



On critical exponential Kirchhoff systems on the Heisenberg group

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Received: 23 June 2022 / Accepted: 1 September 2022 / Published online: 11 September 2022
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Abstract

In this paper, existence of solutions is established for critical exponential Kirchhoff systems on the Heisenberg group by using the variational method. The novelty of our paper is that not only the nonlinear term has critical exponential growth, but also that Kirchhoff function covers the degenerate case. Moreover, our result is new even for the Euclidean case.

Keywords Kirchhoff system · Heisenberg group · Critical exponential growth · Variational method

Mathematics Subject Classification 35J20 · 35R03 · 46E35

1 Introduction

In this paper, we are interested in critical exponential Kirchhoff systems on the Heisenberg group \mathbb{H}^n :

$$\begin{cases} -K_1(\int_{\Omega} |\nabla_{\mathbb{H}^n} u|^Q d\xi) \Delta_Q u = \lambda f_1(\xi, u, v), & \text{for } \xi \in \Omega, \\ -K_2(\int_{\Omega} |\nabla_{\mathbb{H}^n} v|^Q d\xi) \Delta_Q v = \lambda f_2(\xi, u, v), & \text{for } \xi \in \Omega, \\ u = v = 0, & \text{for } \xi \in \partial\Omega, \end{cases} \quad (1.1)$$

where Δ_Q is the Q -Laplacian operator on the Heisenberg group \mathbb{H}^n , defined by

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$$\Delta_Q(\cdot) = \operatorname{div}_{\mathbb{H}^n}(|\nabla_{\mathbb{H}^n}(\cdot)|_{\mathbb{H}^n}^{Q-2} \nabla_{\mathbb{H}^n}(\cdot)),$$

Ω is a bounded open smooth subset of the Heisenberg group \mathbb{H}^n , and $\lambda > 0$ is a positive parameter.

There are already several interesting papers devoted to the study of the Heisenberg group \mathbb{H}^n . For example, Pucci and Temperini [21] studied the existence of entire nontrivial solutions of (p, q) critical systems on the Heisenberg group \mathbb{H}^n , by using the variational methods and the concentration-compactness principle. Pucci and Temperini [20], they accomplished the conclusions of Pucci and Temperini [21] and worked out some kind of elliptic systems involving critical nonlinearities and Hardy terms on the Heisenberg group \mathbb{H}^n . There are additional interesting results in Liang and Pucci [13], Pucci [18], Pucci [19] and Pucci and Temperini [22].

In the Euclidean case, Kirchhoff-type problems have attracted wave after wave of scholars. Kirchhoff [11] established a model given by the hyperbolic equation

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{p_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx \right) \frac{\partial^2 u}{\partial x^2} = 0,$$

where parameters ρ, p_0, h, E, L are constants with some physical meaning, which extends the classical D'Alembert wave equation for free vibrations of elastic strings. In particular, Kirchhoff equation models also appear in physical and biological systems. We refer the reader to Alves et al. [2] for more details. After Kirchhoff's work, Figueiredo and Severo [10] studied the following problem

$$\begin{cases} -m(\int_{\Omega} |\nabla u|^2 dx) \Delta u = f(x, u) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

and addressed the existence of ground state solutions of the problems on \mathbb{R}^2 by using the minimax techniques with the Trudinger–Moser inequality. Mingqi et al. [15] studied the existence and multiplicity of solutions for a class of perturbed fractional Kirchhoff type problems with singular exponential nonlinearity. Alves and Boudjeriou [1] obtained the existence of a nontrivial solution for a class of nonlocal problems by using the dynamical methods. For several interesting results recovering the Kirchhoff-type problems, we refer to Ambrosio et al. [3], Caponi and Pucci [6], Mingli et al. [14], and Pucci et al. [23], and the references therein.

Recently, some authors have focused their attention to the problem with critical exponential growth in the Euclidean case, see Aouaoui [4], Albuquerque et al. [8], Moser [16], and Trudinger [24]. In the Heisenberg group \mathbb{H}^n case, Cohn and Lu [7] have established a new version of the Trudinger–Moser inequality: Let $\Omega \subset \mathbb{H}^n$ and assume that $|\Omega| < \infty$ and $0 < \alpha \leq \alpha_Q$. Then

$$\sup_{u \in W_0^{1,Q}(\Omega), \|\nabla_{\mathbb{H}^n} u\|_{L^Q(\Omega)} \leq 1} \frac{1}{|\Omega|} \int_{\Omega} e^{\alpha|u(\xi)|^{\frac{Q}{Q-1}}} d\xi \leq C_0 < \infty, \tag{1.2}$$

where $C_0 > 0$ is a constant which depends only on $Q = 2n + 2$. Moreover,

$$\alpha_Q = Q\sigma_Q^{1/(Q-1)} \quad \text{and} \quad \sigma_Q = \int_{\rho(z,t)=1} |z|^Q d\mu.$$

Deng and Tian [9] have established the existence of nontrivial solutions for the non-degenerate Kirchhoff elliptic system with nonlinear term have critical exponential growth. Although the study of critical Kirchhoff-type problems is more meaningful, there are some authors working on the degenerate Kirchhoff problem. From a physical point of view, the fact that $M(0) = 0$ means that the base tension of the string is zero, which is a very realistic model. To the best of our knowledge, the existence results for system (1.1) in the degenerate case are not yet known for the Heisenberg group \mathbb{H}^n .

For these reasons, we mainly consider the critical exponential Kirchhoff systems (1.1) on the Heisenberg group. We say that f_i satisfies critical exponential growth at $+\infty$ provided that there exists $\alpha_0 > 0$ such that

$$\lim_{|(u,v)| \rightarrow +\infty} \frac{|f_i(\xi, u, v)|}{e^{\alpha|(u,v)|^{\frac{Q}{Q-1}}}} = \begin{cases} 0 & \text{uniformly on } \xi \in \Omega, \text{ for all } \alpha > \alpha_0; \\ +\infty & \text{uniformly on } \xi \in \Omega, \text{ for all } \alpha < \alpha_0. \end{cases} \tag{1.3}$$

In view of the critical exponential growth of the nonlinear terms f_i , we work out the problem of the ‘‘lack of compactness’’ by using a new version of the Trudinger–Moser inequality for the Heisenberg group \mathbb{H}^n .

Throughout the paper, Kirchhoff-type functions K_i and f_i will satisfy the following conditions:

(\mathcal{K}) (\mathcal{K}_1) There exist $q \in [1, n)$ and $\sigma \in [1, q)$, satisfying

$$\sigma \mathcal{K}_i(t) := \sigma \int_0^t K_i(s) ds \geq K_i(t)t, \quad \text{for all } t \geq 0.$$

(\mathcal{K}_2) There exist $k_1, k_2 > 0$ such that $K_i(t) \geq k_1 t^{\sigma-1}$, where $K(0) = 0$, for all $t \geq 0$.

(\mathcal{F}) (F_1) $\lim_{w \rightarrow 0} \frac{f_i(\xi, w)}{|w|^{Q-1}} = 0$, for all $w = (u, v)$ and $|w| = \sqrt{u^2 + v^2}$.

(F_2) $F \in C^1(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ and there exists $\mu > \sigma Q$ such that

$$0 < \mu F(\xi, w) \leq \nabla F(\xi, w)w, \quad \text{for all } w \in \mathbb{R}^2, \text{ where } \nabla F = (f_1, f_2).$$

(F_3) $\liminf_{w \rightarrow 0^+} \frac{F(\xi, w)}{|w|^\mu} =: \eta > 0$.

Remark 1.1 Note that some typical examples of functions $K_i : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ which are nondecreasing and satisfy conditions (\mathcal{K}_1) and (\mathcal{K}_2), are given by

$$K_i(t) = a_i + b_i t^{\sigma-1}, \quad \text{for all } t \in \mathbb{R}, \text{ where } a \in \mathbb{R}, b \in \mathbb{R} \text{ and } a + b > 0.$$

Clearly, conditions (\mathcal{K}_1) and (\mathcal{K}_2) also cover the degenerate case, that is when $a = 0$.

Remark 1.2 By condition (F_2), we can easily get that $F(\xi, w)/|w|^\mu$ is nondecreasing for all $w > 0$. Thus, for all $w \geq 0$, we obtain $F(\xi, w) \geq \eta|w|^\mu$ by invoking condition (F_3).

The main result of this paper is as follows.

Theorem 1.1 Assume that conditions (\mathcal{K}) and (\mathcal{F}) are satisfied and that the nonlinear terms f_i have critical exponential growth. Then system (1.1) has at least one nontrivial solution for all $\lambda > 0$, when $\eta > 0$ from condition (F_3) is large enough.

In conclusion, we describe the structure of the paper. In Sect. 2 we collect all necessary preliminaries. In Sect. 3 we study the mountain pass geometry. In Sect. 4 we verify the compactness condition. Finally, in Sect. 5 we present the proof of our main result.

2 Preliminaries

We begin by recalling some key facts about the Heisenberg group \mathbb{H}^n , i.e. a Lie group of topological dimension $2n + 1$ and the background manifold \mathbb{R}^{2n+2} . We define

$$\begin{aligned} \xi \circ \xi' &= \tau_\xi(\xi') = (x + x', y + y', t + t' + 2(x'y - y'x)), \text{ for all} \\ \xi &= (x, y, z), \xi' = (x', y', z') \in \mathbb{H}^n. \end{aligned}$$

Furthermore,

$$\delta_s(\xi) = (sx, sy, s^2t), \text{ where } s > 0,$$

gives a natural group of dilations on \mathbb{H}^n , so

$$\delta_s(\xi_0 \circ \xi) = \delta_s(\xi_0) \circ \delta_s(\xi).$$

The Jacobian determinant of dilations $\delta_s : \mathbb{H}^n \rightarrow \mathbb{H}^n$ is constant for all $\xi = (x, y, t) \in \mathbb{H}^n$ and $\delta_s = \mathbb{R}^{2n+2}$. Next,

$$B_R(\xi_0) = \{\xi \in \mathbb{H}^n : d_K(\xi, \xi_0) < R\}$$

denotes the Korányi open ball with radius R , centered at ξ_0 (see Leonardi and Masnou [12]).

The horizontal gradient $\nabla_H = (X, Y)$ and

$$X = \frac{\partial}{\partial x} + 2y \frac{\partial}{\partial t}, Y = \frac{\partial}{\partial y} - 2x \frac{\partial}{\partial t}$$

give rise to the Lie algebra of left-invariant vector fields on \mathbb{H}^n . Finally, $\Delta_H u = \operatorname{div}_H(\nabla_H u)$ represents the Kohn-Laplacian Δ_H . In addition, the degenerate elliptic operator Δ_H satisfies Bony’s maximum principle (see Bony [5]).

Next, we define the classical Sobolev space $W_0^{1,Q}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\| := \|\nabla_{H^n} u\|_{W_0^{1,Q}}.$$

Let

$$W_0^{1,Q}(\Omega, \mathbb{R}^2) := W_0^{1,Q}(\Omega) \times W_0^{1,Q}(\Omega),$$

endowed by the norm

$$\|(u, v)\|_{W_0^{1,Q}(\Omega, \mathbb{R}^2)} = (\|u\|_{W_0^{1,Q}(\Omega)}^Q + \|v\|_{W_0^{1,Q}(\Omega)}^Q)^{1/Q}.$$

System (1.1) is variational and the corresponding energy functional $I_\lambda : W_0^{1,Q}(\Omega, \mathbb{R}^2) \rightarrow \mathbb{R}$ is given by

$$I_\lambda(u, v) := \frac{1}{Q} \mathcal{K}_1 \left(\int_\Omega |\nabla_{\mathbb{H}^n} u|^Q d\xi \right) + \frac{1}{Q} \mathcal{K}_2 \left(\int_\Omega |\nabla_{\mathbb{H}^n} v|^Q d\xi \right) - \lambda \int_\Omega F(\xi, u, v) d\xi.$$

By (F_1) , (F_2) and (1.3), there exists for $\varepsilon > 0, r > Q$, a constant $C = C(\varepsilon, r) > 0$ such that

$$|f_1(\xi, w)| + |f_2(\xi, w)| \leq \varepsilon |w|^{Q-1} + C |w|^{r-1} e^{\alpha |w|^{Q'}}, \quad \text{for all } \alpha > \alpha_0. \tag{2.1}$$

Consequently, we have

$$F(\xi, w) \leq \frac{\varepsilon}{\theta Q} |w|^Q + C |w|^r e^{\alpha |w|^{Q'}}, \quad \text{for all } \alpha > \alpha_0, \tag{2.2}$$

where $Q' = \frac{Q}{Q-1}$. Since

$$(a + b)^m \leq 2^{m-1} (a^m + b^m), \quad \text{for all } a, b \geq 0, m > 0, \tag{2.3}$$

we can obtain the following inequality

$$|w|^{Q'} = (|u|^2 + |v|^2)^{\frac{Q'}{2}} \leq 2^{Q'-1} (|u|^{Q'} + |v|^{Q'}).$$

Thus, invoking the Hölder inequality and (1.2), we get

$$\begin{aligned} \int_\Omega e^{\alpha |w|^{Q'}} d\xi &\leq \int_\Omega e^{(\alpha 2^{Q'-1} |u|^{Q'})} e^{(\alpha 2^{Q'-1} |v|^{Q'})} d\xi \\ &\leq \left(\int_\Omega e^{\alpha 2^{Q'} |u|^{Q'}} d\xi \right)^{1/2} \left(\int_\Omega e^{\alpha 2^{Q'} |v|^{Q'}} d\xi \right)^{1/2} < \infty. \end{aligned}$$

Moreover, we can conclude that $I_\lambda \in C^1(W_0^{1,Q}(\Omega, \mathbb{R}^2), \mathbb{R})$ is well-defined and that the derivative of I_λ is

$$\begin{aligned} \langle I'_\lambda(u, v), (\varphi, \psi) \rangle &= K_1 (\|u\|^Q) \int_\Omega |\nabla_{\mathbb{H}^n} u|_{\mathbb{H}^n}^{Q-2} \nabla_{\mathbb{H}^n} u \nabla_{\mathbb{H}^n} \varphi d\xi \\ &\quad + K_2 (\|v\|^Q) \int_\Omega |\nabla_{\mathbb{H}^n} v|_{\mathbb{H}^n}^{Q-2} \nabla_{\mathbb{H}^n} v \nabla_{\mathbb{H}^n} \psi d\xi \\ &\quad - \lambda \int_\Omega f_1(\xi, u, v) \varphi d\xi - \lambda \int_\Omega f_2(\xi, u, v) \psi d\xi, \quad \text{for all } (u, v), (\varphi, \psi) \in W_0^{1,Q}(\Omega, \mathbb{R}^2). \end{aligned}$$

Therefore the solutions of system (1.1) coincide with the critical points of I_λ .

Lemma 2.1 *Suppose that condition (F_1) is satisfied and that*

$$t \geq 0, \quad \alpha > \alpha_0, \quad \|w\| \leq \rho, \quad \text{with } \alpha \rho^{Q'} < \frac{\alpha_Q}{2Q'}.$$

Then there exists $C = C(t, \alpha, \rho) > 0$ such that

$$\int_\Omega e^{\alpha |w|^{Q'}} |w|^t d\xi \leq C \|w\|^t.$$

Proof Due to the Hölder inequality, we get

$$\int_{\Omega} e^{\alpha|w|^{\rho'}} |w|^t d\xi \leq \left(\int_{\Omega} e^{q\alpha|w|^{\rho'}} d\xi \right)^{\frac{1}{q}} \|w\|_{p'}^t, \text{ where } p, q > 1, \frac{1}{p} + \frac{1}{q} = 1, pt \geq Q.$$

Since

$$\|w\| \leq \rho \text{ with } \alpha\rho^{\rho'} < \frac{\alpha_Q}{2^{\rho'}},$$

there exists $q > 1$ such that

$$2^{\rho'} q\alpha\rho^{\rho'} \leq \alpha_Q.$$

By virtue of (2.3), one has

$$\begin{aligned} \int_{\Omega} e^{q\alpha|w|^{\rho'}} d\xi &\leq \int_{\Omega} e^{(2^{\rho'}-1)q\alpha(|u|^{\rho'}+|v|^{\rho'})} d\xi \leq \left(\int_{\Omega} e^{(2^{\rho'} q\alpha|u|^{\rho'})} d\xi \right)^{\frac{1}{2}} \left(\int_{\Omega} e^{(2^{\rho'} q\alpha|v|^{\rho'})} d\xi \right)^{\frac{1}{2}} \\ &\leq \left(\int_{\Omega} e^{(2^{\rho'} q\alpha\rho^{\rho'} \left(\frac{|u|}{\|w\|}\right)^{\rho'})} d\xi \right)^{\frac{1}{2}} \left(\int_{\Omega} e^{2^{\rho'} q\alpha\rho^{\rho'} \left(\frac{|v|}{\|w\|}\right)^{\rho'}} d\xi \right)^{\frac{1}{2}} \leq C, \end{aligned}$$

which implies that the conclusion of Lemma 2.1 is valid. □

For other background information we refer the reader to the comprehensive monograph by Papageorgiou et al. [17].

3 Mountain pass geometry

In this section we shall prove that I_{λ} satisfies the mountain pass geometry.

Lemma 3.1 *Suppose that conditions $(\mathcal{K}), (\mathcal{F})$ are satisfied and that f_i have exponential critical growth. Then the following properties hold:*

- (I_1) There exist $\iota > 0$ and $\kappa > 0$ such that $I_{\lambda}(u, v) \geq \iota$, for all $\|(u, v)\| = \kappa$.
- (I_2) There exists $(e, e) \in W_0^{1,Q}(\Omega, \mathbb{R}^2)$ with $\|(e, e)\| > \kappa$ such that $I_{\lambda}(e, e) < 0$.

Proof First, we prove assertion (I_1). If $r > \sigma Q$, $\alpha > \alpha_0$, and $0 < \varepsilon < \min \{k_1, k_2\}$, then by virtue of (F_1) and (2.2), we have

$$\begin{aligned} I_{\lambda}(u, v) &\geq \frac{1}{\sigma Q} (\min \{k_1, k_2\} - \lambda\varepsilon) \|(u, v)\|^{\sigma Q} \\ &\quad - C_1 \lambda \int_{\Omega} e^{\alpha|w|^{\rho'}} |(u, v)|^r d\xi, \text{ for all } (u, v) \in W_0^{1,Q}(\Omega, \mathbb{R}^2). \end{aligned}$$

By Lemma 2.1 and the following inequality

$$\|(u, v)\| = \kappa < \left(\frac{\alpha_Q}{2\alpha_{Q'}} \right)^{\frac{1}{Q'}}$$

we obtain

$$I_\lambda(u, v) \geq \frac{1}{\sigma Q} (\min \{k_1, k_2\} - \lambda \epsilon) \kappa^{\sigma Q} - C_2 \kappa^r.$$

Next, we choose

$$\kappa < \left(\frac{\alpha_Q}{2\alpha_{Q'}} \right)^{\frac{1}{Q'}}$$

so small that

$$\frac{1}{\sigma Q} (\min \{k_1, k_2\} - \lambda \epsilon) - C_2 \kappa^{r-\sigma Q} > 0.$$

This implies that

$$I_\lambda(u, v) \geq \iota := \kappa^{\sigma Q} \left[\frac{1}{\sigma Q} (\min \{k_1, k_2\} - \lambda \epsilon) - C_2 \kappa^{r-\sigma Q} \right], \text{ for all } \|(u, v)\| = \kappa.$$

Next, we prove assertion (I_2) . Let $\psi \in C_0^\infty(B_R(\xi_0))$ be such that $\psi \geq 0$ on $B_R(\xi_0)$ and let $K = \text{supp}(\psi)$. Invoking condition (\mathcal{K}) , we obtain

$$\mathcal{K}(t) \leq c_i t^\sigma + d_i, \text{ for all } t \geq 0, \text{ where } c_i, d_i > 0, i = 1, 2.$$

By virtue of Remark 1.2, we get

$$I_\lambda(t\psi, t\psi) \leq \frac{t^{\sigma Q}}{Q} \|\psi\|^{\sigma Q} (c_1 + d_1) + \frac{1}{Q} (c_2 + d_2) - \lambda \eta t^\mu \int_\Omega \psi^\mu d\xi.$$

Since $\mu > \sigma Q$, we can conclude that

$$I_\lambda(t\psi, t\psi) \rightarrow -\infty, \text{ as } t \rightarrow +\infty.$$

Therefore, we get the claim by using $e := t\psi$ for a sufficiently large $t > 0$. □

4 Compactness condition

In this section, we shall prove the following compactness condition.

Lemma 4.1 *Suppose that the following inequality holds*

$$c < \frac{\nu_0(\mu - \sigma Q)\alpha_Q^{\sigma(Q-1)}}{\alpha_0^{\sigma(Q-1)} Q \mu \sigma^{\sigma Q+1}}, \text{ where } \nu_0 := \min \{k_1, k_2\}. \tag{4.1}$$

Then there exists a $(PS)_c$ sequence for $I_\lambda \{ (u_n, v_n) \}$ such that the functional I_λ satisfies the Palais–Smale condition at level c .

Proof First, let

$$\{ (u_n, v_n) \} \subset W_0^{1,Q}(\Omega, \mathbb{R}^2)$$

be a $(PS)_c$ sequence for I_λ . If $\inf_{n \in \mathbb{N}} \|w_n\| = 0$, then

$$w_n \rightarrow 0 \text{ in } W_0^{1,Q}(\Omega, \mathbb{R}^2), \text{ as } n \rightarrow \infty.$$

Therefore, we shall use $\inf_{n \in \mathbb{N}} \|w_n\| > 0$ in the sequel.

By conditions (K) and (F_2) , we have

$$\begin{aligned} c + o_n(1) \left\| (u_n, v_n) \right\| &\geq I_\lambda(u_n, v_n) - \frac{1}{\mu} I'_\lambda(u_n, v_n)(u_n, v_n) \\ &\geq \left(\frac{1}{\sigma Q} - \frac{1}{\mu} \right) K_1 \left(\|u_n\|^Q \right) \|u_n\|^Q + \left(\frac{1}{\sigma Q} - \frac{1}{\mu} \right) K_2 \left(\|v_n\|^Q \right) \|v_n\|^Q \\ &\quad + \frac{\lambda}{\mu} \int [\nabla F(\xi, u_n, v_n)(u_n, v_n) - \mu F(\xi, u_n, v_n)] d\xi, \end{aligned} \tag{4.2}$$

which combined with conditions (H_1) and (F_2) , implies

$$c + o_n(1) \left\| (u_n, v_n) \right\| \geq \left(\frac{\mu - \sigma Q}{\sigma Q \mu} \right) v_0 \left\| (u_n, v_n) \right\|^{\sigma Q}, \text{ where } v_0 := \min \{ k_1, k_2 \}.$$

Hence, (4.2) implies that the sequence $\{ (u_n, v_n) \}$ is bounded on $W_0^{1,Q}(\Omega, \mathbb{R}^2)$ and that

$$\limsup_{n \rightarrow +\infty} \left\| (u_n, v_n) \right\|^{\sigma Q} \leq \frac{\sigma Q \mu}{(\mu - \sigma Q) v_0} c.$$

Next, let $(u_0, v_0) \in W_0^{1,Q}(\Omega, \mathbb{R}^2)$ be such that $(u_n, v_n) \rightharpoonup (u_0, v_0)$ weakly in $W_0^{1,Q}(\Omega, \mathbb{R}^2)$. We shall prove the convergence

$$\int_\Omega f_1(\xi, u_n, v_n)(u_n - u_0) d\xi \rightarrow 0, \int_\Omega f_2(\xi, u_n, v_n)(u_n - u_0) d\xi \rightarrow 0, \text{ as } n \rightarrow +\infty. \tag{4.3}$$

Let $\varepsilon > 0, \alpha > \alpha_0, s > 1$, and $s' = s/(s - 1)$. By (2.1) and Hölder’s inequality, we have

$$\begin{aligned} \left| \int_\Omega f_1(\xi, u_n, v_n)(u_n - u_0) d\xi \right| &\leq \varepsilon \left\| (u_n, v_n) \right\|_Q^{Q-1} \|u_n - u_0\|_{Q'} \\ &\quad + C \left(\int_\Omega e^{\alpha s |w_n|^{Q'}} d\xi \right)^{1/s} \|u_n - u_0\|_{s'}. \end{aligned}$$

Thus, invoking (2.3) and the compactness of the embedding $W_0^{1,Q}(\Omega) \hookrightarrow L^{s'}(\Omega)$, we obtain

$$\begin{aligned}
 \left| \int_{\Omega} f_1(\xi, u_n, v_n)(u_n - u_0) d\xi \right| &\leq \varepsilon C + C \left(\int_{\Omega} e^{s\alpha 2^{Q'} - 1(|u_n|^{Q'} + |v_n|^{Q'})} d\xi \right)^{\frac{1}{s}} o_n(1) \\
 &\leq \varepsilon C + C \left(\int_{\Omega} e^{2^{Q'} s\alpha |u_n|^{Q'}} d\xi \right)^{\frac{1}{2s}} \left(\int_{\Omega} e^{2^{Q'} s\alpha |v_n|^{Q'}} d\xi \right)^{\frac{1}{2s}}.
 \end{aligned}
 \tag{4.4}$$

Due to

$$\int_{\Omega} e^{2^{Q'} s\alpha |u_n|^{Q'}} d\xi \leq \int_{\Omega} e^{\left(\sigma^{Q'} s\alpha |w_n|^{Q'} \left(\frac{|u_n|}{\|w_n\|} \right)^{Q'} \right)} d\xi,$$

we can choose $\delta > 0$ such that

$$\sigma^{Q'} \alpha_0 \left\| (u_n, v_n) \right\|^{Q'} \leq \alpha_Q - \delta, \text{ for sufficiently large } n \in \mathbb{N}.$$

Therefore, we have

$$\sigma^{Q'} s\alpha \left\| (u_n, v_n) \right\|^{Q'} < \alpha_Q, \text{ for sufficiently large } n \in \mathbb{N},$$

where $\alpha > \alpha_0$ is close to α_0 and $s > 1$ is close to 1.

It then follows from (1.2) that it suffices to show that the following holds

$$\int_{\Omega} e^{\sigma^{Q'} s\alpha |u_n|^{Q'}} d\xi \leq C.$$

Similarly, we can get

$$\int_{\Omega} e^{\sigma^{Q'} s\alpha |v_n|^{Q'}} d\xi \leq C.$$

As above, we get (4.3). Finally, we define

$$\Phi(u, v) := \frac{1}{Q} \mathcal{K}_1(\|u\|^Q) + \frac{1}{Q} \mathcal{K}_2(\|v\|^Q),$$

where \mathcal{K}_i is the convexity, when (\mathcal{K}) holds.

Due to weak lower semicontinuity, we have

$$\frac{1}{Q} \mathcal{K}_1(\|u_0\|^Q) + \frac{1}{Q} \mathcal{K}_2(\|v_0\|^Q) \leq \frac{1}{Q} \liminf_{n \rightarrow +\infty} \mathcal{K}_1(\|u_n\|^Q) + \frac{1}{Q} \liminf_{n \rightarrow +\infty} \mathcal{K}_2(\|v_n\|^Q).
 \tag{4.5}$$

Moreover, by virtue of (4.3) and convexity of $\Phi(u, v)$, we obtain

$$\begin{aligned}
 \Phi(u_0, v_0) - \Phi(u_n, v_n) &\geq \Phi'(u_n, v_n)(u_0 - u_n, v_0 - v_n) \\
 &= I'_\lambda(u_n, v_n)(u_0 - u_n, v_0 - v_n) + \lambda \int_{\Omega} f_1(\xi, u_n, v_n)(u_0 - u_n) d\xi \\
 &\quad + \lambda \int_{\Omega} f_2(\xi, u_n, v_n)(v_0 - v_n) d\xi.
 \end{aligned}$$

Therefore, we have

$$\Phi(u_0, v_0) + o_n(1) \geq \Phi(u_n, v_n)$$

and we get

$$\begin{aligned} \frac{1}{Q} \mathcal{K}_1(\|u_0\|^Q) + \frac{1}{Q} \mathcal{K}_2(\|v_0\|^Q) &\geq \limsup_{n \rightarrow +\infty} \Phi(u_n, v_n) \\ &\geq \frac{1}{Q} \liminf_{n \rightarrow +\infty} \mathcal{K}_1(\|u_n\|^Q) + \frac{1}{Q} \liminf_{n \rightarrow +\infty} \mathcal{K}_2(\|v_n\|^Q). \end{aligned}$$

This fact together with (4.5), yields the contradiction. Therefore,

$$\frac{1}{Q} \mathcal{K}_1(\|u_n\|^Q) \rightarrow \frac{1}{Q} \mathcal{K}_1(\|u_0\|^Q) \quad \text{and} \quad \frac{1}{Q} \mathcal{K}_2(\|v_n\|^Q) \rightarrow \frac{1}{Q} \mathcal{K}_2(\|v_0\|^Q), \text{ as } n \rightarrow +\infty.$$

We can conclude that

$$\|u_n\|^Q \rightarrow \|u_0\|^Q \text{ and } \|v_n\|^Q \rightarrow \|v_0\|^Q,$$

since $K_1(t)$ and $K_2(t)$ are increasing for $t > 0$, as $n \rightarrow +\infty$. Therefore, $(u_n, v_n) \rightarrow (u_0, v_0)$ strongly in E , and the proof is complete. \square

5 Proof of Theorem 1.1

We claim that

$$c^* := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I_\lambda(\gamma(t)), \tag{5.1}$$

where

$$\Gamma := \left\{ \gamma \in C([0, 1], W_0^{1,Q}(\Omega, \mathbb{R}^2)) : \gamma(0) = (0, 0) \text{ and } I_\lambda(\gamma(1)) < 0 \right\}.$$

If we assume that (5.1) holds, then Lemmas 3.1 and 4.1 and the Mountain pass lemma yield the existence of nontrivial critical points of I_λ .

Lemma 5.1 *Assume that*

$$\eta > \max \left\{ \eta_1, \frac{(Q\eta_1)^{\frac{\mu}{Q}}}{\mu} \left(\frac{\sigma^{\sigma Q+1} \alpha_0^{\sigma(Q-1)} \lambda \pi (\mu - Q)}{v_0 \alpha_Q^{\sigma(Q-1)} (\mu - \sigma Q)} \right)^{\frac{\mu-Q}{Q}} \right\}, \tag{5.2}$$

where

$$\eta_1 := [\mathcal{K}_1(2\varepsilon^Q \pi) + \mathcal{K}_2(2\varepsilon^Q \pi)] / (Q\lambda \pi) \quad \text{and} \quad v_0 := \min \{k_1, k_2\}.$$

Then the following inequality holds

$$c^* < \frac{v_0(\mu - \sigma Q)\alpha_Q^{\sigma(Q-1)}}{\alpha_0^{\sigma(Q-1)}Q\mu\sigma^{\sigma Q+1}}. \tag{5.3}$$

Proof In order to prove (5.3), let $\varepsilon > 0$ be so small that there exists a cut-off function $\psi_\varepsilon \in C_0^\infty(B_R(\xi_0))$ such that

$$0 \leq \psi_\varepsilon \leq 1, \text{supp}(\psi_\varepsilon) \subset B_\varepsilon(0), \psi_\varepsilon \equiv 1 \text{ on } B_{\frac{\varepsilon}{2}}(0), |\nabla\psi_\varepsilon| \leq \frac{4}{\varepsilon}.$$

Then we have

$$\|\psi_\varepsilon\|^Q = \int_{B_\varepsilon(0)} |\nabla\psi_\varepsilon|^Q d\xi + \int_{B_\varepsilon(0)} \psi_\varepsilon^Q d\xi \leq 2|B_\varepsilon(0)| = 2|B_1(0)|\varepsilon^Q = 2\varepsilon^Q\pi.$$

On the other hand, since $\eta > \eta_1$, we obtain by (F_3) ,

$$I_\lambda(\psi_\varepsilon, \psi_\varepsilon) < \frac{\mathcal{K}_1(2\varepsilon^Q\pi) + \mathcal{K}_2(2\varepsilon^Q\pi)}{Q} - \lambda\eta_1\pi = 0. \tag{5.4}$$

By the definition of $\gamma(t) := (t\psi_\varepsilon, t\psi_\varepsilon)$, we get the path $\gamma : [0, 1] \rightarrow W_0^{1,Q}(\Omega, \mathbb{R}^2)$. Then $\gamma \in \Gamma$ by (5.4), and we obtain

$$\begin{aligned} c^* &\leq \max_{t \in [0,1]} \left[\frac{1}{Q}\mathcal{K}_1(t^Q\|\psi_\varepsilon\|^Q) + \frac{1}{Q}\mathcal{K}_2(t^Q\|\psi_\varepsilon\|^Q) - \lambda \int_\Omega F(t\psi_\varepsilon, t\psi_\varepsilon) d\xi \right] \\ &\leq \max_{t \in [0,1]} \left[\left(\frac{\mathcal{K}_1(2\varepsilon^Q\pi) + \mathcal{K}_2(2\varepsilon^Q\pi)}{Q} \right) t^Q - \lambda\eta t^\mu \int_{B_{\frac{\varepsilon}{2}}(0)} \psi_\varepsilon^\mu d\xi \right] \leq \lambda\pi \max_{t \in [0,+\infty)} [\eta_1 t^Q - \eta t^\mu], \end{aligned}$$

where \mathcal{K}_1 and \mathcal{K}_2 are convex. Consequently, we get

$$\max_{t \in [0,+\infty)} [\eta_1 t^Q - \eta t^\mu] = \frac{1}{\eta^{\frac{Q}{\mu-Q}}} (\mu - Q) Q^{\frac{Q}{\mu-Q}} \left(\frac{\eta_1}{\mu} \right)^{\frac{\mu}{\mu-Q}}.$$

Moreover, we have

$$c^* \leq \frac{\lambda\pi}{\eta^{\frac{Q}{\mu-Q}}} (\mu - Q) Q^{\frac{Q}{\mu-Q}} \left(\frac{\eta_1}{\mu} \right)^{\frac{\mu}{\mu-Q}}.$$

Therefore (5.3) holds when η satisfies (5.2). □

Acknowledgements Li was supported by the Graduate Scientific Research Project of Changchun Normal University (SGSRPCNU [2022], Grant No. 059). Liang was supported by the Foundation for China Postdoctoral Science Foundation (Grant No. 2019M662220), the Research Foundation of Department of

Education of Jilin Province (Grant No. JJKH20211161KJ), and the Natural Science Foundation of Jilin Province (Grant no. YDZJ202201ZYTS582). Repovš was supported by the Slovenian Research Agency Program No. P1-0292 and Grants Nos. N1-0278, N1-0114, and N1-0083. The authors thank the anonymous referees for their suggestions and comments.

Declarations

Conflict of interest The authors have no competing interests to declare that are relevant to the content of this article.

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