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Limits of Manifolds in the Gromov–Hausdorff Metric Space

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Abstract. We apply the Gromov–Hausdorff metric d_G for characterization of certain generalized manifolds. Previously, we have proven that with respect to the metric d_G, generalized n-manifolds are limits of spaces which are obtained by gluing two topological n-manifolds by a controlled homotopy equivalence (the so-called 2-patch spaces). In the present paper, we consider the so-called *manifold-like* generalized n-manifolds X^n, introduced in 1966 by Mardešić and Segal, which are characterized by the existence of δ-mappings f_δ of X^n onto closed manifolds M^n_δ, for arbitrary small $\delta > 0$, i.e., there exist onto maps $f_\delta \colon X^n \to M^n_\delta$ such that for every $u \in M^n_\delta$, $f^{-1}_\delta(u)$ has diameter less than δ. We prove that with respect to the metric d_G, manifold-like generalized n-manifolds X^n are limits of topological n-manifolds M^n_i. Moreover, if topological n-manifolds M^n_i satisfy a certain local contractibility condition $\mathcal{M}(\varrho, n)$, we prove that generalized n-manifold X^n is resolvable.

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Keywords. Gromov–Hausdorff metric, Gromov topological moduli space, manifold-like generalized manifold, absolute neighborhood retract, cell-like map, δ-map, structure map, controlled surgery sequence, ε-homotopy equivalence, 2-patch space, periodic surgery spectrum $\mathbb{L}$.

1. Introduction

This paper is a continuation of our systematic study of the characterization problem for generalized n-manifolds, $n \geq 5$ (see Cavicchioli *et al.* [5,6] and Hegenbarth and Repovš [23–28]). This is a very important class of spaces which in the algebraic sense strongly resemble topological manifolds, whereas in the geometric sense they can fail to be locally Euclidean at any point (see, e.g., Cannon [4], Edwards [11], and Repovš [42–44]).

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Definition 1.1. A *generalized n-manifold* X^n is an n-dimensional metric absolute neighborhood retract (ANR) X^n with local homology

$$H_*(X^n, X^n \setminus \{x\}; \mathbb{Z}) \cong H_*(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}; \mathbb{Z}), \text{ for every } x \in X.$$

We shall only consider *oriented* generalized n-manifolds *without boundary* (i.e., $H_n(X^n, X^n \setminus \{x\}; \mathbb{Z}) \cong \mathbb{Z}$, for every $x \in X^n$). Throughout the paper, we shall be assuming that $n \geq 5$.

Definition 1.2. Given any $\delta > 0$, a continuous map $f_\delta \colon X \to Y$ of a metric space X onto a topological space Y is called a *δ-map* if for every point $y \in Y$, the preimage $f_\delta^{-1}(y)$ has diameter $< \delta$.

More than half a century ago, Mardešić and Segal [30, Theorem 1] proved the following very nice characterization result for generalized manifolds in terms of δ-maps.

Theorem 1.3. *Let X^n be a compact n-dimensional metric ANR such that for every $\delta > 0$, there exists a δ-map $f_\delta \colon X^n \to M^n_\delta$ of X^n onto some (triangulated) oriented closed topological n-manifold M^n_δ. Then, X^n is a generalized n-manifold.*

Definition 1.4. Mardešić and Segal called such a generalized n-manifold X^n *manifold-like*. We shall call such maps $f_\delta \colon X^n \to M^n_\delta$ *structure maps*.

Remark 1.5. Since every topological n-manifold (except for nonsmoothable 4-manifolds) admits a handlebody decomposition (see Quinn [37]), we shall hereafter neglect "triangulated."

Let d_G be the *Gromov–Hausdorff distance* which is a complete metric on the set of all isometry classes of compact metric spaces. (Details are given in Sect. 2, for an overview see Ferry [14, §29].) In our previous paper Hegenbarth–Repovš [25, §4.3], we proved that with respect to metric d_G, every generalized n-manifold X^n is the limit of 2-patch spaces, defined by Bryant *et al.* [3].

In this paper, we shall prove the following new characterization result for manifold-like generalized n-manifolds—an approximation by topological n-manifolds in terms of the Gromov–Hausdorff metric d_G.

Theorem 1.6. *(Approximation Theorem) For every manifold-like generalized n-manifold X^n and every $\delta > 0$, there exists a topological n-manifold M^n_δ such that $d_G(X^n, M^n_\delta) < \delta$.*

Remark 1.7. The metric on generalized n-manifold X^n is induced by a fixed embedding $X^n \hookrightarrow \mathbb{R}^m$ of X^n into some Euclidean m-space $\mathbb{R}^m$, for a sufficiently large dimension $m \in \mathbb{N}$. The metric on topological n-manifold M^n_δ is then induced by an embedding $M^n_\delta \hookrightarrow N^m_{X^n}$ of M^n_δ into a small neighborhood $N^m_{X^n} \subset \mathbb{R}^m$ of X^n in $\mathbb{R}^m$ (see Sect. 2 for more details).

Edwards [11] obtained a fundamental criterion for a generalized n-manifold X^n to be a topological n-manifold. The first (sufficient) condition is the existence of a *cell-like map* $f\colon M^n \to X^n$, where M^n is a closed topological n-manifold, also called the *(cell-like) resolution of* X^n (see, e.g., Mitchell and Repovš [32]). By the uniqueness result of Quinn ([36, Proposition 3.2.3]), any two resolutions $f_1\colon M_1^n \to X^n$ and $f_2\colon M_2^n \to X^n$ of X^n are equivalent, i.e., for every $\varepsilon > 0$, there exists a homeomorphism $h_\varepsilon\colon M_1^n \to M_2^n$ such that $d(f_1, f_2 \circ h_\varepsilon) < \varepsilon$. The second (sufficient) condition is a general position type of property, the so-called *disjoint disks property* of X^n (see, e.g., Cavicchioli *et al.* [6]).

Quinn [38,39] developed a controlled surgery theory and constructed a surgery obstruction $i(X^n) \in \mathbb{Z}$ to existence of resolutions of generalized n-manifolds X^n. It is convenient to the consider $I(X^n) := 1 + 8i(X^n)$, called the *resolution index* (this appears naturally, passing from the quadratic $\mathbb{L}$-spectrum to the symmetric $\mathbb{L}$-spectrum, see Ranicki [40]). So $I(X^n) = 1$ if and only if X^n admits a (cell-like) resolution.

There are no known general methods for calculating Quinn's resolution index $I(X^n)$, like there are for other invariants. In this paper, we shall show that it vanishes for a certain class of manifold-like generalized n-manifolds, and thus, we shall prove that they are resolvable (see Theorem 1.9). First, we need some more notations (see Ferry [14, §29]).

Definition 1.8. A function $\varrho\colon [0, R) \to [0, \infty)$ is called *contractible* if for every t, $\varrho(t) \geq t$, and ϱ is continuous at 0. Let $\mathcal{M}(\varrho, n)$ denote the set of all compact metric spaces M of dimension $\leq n$, such that for every $x \in M$, the r-ball $B_r(x) = \{y \in M \mid d(x, y) \leq r\}$ contracts to $\{x\}$ inside the $\varrho(r)$-ball $B_{\varrho(r)}(x)$.

The following is the second main result of our paper.

Theorem 1.9. *(Resolution Theorem) Let X^n be a generalized n-manifold and fix an embedding $i\colon X^n \hookrightarrow \mathbb{R}^m$ for some $m \geq n \geq 5$. Let $\varrho\colon [0, R) \to [0, \infty)$ be a contractible function and suppose that for every small $\delta > 0$, there is a structure map $f_\delta : X^n \to M_\delta^n$ such that $M_\delta^n \in \mathcal{M}(\varrho, n)$ with respect to the metric defined in Theorem 1.6. Then, X^n is resolvable.*

Remark 1.10. We recall that the metric on generalized n-manifold X^n (resp. topological n-manifold M_δ^n) is induced by the embedding $X^n \hookrightarrow \mathbb{R}^m$ (resp. $M_\delta^n \hookrightarrow N_{X^n}^m \subset \mathbb{R}^m$).

As an application, consider the following nice result of Ferry [14, Proposition 29.38].

Theorem 1.11. *Suppose that $X = \varinjlim\{M_i^n\}$, where $\{M_i^n\} \subset \mathcal{M}(\varrho, n)$, in the Gromov–Hausdorff metric. If* $\dim X < \infty$*, then X is a generalized n-manifold.*

It now follows by our Theorem 1.9 that the space X in Theorem 1.11 is in fact, a *resolvable* generalized n-manifold X. For some related previous results on limits in the Gromov–Hausdorff metric space see Dranishnikov and Ferry [7,8] Dranishnikov *et al.* [9], Engel [12], Ferry [13,15,16], Ferry and Okun [18], Grove *et al.* [22], Kawamura [29], and Moore [33].

We conclude the introduction with the following very interesting open problem related to our Theorem 1.9. Recall that there are plenty of nonresolvable generalized n-manifolds—see, e.g., Cavicchioli *et al.* [5]. How about *manifold-like* generalized n-manifolds?

Question 1.12. *Does there exist, for any $n \geq 5$, a nonresolvable manifold-like generalized n-manifold?*

2. Proof of Theorem 1.6

Let X^n be a manifold-like generalized n-manifold. For any $\delta > 0$, let $f_\delta \colon X^n \to M^n_\delta$ be a *structure map* from Definition 1.4. We shall invoke the following result due to Eilenberg (see, e.g., Ferry [14, Corollary 29.10]).

Proposition 2.1. *For every $\delta > 0$, there exist a structure map $f_\delta \colon X^n \to M^n_\delta$ and a continuous map $g_\delta \colon M^n_\delta \to X^n$ such that $g_\delta \circ f_\delta \colon X^n \to X^n$ is δ-homotopic to $Id_{X^n} \colon X^n \to X^n$.*

This is a special case where also the following fact holds.

Supplement 2.2. *The structure map $f_\delta \colon X^n \to M^n_\delta$ from Proposition 2.1 is a homotopy equivalence with the inverse $g_\delta \colon M^n_\delta \to X^n$.*

Proof of Proposition 2.1. The induced map

$$(f_\delta)_* \colon H_*(X^n;\mathbb{Z}) \to H_*(M^n_\delta;\mathbb{Z})$$

is injective since $g_\delta \circ f_\delta \sim Id_{X^n}$. Therefore, the composition

$$H_n(X^n;\mathbb{Z}) \overset{(f_\delta)_*}{\to} H_n(M^n_\delta;\mathbb{Z}) \overset{(g_\delta)_*}{\to} H_n(X^n;\mathbb{Z})$$

is the identity, $(g_\delta)_* \circ (f_\delta)_* = (Id_{X^n})_*$, and we have

$$H_n(M^n_\delta;\mathbb{Z}) \cong \mathbb{Z}, \quad (g_\delta)_*([M^n_\delta]) = [X^n],$$

if we choose the fundamental class appropriately. It follows by duality that the induced map

$$(f_\delta)_* \colon H_*(X^n;\mathbb{Z}) \to H_*(M^n_\delta;\mathbb{Z})$$

is also surjective and that $f_\delta \colon X^n \to M^n_\delta$ and $g_\delta \colon M^n_\delta \to X^n$ are both of degree 1. In particular, since the map $f_\delta \colon X^n \to M^n_\delta$ is of degree 1, it now follows that the induced map

$$(f_\delta)_* \colon \pi_1(X^n) \to \pi_1(M^n_\delta)$$

is surjective (see Browder [1, Proposition 1.2]). Since $(f_\delta)_* \colon \pi_1(X^n) \to \pi_1(M^n_\delta)$ is also injective, it is in fact, an isomorphism.

Now, arguing as above, we can show that $f_\delta \colon X^n \to M^n_\delta$ induces isomorphisms in homology with coefficients in group rings. It therefore follows by Ferry [13, Theorem 7.4] that $f_\delta \colon X^n \to M^n_\delta$ is indeed a homotopy equivalence with the inverse $g_\delta \colon M^n_\delta \to X^n$. This completes the proof of Proposition 2.1.

□

Definition 2.3. The *Gromov–Hausdorff distance* between any compact metric spaces X and Y is defined as follows: For any closed subsets X and Y of a compact metric space (Z, d), and any $\delta > 0$, define their neighborhoods

$$N_\delta(X) := \{z \in Z \mid d(z, X) < \delta\},$$

and

$$N_\delta(Y) := \{z \in Z \mid d(z, Y) < \delta\}$$

and define the following distances

$$d_Z(X, Y) := \inf\{\delta > 0 \mid X \subset N_\delta(Y) \textit{ and } Y \subset N_\delta(X)\}$$

and

$$d_G(X, Y) := \inf\{d_Z(X, Y) \mid X, Y \textit{ are isometrically embedded in } Z\},$$

where Z ranges over all compact metric spaces.

Remark 2.4. The Gromov–Hausdorff convergence is a notion of convergence of metric spaces which is a generalization of the classical Hausdorff convergence. The Gromov–Hausdorff distance was introduced in 1975 by Edwards [10] and then rediscovered and generalized in 1981 by Gromov [21] (see also Tuzhilin [46]).

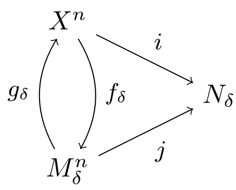
To determine $d_G(X^n, M^n_\delta)$ for a structure map $f_\delta\colon X^n \to M^n_\delta$, the choice of the metric is important. We choose an embedding $X^n \hookrightarrow \mathbb{R}^m$, and take on X^n the metric induced from $\mathbb{R}^m$. It is important to note that the property of $f_\delta\colon X^n \to M^n_\delta$ being a structure map does not depend on the choice of the metric on M^n_δ. It will be appropriately chosen below.

Let $f_\delta\colon X^n \to M^n_\delta$ be a structure map with the inverse $g_\delta\colon M^n_\delta \to X^n$, such that $g_\delta \circ f_\delta$ is δ-homotopic to Id_{X^n} for a given small $\delta > 0$ (see Proposition 2.1). In the sequel, let

$$i\colon X^n \hookrightarrow N_\delta := N_\delta(X^n \hookrightarrow \mathbb{R}^m)$$

denote the inclusion of X^n into a δ-neighborhood N_δ of X^n in $\mathbb{R}^m$.

Since by hypothesis, X^n is manifold-like, it follows that for arbitrary small $\delta' > 0$, there exists an embedding $j\colon M^n_\delta \hookrightarrow N_\delta$ with $d(i \circ g_\delta, j) < \delta'$ (see Rourke and Sanderson [45, General Position Theorem for Maps 5.4]). These maps can be represented by the following diagram



(2.1)

We choose on M^n_δ the metric induced on $j(M^n_\delta) \subset \mathbb{R}^m$. Since

$$d(i \circ g_\delta, j) < \delta',$$

we can deduce the following

$$d(i(x), j(M^n_\delta)) \leq d(i(x), (i \circ g_\delta \circ f_\delta)(x)) + d((i \circ g_\delta \circ f_\delta)(x), j(M^n_\delta)) < \delta + \delta',$$

i.e.,

$$i(X^n) \subset N_{\delta+\delta'}(j(M^n_\delta) \subset \mathbb{R}^m)$$

(see also Remark 2.5).

Of course, N_δ and $N_{\delta+\delta'}(j(M^n_\delta) \subset \mathbb{R}^m)$ belong to a compact subset Z of $\mathbb{R}^m$ with the induced metric. We obtain the following

$$d_G(X^n, M^n_\delta) \leq d_Z(X^n, M^n_\delta) < \delta + \delta'.$$

Now δ and δ' can be chosen to be arbitrarily small; thus, we have completed the proof of Theorem 1.6. □

Remark 2.5. Recall that

$$d(z, A) := \inf\{d(z, a) \mid a \in A\},$$

where $A \subset Z$ is a compact subset of the metric space Z. For $z, z' \in Z$, the inequality

$$d(z', a) \leq d(z, z') + d(z, a)$$

implies the inequality

$$d(z', A) \leq d(z', z) + d(z, A),$$

which was used above.

3. Proof of Theorem 1.9

In this section, we shall apply the controlled surgery sequence to prove Theorem 1.9. For more details on this important subject, we refer to Bryant et al. [2], Cavicchioli et al. [6], Ferry [17,19,20], Mio [31], Pedersen et al. [34], Pedersen and Yamasaki [35], Quinn [38,39], Ranicki and Yamasaki [41], and Yamasaki [47].

Let $\mathbb{L}$ denote the periodic $\mathbb{L}$-spectrum, i. e. $\mathbb{L}_0 = \mathbb{Z} \times G/TOP$, and $\mathbb{L}^+$ is its connected covering spectrum with $\mathbb{L}^+_0 = G/TOP$. Now, if $\mathcal{S}_\varepsilon\begin{pmatrix} X^n \\ \downarrow Id \\ X^n \end{pmatrix} \neq \emptyset$, then there exists an exact sequence

$$\cdots \to H_{n+1}(X^n; \mathbb{L}^+) \to H_{n+1}(X^n; \mathbb{L}) \to \mathcal{S}_\varepsilon\begin{pmatrix} X^n \\ \downarrow Id \\ X^n \end{pmatrix} \to H_n(X^n; \mathbb{L}^+) \to \dots$$

Elements of $\mathcal{S}_\varepsilon\begin{pmatrix} X^n \\ \downarrow Id \\ X^n \end{pmatrix}$ are equivalence classes of ε-homotopy equivalences $M^n \xrightarrow{h} X^n$ (measured in X^n), with M^n a closed (oriented) topological n-manifold.

Definition 3.1. Two elements

$$M^n_1 \xrightarrow{h_1} X^n, M^n_2 \xrightarrow{h_2} X^n \in \mathcal{S}_\varepsilon\begin{pmatrix} X^n \\ \downarrow Id \\ X^n \end{pmatrix}$$

are said to be *ε-related* if there exists a homeomorphism $\varphi\colon M^n_1 \to M^n_2$ such that $h_2 \circ \varphi$ is ε-homotopic to h_1.

Remark 3.2. Being ε-related does not define an equivalence relation, but it is a part of the following assertion: There exists an $\varepsilon_0 > 0$ depending only on X^n, such that for every $\varepsilon \leq \varepsilon_0$, this becomes an equivalence relation.

For $p+q = n+1$, it follows from the spectral sequences

$$E^2_{pq} = H_p(X^n; \pi_q(\mathbb{L})) \Rightarrow H_{p+q}(X^n; \mathbb{L})$$

and

$$E^{+2}_{pq} = H_p(X^n; \pi_q(\mathbb{L}^+)) \Rightarrow H_{p+q}(X^n; \mathbb{L}^+)$$

that

$$E^{+2}_{pq} = E^2_{pq},$$

hence

$$H_{n+1}(X^n; \mathbb{L}^+) \cong H_{n+1}(X^n; \mathbb{L}).$$

Moreover, $H_n(X^n; \mathbb{L}^+) \to H_n(X^n; \mathbb{L})$ must be injective. It follows that if

$$\mathcal{S}_\varepsilon \begin{pmatrix} X^n \\ \downarrow Id \\ X^n \end{pmatrix} \neq \emptyset.$$

then it consists of only one element

$$\text{card}\left[\mathcal{S}_\varepsilon \begin{pmatrix} X^n \\ \downarrow Id \\ X^n \end{pmatrix} \right] = 1.$$

Proposition 3.3. *Let X^n be a generalized n-manifold. Then, $I(X^n) = 1$ if and only if*

$$\mathcal{S}_\varepsilon \begin{pmatrix} X^n \\ \downarrow Id \\ X^n \end{pmatrix} \neq \emptyset,$$

i.e., for every $\varepsilon \leq \varepsilon_0$, there exists an ε-homotopy equivalence $M^n \xrightarrow{h} X^n$.

Proof. The proof is standard, see e.g., Mio [31, §3] or Bryant *et al.* [2, p. 444].

□

In order to prove Theorem 1.9, we have to show that for each $\varepsilon \leq \varepsilon_0$, there exists for every $\mathcal{M}(\varrho, n)$-like generalized manifold X^n, an ε-homotopy equivalence $h_\varepsilon \colon M^n \to X^n$. This follows from Theorem 1.6 and Ferry [14, Theorem 29.20].

Theorem 3.4. *Let $\varrho \colon [0, R) \to [0, \infty)$ be a contractible function and let Y and Z be any compact metric spaces. Then, for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $Y, Z \in \mathcal{M}(\varrho, n)$ and $d_G(Y, Z) < \delta$, then Y and Z are ε-homotopy equivalent. Here, $\delta = \delta(\varepsilon, \varrho)$ depends on ε and ϱ, but not on Y, Z.*

Let us provide some more details: We equip generalized n-manifold X^n with the metric given by an embedding $X^n \hookrightarrow \mathbb{R}^m$ of X^n into some $\mathbb{R}^m$, for a sufficiently large $m \in \mathbb{N}$, see Theorem 1.6 and Remark 1.7. By Ferry [14, Theorem 29.14], X^n with this metric belongs to $\mathcal{M}(\varrho, n)$ for some contractible function $\varrho\colon [0, R) \to [0, \infty)$.

By hypothesis, we can now choose a sequence $\{\varepsilon_i > 0\}_{i\in\mathbb{N}}$ such that

$$\lim_{i\to+\infty} \varepsilon_i = 0, \quad \sum_{i=1}^{\infty} \varepsilon_i < \infty,$$

and then invoking Theorem 3.4, obtain a sequence

$$\{\delta_i := \delta_i(\varepsilon_i, \varrho) > 0\}_{i\in\mathbb{N}}.$$

By Theorem 1.6, then there exists a sequence of closed topological n-manifolds $\{M^n_{\delta_i}\}_{i\in\mathbb{N}} \subset \mathcal{M}(\varrho, n)$, with respect to the metric obtained by embedding $M^n_{\delta_i} \hookrightarrow N^m_{X^n} \subset \mathbb{R}^m$ each $M^n_{\delta_i}$ into a small neighborhood $N^m_{X^n}$ of generalized n-manifold X^n in $\mathbb{R}^m$, such that

$$d_G(M^n_{\delta_i}, X^n) < \delta_i, \quad \text{for every} \quad i \in \mathbb{N}.$$

Therefore, every topological n-manifold $M^n_{\delta_i}$ is ε_i-homotopy equivalent to X^n. This proves Theorem 1.9. □

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