\right) dx. \tag{3.16}$$

Since $u_j \to u$ in $H_0^1(\Omega)$ and $H(\frac{u_j-1}{\delta_j})\chi_{\{u\neq 1\}}$, $F_{\delta_j}(x,(u_j-1)_+)$ are bounded and converge pointwise to $\chi_{\{u:u>1\}}$ and $F(x,(u-1)_+)$, respectively, it follows that the first integral in (3.16) converges to $E(u)$. Moreover,

$$0 \le \int_{\{u:u=1\}} H\left(\frac{u_j-1}{\delta_j}\right) dx \le \big|\{u : u=1\}\big|.$$

This proves (iv). □

4 Free boundary condition

We shall now show that u satisfies the free boundary condition in the generalized sense (refer to condition (1.4)). We choose $\vec{\varphi} \in C_0^1(\Omega,\mathbb{R}^N)$ such that $u \neq 1$ a.e. on the support of $\vec{\varphi}$. Multiplying $\nabla u_j \cdot \vec{\varphi}$ to (3.2) and integrating over the set $\{u : 1-\epsilon^- < u < 1+\epsilon^+\}$ gives

$$\begin{aligned}&\int_{\{u:1-\epsilon^-<u<1+\epsilon^+\}} \left[-\Delta u_j + \frac{1}{\delta_j} h\left(\frac{u_j-1}{\delta_j}\right)\right]\nabla u_j \cdot \vec{\varphi}\, dx \\ &\quad = \int_{\{u:1-\epsilon^-<u<1+\epsilon^+\}} \big(\alpha f_{\delta_j}\big(x,(u_j-1)_+\big) + \beta u_j^{-\gamma}\big)\nabla u_j \cdot \vec{\varphi}\, dx.\end{aligned} \tag{4.1}$$

The term on the left-hand side of (4.1) can be expressed as follows:

$$\begin{aligned}&\nabla \cdot \left(\frac{1}{2}|\nabla u_j|^2 \vec{\varphi} - (\nabla u_j \cdot \vec{\varphi})\nabla u_j\right) + \nabla u_j \cdot (\nabla \vec{\varphi} \cdot \nabla u_j) \\ &\quad - \frac{1}{2}|\nabla u_j|^2 \nabla \cdot \vec{\varphi} + \nabla H\left(\frac{u_j-1}{\delta_j}\right) \cdot \vec{\varphi}.\end{aligned} \tag{4.2}$$

Using this, we integrate by parts to obtain

$$\begin{aligned}&\int_{\{u:u=1+\epsilon^+\}\cup\{u=1-\epsilon^-\}} \left[\frac{1}{2}|\nabla u_j|^2\vec{\varphi} - (\nabla u_j \cdot \vec{\varphi})\nabla u_j + H\left(\frac{u_j-1}{\delta_j}\vec{\varphi}\right)\right] \cdot \hat{n}\, dx \\ &\quad = \int_{\{u:1-\epsilon^-<u<1+\epsilon^+\}} \left(\frac{1}{2}|\nabla u_j|^2\vec{\varphi} - (\nabla u_j \cdot \vec{\varphi})\nabla u_j\right) dx \\ &\qquad + \int_{\{u:1-\epsilon^-<u<1+\epsilon^+\}} \left[H\left(\frac{u_j-1}{\delta_j}\right)\nabla \cdot \vec{\varphi} + \alpha f_{\delta_j}\big(x,(u_j-1)_+\big)\nabla u_j \cdot \vec{\varphi} \right. \\ &\qquad \left. + \beta u_j^{-\gamma}\nabla u_j \cdot \vec{\varphi}\right] dx.\end{aligned} \tag{4.3}$$

By using (ii), the integral on the left of equation (4.3) converges to

$$\int_{\{u:u=1+\epsilon^+\}\cup\{u=1-\epsilon^-\}} \left(\frac{1}{2}|\nabla u|^2 \varphi - (\nabla u \cdot \vec{\varphi})\nabla u\right) \cdot \hat{n}\, dS + \int_{\{u:u=1+\epsilon^+\}} \vec{\varphi} \cdot \hat{n}\, dS. \tag{4.4}$$

Equation (4.4) is further equal to

$$\int_{\{u:u=1+\epsilon^+\}}\left(1-\frac{1}{2}|\nabla u|^2\right)\vec{\varphi}\cdot\hat{n}\,dS-\int_{\{u:u=1-\epsilon^-\}}\frac{1}{2}|\nabla u|^2\vec{\varphi}\cdot\hat{n}\,dS. \tag{4.5}$$

This is because $\hat{n}=\pm\frac{\nabla u}{|\nabla u|}$ on the set $\{u:u=1+\epsilon^{\pm}\}\cup\{u:u=1-\epsilon^{\pm}\}$. By using (iii), the first integral on the right-hand side of (4.3) converges to

$$\int_{\{u:1-\epsilon^-<u<1+\epsilon^+\}}\left(\frac{1}{2}|\nabla u|^2\nabla\cdot\vec{\varphi}-\nabla uD\vec{\varphi}\cdot\nabla u\right)dx, \tag{4.6}$$

whereas the second integral of (4.3) is bounded by

$$\int_{\{u:1-\epsilon^-<u<1+\epsilon^+\}}\big(|\nabla\cdot\vec{\varphi}|+C|\vec{\varphi}|\big)\,dx \tag{4.7}$$

for some constant $C>0$. The last two integrals (4.6)–(4.7) vanish as $\epsilon^{\pm}\to 0$ since $|\operatorname{supp}(\vec{\varphi})\cap\{u:u=1\}|=0$. Therefore we first let $j\to\infty$ and then we let $\epsilon^{\pm}\to 0$ in (4.3) to prove that u satisfies the free boundary condition.

Using (f_1),

$$E_\delta(u)\geq\int_\Omega\left\{\frac{1}{2}|\nabla u|^2-\alpha\left(c_0(u-1)_+ +\frac{c_1}{p}(u-1)_+^p\right)-\frac{\beta}{1-\gamma}u^{1-\gamma}\right\}dx. \tag{4.8}$$

Clearly, since $1<p<2$, we have that E_δ is bounded from below and coercive. Thus E_δ satisfies the (PS) condition (see Definition 2.1). It is easy to see that every (PS) sequence is bounded by coercivity and hence contains a convergent subsequence by a standard argument—we extract weakly convergent subsequence and show that this weak limit is the strong limit of, possibly, a different subsequence. Let us show that E_δ has a minimizer, say, u_1^δ. By (f_2), we have $F(x,t)>0$ for all $x\in\Omega$ and $t>0$. Thus, for any $u\in H_0^1(\Omega)$ with $u>1$ on a set of positive measure, we have

$$\int_\Omega F\big(x,(u-1)_+\big)\,dx>0. \tag{4.9}$$

Therefore, $E(u)\to-\infty$ as $\alpha\to\infty$. Thus, there exists $\Lambda>0$ such that for all $\alpha>\Lambda$ we have

$$m_1(\alpha)=\inf_{u\in H_0^1(\Omega)}\big\{E(u)\big\}<-|\Omega|. \tag{4.10}$$

Set

$$\delta_0(\alpha)=\min\left\{\frac{|m_1(\alpha)|}{2\alpha c_0|\Omega|},\left(\frac{pc_0}{c_1}\right)^{\frac{1}{p-1}}\right\}.$$

5 Auxiliary lemmas

We shall now establish the existence of the first solution of problem (1.1), which also is a minimizer for the functional E. Let us begin with the following lemma.

Lemma 5.1 *For all $\alpha > \Lambda$, $0 < \beta < \beta_*$, $\delta < \delta_0(\alpha)$, the functional E_δ has a minimizer $u_1^\delta > 0$ that satisfies*

$$E_\delta(u_1^\delta) \le m_1(\alpha) + 2\alpha\delta c_0|\Omega| < 0. \tag{5.1}$$

Proof Since E_δ is bounded below and satisfies the (PS) condition, it possesses a minimizer u_1^δ. Also, since $H(\frac{t-1}{\delta}) \le \chi_{(1,\infty)}(t)$ for all t, we have

$$\begin{aligned} E_\delta(u) - E(u) &\le \alpha \int_\Omega \bigl[F\bigl(x,(u-1)_+\bigr) - F_\delta\bigl(x,(u-1)_+\bigr)\bigr]\,dx \\ &= \alpha \int_\Omega \int_0^{(u-1)_+} \left[1 - H\left(\frac{t}{\delta} f(x,t)\right)\right] dt\,dx \\ &\le \alpha \int_\Omega \int_0^\delta f(x,t)\,dt\,dx \\ &\le \alpha \left(c_0\delta + \frac{c_1}{p}\delta^p\right)|\Omega| \quad \text{by } (f_1). \end{aligned} \tag{5.2}$$

Further, for $\delta < \delta_0(\alpha)$ we obtain (5.1). Since $E_\delta(u_1^\delta) < 0 = E_\delta(0)$, this implies that u_1^δ is a nontrivial solution of problem (3.2). This solution is positive since it is a minimizer. □

We shall now prove that the functional E_δ has a second nontrivial critical point, say u_2^δ.

Lemma 5.2 *For any $\alpha > \Lambda$ and $0 < \beta < \beta_*$, there exists a constant $c_3(\alpha)$ such that for all $\delta < \delta_0(\alpha)$ the functional E_δ has a second critical point $0 < u_2^\delta \le u_1^\delta$ that obeys*

$$c_3(\alpha) \le E_\delta(u_2^\delta) \le \frac{1}{2}\|u_1^\delta\|^2 + |\Omega|.$$

Furthermore, $\emptyset \ne \{u_2^\delta : u_2^\delta > 1\} \subset \{u_1^\delta : u_1^\delta > 1\}$.

Proof Choose some $\delta < \delta_0(\alpha)$. Consider

$$h_\delta(x,t) = \frac{1}{\delta} h\left(\frac{\min\{t, u_1^\delta(x)\} - 1}{\delta}\right), \qquad H_\delta(x,t) = \int_0^t h_\delta(x,t)\,dt,$$

$$\tilde{f}_\delta(x,t) = f_\delta\bigl(x, \bigl(\min\{t, u_1^\delta(x)\} - 1\bigr)_+\bigr), \qquad \tilde{F}_\delta(x,t) = \int_0^t \tilde{f}_\delta(x,t)\,dt.$$

Further, we set

$$\tilde{E}_\delta(u) = \int_\Omega \left[\frac{1}{2}|\nabla u|^2 + H_\delta(x,u) - \alpha \tilde{F}_\delta(x,u) - \beta\phi_\beta(u)\right] dx, \quad u \in H_0^1(\Omega).$$

The functional $\tilde{E}_\delta$ is of C^1 class and its critical points coincide with the weak solutions of the following problem:

$$\begin{cases} -\Delta u = -h_\delta(x,u) + \alpha \tilde{f}_\delta(x,u) + \beta\phi_\beta(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.3}$$

By the elliptic (Schauder) regularity, a weak solution of (5.3) is also a classical solution. Also, by the maximum principle, we have that $u \leq u_1^\delta$. Thus u is a weak solution of problem (3.3) and hence is a critical point of $\tilde{E}_\delta$ with $\tilde{E}_\delta(u) = E_\delta(u)$. We shall now show that $\tilde{E}_\delta$ has a critical point, say u_2^δ, that satisfies

$$m_2(\alpha) \leq \tilde{E}_\delta\left(u_2^\delta\right) \leq \frac{1}{2}\left\|u_1^\delta\right\|^2 + |\Omega| \quad \text{for some } m_2(\alpha) > 0. \tag{5.4}$$

This enables us to conclude that $E_\delta(u_2^\delta) = \tilde{E}_\delta(u_2^\delta) > 0 > E_\delta(u_1^\delta)$, which in turn will imply that $u_2^\delta > 0$ and different from u_1^δ.

By the mountain pass theorem (see Lemma 2.2), the functional $\tilde{E}_\delta$ that is coercive (owing to its sublinear nature) satisfies the (PS) condition. Clearly, for any $t \leq 1$, we have

$$\tilde{f}_\delta(x,t) = f_\delta(x,0)$$

and

$$\tilde{f}_\delta(x,t) \leq c_0 + c_1\left(\min\left\{t, u_1^\delta(x)\right\} - 1\right)_+^{p-1} \leq c_0 + c_1(t-1)^{p-1} \quad \text{for } t > 1.$$

By (f_1), we get

$$\tilde{F}_\delta(x,t) \leq c_0(t-1)_+ + \frac{c_1}{p}(t-1)_+^p \leq \left(c_0 + \frac{c_1}{p}\right)|t|^q$$

for all t with $q > 2$ if $N = 2$ and $2 < q \leq \frac{2N}{N-2}$ if $N \geq 3$. We observe that

$$\tilde{E}_\delta(u) \geq \int_\Omega \left[\frac{1}{2}|\nabla u|^2 - \alpha\left(c_0 + \frac{c_1}{p}\right)|u|^q - \beta|u|^{1-\gamma}\right] dx \tag{5.5}$$

$$\geq \frac{1}{2}\|u\|^2 - \lambda c_4\left(c_0 + \frac{c_1}{p}\right)\|u\|^q - \beta c_5\|u\|^{1-\gamma}. \tag{5.6}$$

By the embedding result $H_0^1(\Omega) \hookrightarrow L^q(\Omega)$ for $q > 2$, the integral in (5.5) is positive if $\|u\| = r$, i.e., when $u \in \partial B_r(0)$ for sufficiently small $r > 0$, where $B_r(0) = \{u \in H_0^1(\Omega) : \|u\| < r\}$. Furthermore, since $\tilde{E}_\delta(u_1^\delta) = E_\delta(u_1^\delta) < 0 = \tilde{E}_\delta(0)$, we choose $r < \|u_1^\delta\|$, and then applying the mountain pass theorem (Lemma 2.2), we get a critical point u_2^δ of $\tilde{E}_\delta$ with

$$\tilde{E}_\delta\left(u_2^\delta\right) = \inf_{\psi \in \Gamma} \max_{u \in \psi([0,1])} \tilde{E}_\delta(u) \geq m_2(\alpha),$$

where $\Gamma = \{\psi \in C([0,1], H_0^1(\Omega)) : \psi(0) = 0, \psi(1) = u_1^\delta\}$ is the class of paths joining 0 and u_1^δ. For the path $\psi_0(t) = tu_1^\delta$, $t \in [0,1]$, we have

$$\tilde{E}_\delta\left(\psi_0(t)\right) \leq \int_\Omega \left(\frac{1}{2}\left|\nabla u_1^\delta\right|^2 + H_\delta\left(x, u_1^\delta\right)\right) dx \tag{5.7}$$

since $H_\delta(x,t)$ is nondecreasing in t and $\tilde{F}_\delta(x,t) \geq 0$ for all t by condition (f_2). Since

$$H_\delta\left(x, u_1^\delta(x)\right) = \int_0^{u_1^\delta} \frac{1}{\delta} h\left(\frac{t-1}{\delta}\right) dt = H\left(\frac{u_1^\delta(x) - 1}{\delta}\right) \leq 1, \tag{5.8}$$

it follows by (5.7) and (5.8) that

$$\begin{aligned} \tilde{E}_\delta\left(u_2^\delta\right) &\le \max_{u\in\psi_0([0,1])} \tilde{E}_\delta(u) \le \int_\Omega \left(\frac{1}{2}\left|\nabla u_1^\delta\right|^2 + 1\right) dx \\ &= \frac{1}{2}\left\|u_1^\delta\right\|^2 + |\Omega|. \end{aligned} \tag{5.9}$$

□

6 Positive Radon measure

We shall now prove two more results that will be needed in the last section.

Lemma 6.1 *Let $0 < \beta < \beta_*$. Then a solution of the problem*

$$\begin{cases} -\Delta v = \beta v^{-\gamma} & \text{in } \Omega, \\ v > 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{6.1}$$

say u_β, satisfies $u_\beta < u$ a.e. in Ω, where u is a solution of problem (1.1).

Proof Let $u \in H_0^1(\Omega)$ be a positive solution of problem (1.1) and $u_\beta > 0$ be a solution of problem (6.1). For any $0 < \beta < \beta_*$, define a weak solution u_β of problem (6.1) as follows:

$$0 = \int_\Omega \nabla u_\beta \cdot \nabla\varphi\, dx - \beta \int_\Omega u_\beta^{-\gamma} \varphi\, dx \quad \text{for all } \varphi \in H_0^1(\Omega). \tag{6.2}$$

By the *Schauder estimates*, we have $u \in C^{2,a}(\Omega)$, and by Lazer and McKenna [16] we have $u_\beta \in C^{2,a}(\Omega) \cap C(\bar{\Omega})$. We shall show that $u \ge u_\beta$ a.e. in Ω. We let $\tilde{\Omega} = \{x \in \Omega : u(x) < u_\beta(x)\}$. Thus, from the weak formulations satisfied by u, u_β and testing with the function $\varphi = (u_\beta - u)_+$, we have

$$\begin{aligned} 0 &\le \int_\Omega \nabla(u_\beta - u) \cdot \nabla(u_\beta - u)_+\, dx \\ &= -\alpha \int_\Omega \chi_{\{u>1\}} g\big(x, (u-1)_+\big)(u_\beta - u)_+\, dx \\ &\quad + \beta \int_\Omega \big(u_\beta^{-\gamma} - u^{-\gamma}\big)(u_\beta - u)_+\, dx \le 0. \end{aligned} \tag{6.3}$$

Thus, $\|(u_\beta - u)_+\| = 0$ and hence $|\tilde{\Omega}| = 0$. However, since the functions u, u_β are continuous, it follows that $\tilde{\Omega} = \emptyset$. Hence, by (6.3), we obtain $u \ge \underline{u}_\beta$ in Ω.

Let $W = \{x \in \Omega : u(x) = u_\beta(x)\}$. Since W is a measurable set, it follows that for any $\eta > 0$ there exists a closed subset V of W such that $|W \setminus V| < \eta$. Further assume that $|W| > 0$. We now define a test function $\varphi \in C_c^1(\mathbb{R}^N)$ such that

$$\varphi(x) = \begin{cases} 1 & \text{if } x \in V, \\ 0 < \varphi < 1 & \text{if } x \in W \setminus V, \\ 0 & \text{if } x \in \Omega \setminus W. \end{cases} \tag{6.4}$$

Since u is a weak solution to (1.1), we have

$$\begin{aligned} 0 &= \int_\Omega -\Delta u\varphi\,dx - \beta\int_V u^{-\gamma}\,dx - \beta\int_{W\setminus V} u^{-\gamma}\varphi\,dx \\ &\quad - \int_V f\big(x,(u-1)_+\big)\,dx - \int_{W\setminus V} f\big(x,(u-1)_+\big)\varphi\,dx \\ &= -\int_V f\big(x,(u-1)_+\big)\,dx - \int_{W\setminus V} f\big(x,(u-1)_+\big)\varphi\,dx < 0. \end{aligned} \tag{6.5}$$

This is a contradiction. Therefore, $|W| = 0$, which implies that $W = \emptyset$. Hence, $u > u_\beta$ in Ω. □

Lemma 6.2 *Function u is in $H^{1,2}_{\mathrm{loc}}(\Omega)$ and the Radon measure $\mu = \Delta u + \beta u^{-\gamma}$ is nonnegative and supported on $\Omega \cap \{u : u < 1\}$ for $\beta \in (0,\beta_*)$.*

Proof We follow the proof due to Alt and Caffarelli [2]. Choose $\delta > 0$, $\beta \in (0,\beta_*)$, and a test function $\varphi^2\chi_{\{u:u<1-\delta\}}$, where $\varphi \in C_0^\infty(\Omega)$. Therefore,

$$\begin{aligned} 0 &= -\int_\Omega \nabla u\cdot\nabla\big(\varphi^2\min\{u-1+\delta,0\}\big)\,dx \\ &\quad + \beta\int_\Omega u^{-\gamma}\varphi^2\min\{u-1+\delta,0\}\,dx \\ &= \int_{\Omega\cap\{u:u<1-\delta\}} \nabla u\cdot\nabla\big(\varphi^2(u-1+\delta)\big)\,dx \\ &\quad + \beta\int_{\Omega\cap\{u:u<1-\delta\}} u^{-\gamma}\big(\varphi^2(u-1+\delta)\big)\,dx \\ &= \int_{\Omega\cap\{u:u<1-\delta\}} |\nabla u|^2\varphi^2\,dx + 2\int_{\Omega\cap\{u:u<1-\delta\}} \varphi\nabla u\cdot\nabla\varphi(u-1+\delta)\,dx \\ &\quad + \beta\int_{\Omega\cap\{u:u<1-\delta\}} u^{-\gamma}\big(\varphi^2(u-1+\delta)\big)\,dx. \end{aligned} \tag{6.6}$$

By an application of integration by parts to the second term of (6.6), we get

$$\begin{aligned} &\int_{\Omega\cap\{u:u<1-\delta\}} |\nabla u|^2\varphi^2\,dx \\ &\quad = -2\int_{\Omega\cap\{u:u<1-\delta\}} \varphi\nabla u\cdot\nabla\varphi(u-1+\delta)\,dx \\ &\qquad + \beta\int_{\Omega\cap\{u:u<1-\delta\}} u^{-\gamma}\big(\varphi^2(u-1+\delta)\big)\,dx \\ &\quad \le 4\int_\Omega u^2|\nabla\varphi|^2\,dx - \beta\int_\Omega u^{1-\gamma}\varphi^2\,dx \\ &\quad \le 4\int_\Omega u^2|\nabla\varphi|^2\,dx. \end{aligned} \tag{6.7}$$

On passing to the limit $\delta \to 0$, we conclude that $u \in H^{1,2}_{\mathrm{loc}}(\Omega)$.

Furthermore, for nonnegative $\zeta \in C_0^\infty(\Omega)$, we have

$$
\begin{aligned}
&-\int_\Omega \nabla\zeta\cdot\nabla u\,dx+\beta\int_\Omega u^{-\gamma}\zeta\,dx\\
&=\left(\int_{\Omega\cap\{u:0<u<1-2\delta\}}+\int_{\Omega\cap\{u:1-2\delta<u<1-\epsilon\}}+\int_{\Omega\cap\{u:1-\delta<u<1\}}\right.\\
&\quad\left.+\int_{\Omega\cap\{u:u>1\}}\right)\\
&\quad\times\left[\nabla\left(\zeta\max\left\{\min\left\{2-\frac{1-u}{\delta},1\right\},0\right\}\right)\cdot\nabla u\right.\\
&\quad\left.+\beta u^{-\gamma}\zeta\right]dx\\
&\ge\int_{\Omega\cap\{u:1-2\delta<u<1-\delta\}}\left[\left(2-\frac{1-u}{\delta}\right)\nabla\zeta\cdot\nabla u+\frac{\zeta}{\delta}|\nabla u|^2\right.\\
&\quad\left.+\beta u^{-\gamma}\zeta\right]dx.
\end{aligned}
\tag{6.8}
$$

On passing to the limit $\delta\to 0$, we obtain $\Delta(u-1)_-\ge 0$ in the distributional sense, and hence there exists a Radon measure μ (say) such that $\mu=\Delta(u-1)_-\ge 0$. □

7 Proof of the main theorem

Finally, we are in a position to prove Theorem 1.1.

Proof of Theorem 1.1 Choose $\alpha>\lambda$ and a sequence $\delta_j\to 0$ such that $\delta_j<\delta_0(\alpha)$. For each j, Lemma 5.1 gives a minimizer $u_1^\delta>0$ of E_{δ_j} that obeys

$$E_{\delta_j}\big(u_1^{\delta_j}\big)\le m_1(\alpha)+2\alpha\delta_j c_0|\Omega|<0. \tag{7.1}$$

Further, by Lemma 5.2, we can guarantee the existence of the second critical point $0<u_2^\delta\le u_1^{\delta_j}$ such that

$$m_2(\alpha)\le E_{\delta_j}\big(u_2^{\delta_j}\big)\le\frac{1}{2}\big\|u_1^{\delta_j}\big\|^2+|\Omega|. \tag{7.2}$$

The next step is to show that $(u_1^{\delta_j})$, $(u_2^{\delta_j})$ are bounded in $H_0^1(\Omega)\cap L^\infty(\Omega)$. We shall then apply Lemma 3.1.

Since $H\ge 0$ and

$$H_\delta\big(x,(t-1)_+\big)\le c_0(t-1)_++\frac{c_1}{p}(t-1)_+^p\le\left(c_0+\frac{c_1}{p}\right)|t|^p$$

for all t by (f_1), it follows that

$$\frac{1}{2}\big\|u_1^\delta\big\|^2\le E_\delta\big(u_1^\delta\big)+\alpha\left(c_0+\frac{c_1}{p}\right)\int_\Omega\big(u_1^\delta\big)^p\,dx+\beta\int_\Omega\big(u_1^\delta\big)^{1-\gamma}\,dx. \tag{7.3}$$

Since $E_{\delta_j}(u_1^\delta)<0$ by (7.1) and $p<2$, we have that $(u_1^{\delta_j})$ is bounded in $H_0^1(\Omega)$.

Since $f_\delta(x,(t-1)_+) = f_\delta(x,0) = 0$ for any $t \le 1$ and

$$f_\delta\big(x,(t-1)_+\big) \le c_0 + c_1(t-1)^{p-1} \le (c_0+c_1)t^{p-1}$$

whenever $t > 1$ by (f_1), we get

$$\begin{aligned} -\Delta u_1^{\delta_j} &= -\frac{1}{\delta_j} h\left(\frac{u_1^{\delta_j}-1}{\delta_j}\right) + \alpha f_{\delta_j}\big(x,\big(u_1^{\delta_j}-1\big)_+\big) + \beta\big(u_1^{\delta_j}\big)^{-\gamma} \\ &\le \alpha(c_0+c_1)\big(u_1^{\delta_j}\big)^{p-1} + \beta\big(u_1^{\delta_j}\big)^{-\gamma}. \end{aligned} \tag{7.4}$$

However, when $u_1^{\delta_j} < 1$,

$$-\Delta u_1^{\delta_j} = \beta\big(u_1^{\delta_j}\big)^{-\gamma}, \tag{7.5}$$

in which case $u_1^{\delta_j} = u_\beta|_{\{u_1^{\delta_j}<1\}}$. □

The sublinearity of (7.5) together with the boundedness of $(u_1^{\delta_j})$ in $H_0^1(\Omega)$ implies by the *Moser* iteration method that $(u_1^{\delta_j})$ in $L^\infty(\Omega)$. By a similar argument, $(u_2^{\delta_j})$ is also bounded in $L^\infty(\Omega)$ since $0 < u_1^{\delta_j} \le u_2^{\delta_j}$ in Ω. On renaming the subsequence of (δ_j), the sequences $(u_1^{\delta_j})$, $(u_2^{\delta_j})$ converge uniformly to a Lipschitz continuous functions, say $u_1, u_2 \in H_0^1(\Omega) \cap C^2(\bar{\Omega} \setminus G(u))$ respectively, of problem (1.1) that satisfies

$$-\Delta u = \alpha\chi_{\{u>1\}} f\big(x,(u-1)_+\big) + \beta u^{-\gamma}$$

classically in the region $\Omega \setminus G(u)$, the free boundary condition in the generalized sense and furthermore continuously vanishes on $\partial\Omega$. We also have that

$$E(u_1) \le \liminf E_{\delta_j}\big(u_1^{\delta_j}\big) \le \limsup E_{\delta_j}\big(u_1^{\delta_j}\big) \le E(u_1) + \big|\{u_1 : u_1 = 1\}\big| \tag{7.6}$$

and

$$E(u_2) \le \liminf E_{\delta_j}\big(u_2^{\delta_j}\big) \le \limsup E_{\delta_j}\big(u_2^{\delta_j}\big) \le E(u_2) + \big|\{u_2 : u_2 = 1\}\big|. \tag{7.7}$$

Using (7.6) in combination with (7.1) and (4.10) yields

$$E(u_1) \le \limsup E_{\delta_j}\big(u_1^{\delta_j}\big) \le m_1(\alpha) \le E(u_1).$$

Therefore,

$$E(u_1) = m_1(\alpha) < -|\Omega|. \tag{7.8}$$

Similarly, combining (7.7) with (7.2) yields

$$0 < m_2(\alpha) \le \liminf E_{\delta_j}\big(u_2^{\delta_j}\big) \le E(u_2) + \big|\{u_2 : u_2 = 1\}\big|.$$

Thus,

$$E(u_2) > -\big|\{u_2 : u_2 = 1\}\big| \geq -|\Omega|. \tag{7.9}$$

So, from (7.8) and (7.9) we can conclude that u_1, u_2 are distinct and nontrivial solutions of problem (1.1). Here u_1 is a minimizer, whereas u_2 is not. Also, since $u_2^{\delta_j} \leq u_1^{\delta_j}$ for each j, we have $u_2 \leq u_1$. Since u_2 is a nontrivial solution, it follows that $0 < u_2 \leq u_1$ and the sets $\{u_1 : u_1 < 1\} \subset \{u_2 : u_2 < 1\}$ are connected if $\partial\Omega$ is connected. Moreover, the sets $\{u_2 : u_2 > 1\} \subset \{u_1 : u_1 > 1\}$ are nonempty.

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