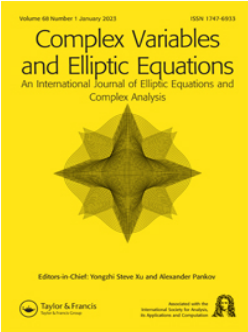

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The Nehari manifold approach for singular equations involving the p(x)-Laplace operator

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ABSTRACT
We study the following singular problem involving the p(x)-Laplace operator $\Delta_{p(x)}u = div(|\nabla u|^{p(x)-2}\nabla u)$, where $p(x)$ is a nonconstant continuous function,

$$(\mathrm{P}_\lambda)\begin{cases} -\Delta_{p(x)}u = a(x)|u|^{q(x)-2}u(x) + \dfrac{\lambda b(x)}{u^{\delta(x)}} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Here, Ω is a bounded domain in $\mathbb{R}^{N\geq 2}$ with C^2-boundary, λ is a positive parameter, $a(x), b(x) \in C(\bar{\Omega})$ are positive weight functions with compact support in Ω, and $\delta(x)$, $p(x)$, $q(x) \in C(\bar{\Omega})$ satisfy certain hypotheses (A_0) and (A_1). We apply the Nehari manifold approach and some new techniques to establish the multiplicity of positive solutions for problem (P_λ).

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1. Introduction

The aim of this paper is to study the following inhomogeneous equation

$$(\mathrm{P}_\lambda)\begin{cases} -\Delta_{p(x)}u = a(x)|u|^{q(x)-2}u(x) + \dfrac{\lambda b(x)}{u^{\delta(x)}} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Here, operator $\Delta_{p(x)}u := div(|\nabla u|^{p(x)-2}\nabla u)$ is the $p(x)$-Laplacian, $p(x)$ is a nonconstant continuous function, Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 2)$ with C^2-boundary, λ is a positive parameter, $a(x), b(x) \in C(\overline{\Omega})$ are positive weight functions with compact support in Ω, and $\delta(x), p(x), q(x) \in C(\overline{\Omega})$ satisfy the following conditions

(A_0) $0 < 1 - \delta(x) < p(x) < q(x) < p^*(x)$;
(A_1) $0 < 1 - \delta^- \leq 1 - \delta^+ < p^- \leq p^+ < q^- \leq q^+$.

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Here, $p^*(x) := Np(x)/(N - p(x)), \delta^+ := \operatorname{ess\,sup} \delta(x), \delta^- := \operatorname{ess\,inf} \delta(x)$, and analogous definitions hold for p^-, p^+, q^-, and q^+.

Partial differential equations with variable exponents are very interesting and active topics. The motivation for this type of problems was stimulated by their various applications in physics – for more details, see [1,2], and in particular the book by Rădulescu–Repovš [3], and the references therein.

Before stating our main result, we review the key literature concerning singular partial differential equations with variable exponents. Zhang [4] proved the existence of solutions for the purely singular problem. Using variational methods, Saoudi [5] proved the existence of a superlinear singular equation with variable exponent. Fan [6] investigated the multiplicity of solutions using topological methods. In [7,8], variational methods were used to establish the multiplicity of solutions for singular problems with Dirichlet and Neumann conditions, respectively (see also [9]).

The case when p is constant in problem (P_λ) has received more attention and has been approached by different techniques. For a more general presentation, we refer to [10–16] and the references therein.

Some interesting papers on the applications of the Nehari manifold method in a variable exponent problem have recently been published (see, e.g. [17–19]). In the present paper, we generalize the results of Giacomoni et al. [14] and Saoudi [15] to the problem with variable exponent, by using topological methods. Here is the main result of this paper.

Theorem 1.1: *Suppose that conditions* (A_0) *and* (A_1) *are fulfilled. Then there exists* $\lambda_0 > 0$ *such that for every* $\lambda \in (0, \lambda_0)$, *problem* (P_λ) *has at least two positive solutions.*

This paper is organized as follows. In Section 2, we briefly review the properties of generalized Lebesgue–Sobolev spaces. In Section 3, we prove the necessary lemmas. In Section 4, we prove the existence of a minimum for the functional energy E_λ in $\mathcal{N}_\lambda^+$. In Section 5, we prove the existence of a minimum for the functional energy E_λ in $\mathcal{N}_\lambda^-$. Finally, in Section 6, we present the proof of our main result.

2. Generalized Lebesgue–Sobolev spaces

In this section, we recall definitions of functional spaces with variable exponents and properties of the $p(x)$-Laplacian operator which will be used later (for more on this topics, see [3], and for other additional information, see [20]). Let

$$L^{p(\cdot)}(\Omega) = \left\{ u \in S(\Omega) : \int_\Omega |u(x)|^{p(x)}\, \mathrm{d}x < \infty \right\},$$

with the norm

$$|u|_{p(\cdot)} = |u|_{L^{p(\cdot)}(\Omega)} = \inf \left\{ \lambda > 0 : \int_\Omega \left| \frac{u(x)}{\lambda} \right|^{p(x)} \mathrm{d}x \le 1 \right\}.$$

Then $(L^{p(\cdot)}(\Omega), |\cdot|_{p(\cdot)})$ is a reflexive, uniform convex Banach, separable space – for details, see [3,21].

The variable exponent Sobolev space

$$W^{1,p(\cdot)}(\Omega) = \left\{ u \in L^{p(\cdot)}(\Omega) : |\nabla u| \in L^{p(\cdot)}(\Omega) \right\},$$

can be equipped with the norm

$$\|u\| = |u|_{p(\cdot)} + |\nabla u|_{p(\cdot)}, \quad \text{for all } u \in W^{1,p(\cdot)}(\Omega).$$

Note that $W_0^{1,p(\cdot)}(\Omega)$ is the closure of $C_0^{\infty}(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$.

We denote by $L^{q(x)}(\Omega)$ the conjugate space of $L^{p(x)}(\Omega)$, where $\frac{1}{q(x)} + \frac{1}{p(x)} = 1$. For $u \in L^{p(x)}(\Omega)$ and $v \in L^{q(x)}(\Omega)$, the Hölder-type inequality

$$\left| \int_{\Omega} u(x)v(x)\,\mathrm{d}x \right| \le \left(\frac{1}{p^-} + \frac{1}{q^-} \right) |u|_{p(x)} |v|_{q(x)} \tag{1}$$

holds. Recall the following result.

Lemma 2.1: *Consider the mapping $\rho_{p(x)} : L^{p(x)}(\Omega) \to \mathbb{R}$ defined by*

$$\rho_{p(x)}(u) = \int_{\Omega} |u|^{p(x)}\,\mathrm{d}x,$$

where $(u_n), u \in L^{p(x)}(\Omega)$, and $p^+ < \infty$. Then the following relations hold

$$\|u\|_{L^{p(x)}} > 1 \Rightarrow \|u\|_{L^{p(x)}}^{p^-} \le \rho_{p(x)}(u) \le \|u\|_{L^{p(x)}}^{p^+}, \tag{2}$$

$$\|u\|_{L^{p(x)}} < 1 \Rightarrow \|u\|_{L^{p(x)}}^{p^+} \le \rho_{p(x)}(u) \le \|u\|_{L^{p(x)}}^{p^-}, \tag{3}$$

$$\|u_n - u\|_{L^{p(x)}} \to 0 \quad \text{if and only if } \rho_{p(x)}(u_n - u) \to 0. \tag{4}$$

We state the Sobolev embedding theorem.

Theorem 2.1 (see [22,23]): *Let $p \in C(\bar{\Omega})$ with $p(x) > 1$ for each $x \in \bar{\Omega}$ where $\Omega \subset \mathbb{R}^N$ is an open bounded domain with Lipschitz boundary and suppose that $p(x) \le r(x) \le p^*(x)$ and $r \in C(\bar{\Omega})$, for all $x \in \overline{\Omega}$. Then the embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{r(x)}(\Omega)$ is continuous. Also, if $r(x) < p^*(x)$ almost everywhere in $\overline{\Omega}$, then this embedding is compact.*

Let $\rho(x,s)$ be a Carathéodory function satisfying the following condition

$$|\rho(x,s)| \le A \quad \text{for a.e. x} \in \Omega \quad \text{and all } s \in [-s_0, s_0], \tag{5}$$

where $s_0 > 0$ and A is a constant. Recall the following comparison principle.

Lemma 2.2 ([24, Lemma 2.3]): *Let $\rho(x,t)$ be a function satisfying* (5) *and increasing in t. Let $u, v \in W^{1,p(\cdot)}(\Omega)$ satisfy*

$$-\Delta_{p(x)} u + \rho(x,u) \le -\Delta_{p(x)} v + \rho(x,v), \quad \text{for all } x \in \Omega,$$

and assume that $u \le v$ on $\partial\Omega$. Then $u \le v$ in Ω.

Next, we recall the following strong maximum principle.

Theorem 2.2 ([7, Theorem 3.2]): *Suppose that for some* $0 < \alpha < 1$, $u, v \in C^{1,\alpha}(\overline{\Omega})$ *we have* $0 \lneqq u$, $0 \lneqq v$, *and*

$$- \Delta_{p(x)}u - \frac{\lambda}{u^{\delta(x)}} = h(x) \geq g(x) = -\Delta_{p(x)}v - \frac{\lambda}{v^{\delta(x)}}, \tag{6}$$

with $u = v = 0$ *on* $\partial\Omega$, *where* $g, h \in L^{\infty}(\Omega)$ *are such that* $0 \leq g < h$ *pointwise everywhere in* Ω. *Assume that*

$$\frac{\partial u}{\partial \mathbf{n}} > 0 \quad \frac{\partial v}{\partial \mathbf{n}} > 0 \quad \text{on } \partial\Omega, \tag{7}$$

where $\mathbf{n}$ *is the inward unit normal on* $\partial\Omega$. *Then the following strong comparison principle holds:*

$$u > v \quad \textit{in } \Omega, \quad \text{and there is a positive } \epsilon \quad \text{such that } \frac{\partial(u-v)}{\partial \mathbf{n}} \geq \epsilon \quad \textit{on } \partial\Omega. \tag{8}$$

We shall now prove the following result.

Theorem 2.3: *Suppose that the domain* Ω *has the cone property and consider* $p \in C(\overline{\Omega})$. *Assume that* $b \in L^{\alpha(x)}$, $b(x) > 0$ *for* $x \in \Omega$, $\alpha \in C(\overline{\Omega})$ *and* $\alpha^- > 1$, $\alpha_0^- \leq \alpha_0(x) \leq \alpha_0^+$ $(\frac{1}{\alpha(x)} + \frac{1}{\alpha_0(x)} = 1)$, $\delta \in C(\overline{\Omega})$, *and*

$$0 < 1 - \delta(x) < \frac{\alpha(x) - 1}{\alpha(x)} p^*(x), \quad \textit{for all } x \in \overline{\Omega}. \tag{9}$$

Then the embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{1-\delta(x)}_{b(x)}(\Omega)$ *is compact. Moreover, there is a constant* $c_2 > 0$ *such that the following inequality holds*

$$\int_{\Omega} b(x)|u|^{1-\delta(x)}\,\mathrm{d}x \leq c_2(||u||^{1-\delta^-} + ||u||^{1-\delta^+}). \tag{10}$$

Proof: The proof of the first assertion is adopted from [6]. Let $u \in W^{1,p(x)}(\Omega)$ and

$$r(x) = \frac{\alpha(x)}{\alpha(x) - 1}(1 - \delta(x)) = \alpha_0(x)(1 - \delta(x)).$$

Hence, (9) implies that $r(x) < p^*(x)$. Therefore, using Theorem (2.1), we obtain $W^{1,p(x)}(\Omega) \hookrightarrow L^{r(x)}(\Omega)$. So, for $u \in W^{1,p(x)}(\Omega)$, we get $|u|^{1-\delta(x)} \in L^{\alpha_0(x)}(\Omega)$. By (1),

$$\int_{\Omega} b(x)|u|^{1-\delta(x)}\,\mathrm{d}x \leq c_1 |b|_{\alpha(x)} \left||u|^{1-\delta(x)}\right| < \infty.$$

This means that $W^{1,p(x)}(\Omega) \subset L^{1-\delta(x)}(\Omega)$.

On the other hand, if $u_n \rightharpoonup 0$ weakly in $W^{1,p(x)}(\Omega)$, then we have that $u_n \to 0$ strongly in $L^{r(x)}(\Omega)$. Therefore,

$$\int_\Omega b(x)|u_n|^{1-\delta(x)}\,\mathrm{d}x \le c_1|b|_{\alpha(x)}\left||u_n|^{1-\delta(x)}\right| \to 0,$$

hence $|u_n|_{1-\delta(x),b(x)} \to 0$ and we can conclude that

$$W^{1,p(x)}(\Omega) \hookrightarrow L^{1-\delta(x)}_{b(x)}(\Omega).$$

Next, we shall prove inequality (10). First, we have from above

$$\int_\Omega b(x)|u|^{1-\delta(x)}\,\mathrm{d}x \le c_1|b|_{\alpha(x)}\left||u|^{1-\delta(x)}\right| < \infty.$$

Since $1-\delta^- \le 1-\delta(x) \le 1-\delta^+$ and $|u|^{1-\delta(x)} \le |u|^{1-\delta^-} + |u|^{1-\delta^+}$, we obtain

$$\int_\Omega b(x)|u|^{1-\delta(x)}\,\mathrm{d}x \le \int_\Omega b(x)|u|^{1-\delta^-}\,\mathrm{d}x + \int_\Omega b(x)|u|^{1-\delta^+}\,\mathrm{d}x.$$

On the other hand, using (1), (2), (3), (4), and condition $p(x) < (1-\delta^-)\alpha_0(x) \le (1-\delta^+)\alpha_0(x) < p^*(x)$, we get

$$\int_\Omega b(x)|u|^{1-\delta^-}\,\mathrm{d}x \le c_2|b|_{\alpha(x)}\left||u|^{1-\delta(x)}\right|_{\alpha_0(x)} = c_2|b|_{\alpha(x)}|u|^{1-\delta^-}_{(1-\delta^-)\alpha_0(x)} \le c_3||u||^{1-\delta^-}. \tag{11}$$

In the same way, one gets

$$\int_\Omega b(x)|u|^{1-\delta^+}\,\mathrm{d}x \le c_4||u||^{1-\delta^+}. \tag{12}$$

Hence, using (11) and (12), we have

$$\int_\Omega b(x)|u|^{1-\delta(x)}\,\mathrm{d}x \le c_5(||u||^{1-\delta^-} + ||u||^{1-\delta^+})$$

which completes the proof of Theorem 2.3. ■

Theorem 2.4: *Let $p \in C(\overline{\Omega})$, suppose the boundary of domain Ω has the cone property and let $u \in W^{1,p(x)}(\Omega)$. Then there exist nonnegative constants $c_6, c_7, c_8, c_9 > 0$ such that the following inequalities hold:*

$$\int_\Omega a(x)|u|^{q(x)}\,\mathrm{d}x \le \begin{cases} c_6||u||^{q^+} & \text{if } ||u|| > 1,\\ c_7||u||^{q^-} & \text{if } ||u|| < 1,\end{cases}$$

$$\int_\Omega b(x)|u|^{1-\delta(x)}\,\mathrm{d}x \le \begin{cases} c_8||u||^{1-\delta^-} & \text{if } ||u|| > 1,\\ c_9||u||^{1-\delta^+} & \text{if } ||u|| < 1.\end{cases}$$

Proof: Theorem 2.4 follows immediately by Mashiyev et al. [17, Theorem 2.3] and Theorem 2.3. ■

3. Some necessary lemmas

Let us define the functional $E_\lambda : W_0^{1,p(x)}(\Omega) \to \mathbb{R}$ by

$$E_\lambda(u) \stackrel{\text{def}}{=} \int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)}\,dx - \int_\Omega \frac{a(x)|u|^{q(x)}}{q(x)}\,dx - \lambda \int_\Omega \frac{b(x)(u^+)^{1-\delta(x)}}{1-\delta(x)}\,dx. \tag{13}$$

Definition 3.1: We say that $u \in W_0^{1,p(x)}(\Omega)$ is a generalized solution of the equation

$$-\Delta_{p(x)} u = a(x)|u|^{q(x)-2}u(x) + \frac{\lambda b(x)}{u^{\delta(x)}} \tag{14}$$

if for all $\varphi \in C_0^\infty(\Omega)$ and $\operatorname*{ess\,inf}_K u > 0$ for every compact set $K \subset \Omega$,

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla\varphi\,dx = \int_\Omega a(x)|u|^{q(x)-1}\varphi\,dx + \lambda\int_\Omega b(x)u^{-\delta(x)}\varphi\,dx \tag{15}$$

for all $\varphi \in C_0^\infty(\Omega)$.

Obviously, every weak solution of problem (P_λ) is also a generalized solution of Equation (14).

In many problems, such as (P_λ), E_λ is not bounded below on $W_0^{1,p(x)}(\Omega)$, but it is bounded below on the corresponding Nehari manifold which is defined by

$$\mathcal{N}_\lambda := \{u \in W_0^{1,p(x)}(\Omega) \setminus \{0\} : \langle E'_\lambda(u), u\rangle = 0\}.$$

Then $u \in \mathcal{N}_\lambda$ if and only if

$$\int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)}\,dx - \int_\Omega \frac{a(x)|u|^{q(x)}}{q(x)}\,dx - \lambda\int_\Omega \frac{b(x)|u|^{1-\delta(x)}}{1-\delta(x)}\,dx = 0. \tag{16}$$

We note that $\mathcal{N}_\lambda$ contains every solution of problem (P_λ).

It is well known that the Nehari manifold is closely related to the behaviour of the functions $\Phi_u : [0,\infty) \to \mathbb{R}$ defined as $\Phi_u(t) = E_\lambda(tu)$. Such maps are called fibre maps and were introduced by Drabek–Pohozaev [25]. For $u \in W_0^{1,p(x)}(\Omega) \setminus \{0\}$, we define

$$\Phi_u(t) = \int_\Omega \frac{t^{p(x)}|\nabla u|^{p(x)}}{p(x)}\,dx - \int_\Omega \frac{a(x)t^{q(x)}}{q(x)}|u|^{q(x)}\,dx - \lambda\int_\Omega \frac{b(x)t^{1-\delta(x)}|u|^{1-\delta(x)}}{1-\delta(x)}\,dx,$$

$$\Phi'_u(t) = \int_\Omega t^{p(x)-1}|\nabla u|^{p(x)}\,dx - \int_\Omega a(x)t^{q(x)-1}|u|^{q(x)}\,dx - \lambda\int_\Omega b(x)t^{-\delta(x)}|u|^{1-\delta(x)}\,dx,$$

$$\Phi''_u(t) = \int_\Omega (p(x)-1)t^{p(x)-2}|\nabla u|^{p(x)}\,dx - \int_\Omega a(x)(q(x)-1)t^{q(x)-2}|u|^{q(x)}\,dx + \lambda\int_\Omega b(x)\delta(x)t^{-\delta(x)-1}|u|^{1-\delta(x)}\,dx.$$

It is easy to see that $tu \in \mathcal{N}_\lambda$ if and only if $\Phi'_u(t) = 0$ and in particular, $u \in \mathcal{N}_\lambda$ if and only if $\Phi'_u(1) = 0$. Thus it is natural to split $\mathcal{N}_\lambda$ into three parts corresponding to local minima, local maxima and points of inflection defined as follows:

$$\mathcal{N}_\lambda^+ := \{u \in \mathcal{N}_\lambda : \Phi_u''(1) > 0\} = \left\{tu \in W_0^{1,p(x)}(\Omega) \setminus \{0\} : \Phi_u'(t) = 0, \Phi_u''(t) > 0\right\},$$

$$\mathcal{N}_\lambda^- := \{u \in \mathcal{N}_\lambda : \Phi_u''(1) < 0\} = \left\{tu \in W_0^{1,p(x)}(\Omega) \setminus \{0\} : \Phi_u'(t) = 0, \Phi_u''(t) < 0\right\},$$

$$\mathcal{N}_\lambda^0 := \{u \in \mathcal{N}_\lambda : \Phi_u''(1) = 0\} = \left\{tu \in W_0^{1,p(x)}(\Omega) \setminus \{0\} : \Phi_u'(t) = 0, \Phi_u''(t) = 0\right\}.$$

Our first result is the following.

Lemma 3.1: *E_λ is coercive and bounded below on $\mathcal{N}_\lambda$.*

Proof: Let $u \in \mathcal{N}_\lambda$ and $||u|| > 1$. Then, using (2)–(4) and the embeddings from Theorem 2.1, we estimate $E_\lambda(u)$ as follows:

$$\begin{aligned} E_\lambda(u) &= \int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)}\,\mathrm{d}x - \int_\Omega \frac{a(x)|u|^{q(x)}}{q(x)}\,\mathrm{d}x - \lambda \int_\Omega \frac{b(x)|u|^{1-\delta(x)}}{1-\delta(x)}\,\mathrm{d}x \\ &\geq \left(\frac{1}{p^+} - \frac{1}{q^-}\right) \int_\Omega |\nabla u|^{p(x)}\,\mathrm{d}x - \lambda \left(\frac{1}{1-\delta^+} - \frac{1}{q^-}\right) \int_\Omega b(x)|u|^{1-\delta(x)}\,\mathrm{d}x \\ &\geq \left(\frac{1}{p^+} - \frac{1}{q^-}\right) \| u \|^{p^-} - \lambda c_8 \left(\frac{1}{1-\delta^+} - \frac{1}{q^-}\right) ||u||^{1-\delta^+}. \end{aligned}$$

Note that since $0 < \delta^+ < 1$ and $1 - \delta^+ < p^-$, it follows that $E_\lambda(u) \to \infty$ as $||u|| \to \infty$. Therefore, E_λ is coercive and bounded below. ■

Lemma 3.2: *Let u be a local minimizer for E_λ on subsets $\mathcal{N}_\lambda^+$ or $\mathcal{N}_\lambda^-$ of $\mathcal{N}_\lambda$ such that $u \notin \mathcal{N}_\lambda^0$. Then u is a critical point of E_λ.*

Proof: Recall u is a local minimizer for E_λ under the constraint

$$I_\lambda(u) := \langle E_\lambda'(u), u\rangle = 0. \tag{17}$$

Hence, using the theory of Lagrange multipliers, we obtain the existence of $\mu \in \mathbb{R}$ such that

$$E_\lambda'(u) = \mu I_\lambda'(u).$$

Therefore,

$$\langle E_\lambda'(u), u\rangle = \mu\langle I_\lambda'(u), u\rangle = \mu\Phi_u''(1) = 0.$$

So, $u \notin \mathcal{N}_\lambda^0$, hence $\Phi_u''(1) \neq 0$. Consequently, $\mu = 0$. The proof of Lemma 3.2 is thus complete. ■

Lemma 3.3: *There exists λ_0 such that for every $0 < \lambda < \lambda_0$, we have $\mathcal{N}_\lambda^\pm \neq \emptyset$ and $\mathcal{N}_\lambda^0 =$ $\{0\}$.*

Proof: First, by Lemma 3.2, we deduce that $\mathcal{N}_\lambda^{\pm}$ are nonempty for $\lambda \in (0, \lambda_0)$. Now, suppose that there exists $u \in \mathcal{N}_\lambda^0$ such that $||u|| > 1$. Using the definition of $\mathcal{N}_\lambda^0$, we obtain

$$\int_\Omega |\nabla u|^{p(x)}\,\mathrm{d}x - \int_\Omega a(x)|u|^{q(x)}\,\mathrm{d}x - \lambda \int_\Omega b(x)|u|^{1-\delta(x)}\,\mathrm{d}x = 0.$$

Combining the above equality with (17) and Theorem 2.3 in [17], we get

$$\begin{aligned} 0 = \langle I_\lambda'(u), u\rangle &= \int_\Omega p(x)|\nabla u|^{p(x)}\,\mathrm{d}x - \int_\Omega a(x)q(x)|u|^{q(x)}\,\mathrm{d}x \\ &\quad - \lambda \int_\Omega b(x)(1-\delta(x))|u|^{1-\delta(x)}\,\mathrm{d}x \\ &\geq p^- \int_\Omega |\nabla u|^{p(x)}\,\mathrm{d}x - q^+ \int_\Omega a(x)|u|^{q(x)}\,\mathrm{d}x \\ &\quad - (1-\delta^+)\left(\int_\Omega |\nabla u|^{p(x)}\,\mathrm{d}x - \int_\Omega a(x)|u|^{q(x)}\,\mathrm{d}x\right). \\ &\geq (p^- - (1-\delta^+))\int_\Omega |\nabla u|^{p(x)}\,\mathrm{d}x + (1-\delta^+ - q^+)\int_\Omega a(x)|u|^{q(x)}\,\mathrm{d}x. \end{aligned}$$

It now follows from Theorem 2.4 that

$$(p^- - (1-\delta^+))||u||^{p^-} + c_{10}(1-\delta^+ - q^+)||u||^{q^+} \geq 0,$$

hence

$$||u|| \geq c_{10}\left(\frac{p^- + \delta^+ - 1}{1-\delta^+ - q^+}\right)^{\frac{1}{q^+ - p^-}}. \tag{18}$$

In the same way, since $u \in \mathcal{N}_\lambda$, we obtain

$$\int_\Omega |\nabla u|^{p(x)}\,\mathrm{d}x - \int_\Omega a(x)|u|^{q(x)}\,\mathrm{d}x - \lambda \int_\Omega b(x)|u|^{1-\delta(x)}\,\mathrm{d}x = 0$$

and since $u \in \mathcal{N}_\lambda^0$, we have

$$p^+ \int_\Omega |\nabla u|^{p(x)}\,\mathrm{d}x - q^- \int_\Omega a(x)|u|^{q(x)}\,\mathrm{d}x - \lambda(1-\delta^+)\int_\Omega b(x)|u|^{1-\delta(x)}\,\mathrm{d}x \geq 0.$$

Therefore,

$$\begin{aligned} &p^+ \int_\Omega |\nabla u|^{p(x)}\,\mathrm{d}x - q^- \int_\Omega a(x)|u|^{q(x)}\,\mathrm{d}x \\ &\quad - \lambda(1-\delta^+)\left(\int_\Omega |\nabla u|^{p(x)}\,\mathrm{d}x - \lambda\int_\Omega a(x)|u|^{q(x)}\,\mathrm{d}x\right) \geq 0. \\ &= (p^+ - q^+)\int_\Omega |\nabla u|^{p(x)}\,\mathrm{d}x + \lambda(q^+ + \delta^+ - 1)\int_\Omega b(x)|u|^{1-\delta(x)}\,\mathrm{d}x \geq 0. \end{aligned}$$

Now since $||u|| > 1$, by Theorem 2.4, one has

$$(p^+ - q^+)||u||^{p^-} + c_{11}\lambda(q^+ + \delta^+ - 1)||u||^{1-\delta^+} \geq 0,$$

and therefore,

$$||u|| \leq c_{11}\left(\lambda \frac{q^+ + \delta^+ - 1}{q^+ - p^-}\right)^{\frac{1}{p^- + \delta^+ - 1}}. \tag{19}$$

Using (18) and (19),

$$c_{11}\left(\lambda \frac{q^+ + \delta^+ - 1}{q^+ - p^-}\right)^{\frac{1}{p^- + \delta^+ - 1}} \geq c_{10}\left(\frac{p^- + \delta^+ - 1}{1 - \delta^+ - q^+}\right)^{\frac{1}{q^+ - p^-}}.$$

we get

$$\lambda \geq \frac{c_{10}}{c_{11}}\left(\frac{p^- + \delta^+ - 1}{1 - \delta^+ - q^+}\right)^{\frac{p^- + \delta^+ - 1}{q^+ - p^-}}\left(\frac{q^+ + \delta^+ - 1}{q^+ - p^-}\right).$$

Then, if λ is small enough,

$$\lambda = \frac{c_{10}}{c_{11}}\left(\frac{p^- + \delta^+ - 1}{1 - \delta^+ - q^+}\right)^{\frac{p^- + \delta^+ - 1}{q^+ - p^-}}\left(\frac{q^+ + \delta^+ - 1}{q^+ - p^-}\right),$$

we obtain $||u|| < 1$, which is impossible. Therefore, $\mathcal{N}_\lambda^0 = \{0\}$ for all $\lambda \in (0, \lambda_0)$. Hence, this completes the proof of Lemma 3.3. ■

4. Existence of minimizers on $\mathcal{N}_\lambda^+$

In this section, we shall prove the existence of a minimum for the functional energy E_λ in $\mathcal{N}_\lambda^+$. We shall also prove that this minimizer is a solution to problem (P_λ).

Theorem 4.1: *There exists $u_\lambda \in \mathcal{N}_\lambda^+$ satisfying*

$$E_\lambda(u_\lambda) = \inf_{u \in \mathcal{N}_\lambda^+} E_\lambda(u),$$

for all $\lambda \in (0, \lambda_0)$.

Proof: Suppose that $\lambda \in (0, \lambda_0)$. Now, E_λ is bounded below on $\mathcal{N}_\lambda$ and hence also on $\mathcal{N}_\lambda^+$. Therefore, there exists a sequence $\{u_n\} \subset \mathcal{N}_\lambda^+$, satisfying $E_\lambda(u_n) \to \inf_{u \in \mathcal{N}_\lambda^+} E_\lambda(u)$, as $n \to \infty$.

Since E_λ is coercive, $\{u_n\}$ is bounded in $W_0^{1,p(x)}(\Omega)$. Therefore, we can assume that $u_n \rightharpoonup u_0$ weakly in $W_0^{1,p(x)}(\Omega)$ and by the compact embedding, we obtain

$$u_n \rightharpoonup u_0 \quad \text{in } L_{b(x)}^{1-\delta(x)}(\Omega)$$

and

$$u_n \rightharpoonup u_0 \quad \text{in } L_{a(x)}^{q(x)}(\Omega).$$

Now, we shall show that $u_n \to u_0$ strongly in $W_0^{1,p(x)}(\Omega)$. First, we shall prove that

$$\inf_{u \in \mathcal{N}_\lambda^+} E_\lambda(u) < 0.$$

Let $u_0 \in \mathcal{N}_\lambda^+$. Then $\phi''_{u_0}(1) > 0$ which gives

$$p^+ \int_\Omega |\nabla u|^{p(x)}\, dx - q^- \int_\Omega a(x)|u|^{q(x)}\, dx - \lambda(1-\delta^+) \int_\Omega b(x)|u|^{1-\delta(x)}\, dx > 0. \quad (20)$$

Moreover, by the definition of the functional energy E_λ, we can write

$$E_\lambda(u) \leq \frac{1}{p^-} \int_\Omega |\nabla u|^{p(x)}\, dx - \frac{1}{q^+} \int_\Omega a(x)|u|^{q(x)}\, dx - \frac{\lambda}{1-\delta^+} \int_\Omega b(x)|u|^{1-\delta(x)}\, dx. \quad (21)$$

Now, we multiply (17) by $(-(1-\delta^+))$ and get

$$\begin{aligned} &-(1-\delta^+) \int_\Omega |\nabla u|^{p(x)}\, dx + (1-\delta^+) \int_\Omega a(x)|u|^{q(x)}\, dx \\ &+ \lambda(1-\delta^+) \int_\Omega b(x)|u|^{1-\delta(x)}\, dx = 0. \end{aligned}$$

Invoking the above equality and (20), one gets

$$\int_\Omega a(x)|u|^{q(x)}\, dx < \frac{p^+ + \delta^+ - 1}{q^- + \delta^+ - 1} \int_\Omega |\nabla u|^{p(x)}\, dx. \quad (22)$$

On the other hand, from (17) and (21), we obtain

$$E_\lambda(u) \leq \left(\frac{1}{p^-} - \frac{1}{1-\delta^+}\right) \int_\Omega |\nabla u|^{p(x)}\, dx - \left(\frac{1}{q^-} - \frac{1}{1-\delta^+}\right) \int_\Omega a(x)|u|^{q(x)}\, dx. \quad (23)$$

Then, by (22) and (23), we get

$$E_\lambda(u) < -\frac{(p^- + \delta^+ - 1)(q^+ - p^-)}{p^- q^+ (1-\delta^+)} ||u||^{p^-} < 0.$$

Now, let us assume that $u_n \nrightarrow u_0$ strongly in $W_0^{1,p(x)}(\Omega)$. Then

$$\int_\Omega |\nabla u_0|^{p(x)}\, dx \leq \liminf_{n\to\infty} \int_\Omega |\nabla u_n|^{p(x)}\, dx.$$

Using the compactness of embeddings, we obtain

$$\begin{aligned} \int_\Omega a(x) u_0^{q(x)}\, dx &= \liminf_{n\to\infty} \int_\Omega b(x) u_n^{q(x)}\, dx, \\ \int_\Omega b(x) u_0^{1-\delta(x)}\, dx &= \liminf_{n\to\infty} \int_\Omega a(x) u_n^{1-\delta(x)}\, dx. \end{aligned}$$

Now, by (17) and Theorem 2.3 in [17], one has

$$E_\lambda(u_n) \geq \left(\frac{1}{p^-} - \frac{1}{q^+}\right) \int_\Omega |\nabla u_n|^{p(x)}\, dx + \lambda\left(\frac{1}{q^+} - \frac{1}{1-\delta^+}\right) \int_\Omega b(x)|u_n|^{1-\delta(x)}\, dx.$$

Passing to the limit when n goes to ∞, we obtain

$$\lim_{n\to\infty} E_\lambda(u_n) \geq \left(\frac{1}{p^-} - \frac{1}{q^+}\right) \lim_{n\to\infty} \int_\Omega |\nabla u_n|^{p(x)}\, dx + \lambda \left(\frac{1}{q^+} - \frac{1}{1-\delta^+}\right) \lim_{n\to\infty} \int_\Omega b(x)|u_n|^{1-\delta(x)}\, dx.$$

Hence, using Theorem 2.3 in [17], we get

$$\inf_{u\in\mathcal{N}^+} E_\lambda(u) > \left(\frac{1}{p^-} - \frac{1}{q^+}\right) ||u_0||^{p^-} + \lambda c_5 \left(\frac{1}{q^+} - \frac{1}{1-\delta^+}\right) (||u_0||^{1-\delta^-} + ||u_0||^{1-\delta^+}) > 0$$

since $p^- > 1 - \delta^+ \geq 1 - \delta^-$ and $||u_0|| > 1$, which gives a contradiction. Therefore, $u_n \to u_0$ strongly in $W_0^{1,p(x)}(\Omega)$ and $E_\lambda(u_0) = \inf_{u\in\mathcal{N}_\lambda^+} E_\lambda(u)$. This completes the proof of Theorem 4.1. ■

5. Existence of minimizers on $\mathcal{N}_\lambda^-$

In this section, we shall prove the existence of a minimum for the functional energy E_λ in $\mathcal{N}_\lambda^-$. We shall also prove that this minimizer is a solution to problem (P_λ).

Theorem 5.1: *There exists $v_\lambda \in \mathcal{N}_\lambda^-$ such that*

$$E_\lambda(v_\lambda) = \inf_{v\in\mathcal{N}_\lambda^-} E_\lambda(v),$$

for all $\lambda \in (0, \lambda_0)$.

Proof: Suppose that $\lambda \in (0, \lambda_0)$. Since E_λ is bounded below on $\mathcal{N}_\lambda$ hence also on $\mathcal{N}_\lambda^-$. Therefore, there exists a sequence $\{v_n\} \subset \mathcal{N}_\lambda^-$, satisfying $E_\lambda(v_n) \to \inf_{u\in\mathcal{N}_\lambda^-} E_\lambda(u)$, as $n \to \infty$. Since E_λ is coercive, $\{v_n\}$ is bounded in $W_0^{1,p(x)}(\Omega)$.

Therefore, we can assume that $v_n \rightharpoonup v_0$ weakly in $W_0^{1,p(x)}(\Omega)$ and by the compact embedding, we get

$$v_n \rightharpoonup v_0 \text{ in } L_{b(x)}^{1-\delta(x)}(\Omega)$$

and

$$v_n \rightharpoonup v_0 \text{ in } L_{a(x)}^{q(x)}(\Omega).$$

Now, we shall show $v_n \to v_0$ strongly in $W_0^{1,p(x)}(\Omega)$. First, we shall prove that

$$\inf_{v\in\mathcal{N}_\lambda^-} E_\lambda(v) > 0.$$

Let $v_0 \in \mathcal{N}_\lambda^-$. Then we have from (17),

$$\int_\Omega |\nabla u|^{p(x)}\, dx - \int_\Omega a(x)|u|^{q(x)}\, dx - \lambda \int_\Omega b(x)|u|^{1-\delta(x)}\, dx = 0. \tag{24}$$

Moreover, by the definition of the functional energy E_λ, we can write

$$E_\lambda(v) \geq \frac{1}{p^-}\int_\Omega |\nabla v|^{p(x)}\,\mathrm{d}x - \frac{1}{q^+}\int_\Omega a(x)|v|^{q(x)}\,\mathrm{d}x - \frac{\lambda}{1-\delta^+}\int_\Omega b(x)|v|^{1-\delta(x)}\,\mathrm{d}x. \quad (25)$$

Hence, from (24) and (25), one has

$$\begin{aligned} E_\lambda(v) &\geq \frac{1}{p^-}\int_\Omega |\nabla v|^{p(x)}\,\mathrm{d}x - \frac{\lambda}{1-\delta^+}\int_\Omega b(x)|v|^{1-\delta(x)}\,\mathrm{d}x \\ &\quad - \frac{1}{q^+}\left(\int_\Omega |\nabla v|^{p(x)}\,\mathrm{d}x - \lambda\int_\Omega b(x)|v|^{1-\delta(x)}\,\mathrm{d}x\right) \\ &\geq \left(\frac{1}{p^-}-\frac{1}{q^+}\right)\int_\Omega |\nabla v|^{p(x)}\,\mathrm{d}x + \left(\frac{1}{q^+}-\frac{1}{1-\delta^+}\right)\int_\Omega b(x)|v|^{1-\delta(x)}\,\mathrm{d}x \\ &\geq \left(\frac{1}{p^-}-\frac{1}{q^+}\right)||v||^{p^-} + \lambda c_8\left(\frac{1}{q^+}-\frac{1}{1-\delta^+}\right)||v||^{1-\delta^+} \\ &\geq \left[\left(\frac{1}{p^-}-\frac{1}{q^+}\right) + \lambda c_8\left(\frac{1}{q^+}-\frac{1}{1-\delta^+}\right)\right]||v||^{p-} \end{aligned}$$

since $p^- > 1-\delta^+$.

Hence, if we choose

$$\lambda < \frac{(1-\delta^+)(p^- - q^+)}{c_8 p^+(1-\delta^+ - q^+)},$$

we obtain $E_\lambda(v) > 0$. Moreover, since $\mathcal{N}_\lambda^+ \cap \mathcal{N}_\lambda^- = \emptyset$ and $\inf_{v\in\mathcal{N}_\lambda^+} E_\lambda(v) < 0$, we see that $v \in \mathcal{N}_\lambda^-$.

In the same way, if $v_0 \in \mathcal{N}_\lambda^-$, there exists t_0 satisfying $t_0 v_0 \in \mathcal{N}_\lambda^-$ and so $E_\lambda(t_0 v_0) \leq E_\lambda(v_0)$. Moreover, since

$$I'_\lambda(v) = \int_\Omega p(x)|\nabla v|^{p(x)}\,\mathrm{d}x - \int_\Omega a(x)q(x)|v|^{q(x)}\,\mathrm{d}x - \lambda\int_\Omega b(x)(1-\delta(x))|v|^{1-\delta(x)}\,\mathrm{d}x,$$

we get

$$\begin{aligned} I'_\lambda(t_0 v_0) &= \int_\Omega p(x)|\nabla t_0 v_0|^{p(x)}\,\mathrm{d}x - \int_\Omega a(x)q(x)|t_0 v_0|^{q(x)}\,\mathrm{d}x \\ &\quad - \lambda\int_\Omega b(x)(1-\delta(x))|t_0 v_0|^{1-\delta(x)}\,\mathrm{d}x \\ &\leq t_0^{p^+} p^+ \int_\Omega |\nabla v_0|^{p(x)}\,\mathrm{d}x - t_0^{q^-} q^- \int_\Omega a(x)|v_0|^{q(x)}\,\mathrm{d}x \\ &\quad - \lambda(1-\delta^+)t_0^{1-\delta^+}\int_\Omega b(x)|v_0|^{1-\delta(x)}\,\mathrm{d}x, \end{aligned}$$

since $1-\delta^+ < p^+ < q^-$. By the conditions on a and b, it follows that $I'_\lambda(t_0 v_0) < 0$, so by definition $\mathcal{N}_\lambda^-$ $t_0 v_0 \in \mathcal{N}_\lambda^-$.

Now, let us assume that $v_n \not\to v_0$ strongly in $W_0^{1,p(x)}(\Omega)$. Using the fact that

$$\int_\Omega |\nabla v_0|^{p(x)}\,\mathrm{d}x \le \liminf_{n\to\infty} \int_\Omega, |\nabla v_n|^{p(x)}\,\mathrm{d}x$$

one gets

$$\begin{aligned} E_\lambda(tv_0) &\le \int_\Omega \frac{t^{p(x)}|\nabla v_0|^{p(x)}}{p(x)}\,\mathrm{d}x - \int_\Omega \frac{t^{q(x)}}{q(x)}|v_0|^{q(x)}\,\mathrm{d}x - \lambda\int_\Omega \frac{t^{1-\delta(x)}|v_0|^{1-\delta(x)}}{1-\delta(x)}\,\mathrm{d}x, \\ &\le \lim_{n\to\infty}\left[\frac{t^{p^+}}{p^-}\int_\Omega |\nabla v_n|^{p(x)}\,\mathrm{d}x - \frac{t^{q^-}}{q^+}\int_\Omega |v_n|^{q(x)}\,\mathrm{d}x - \lambda\frac{t^{1-\delta^+}}{1-\delta^+}\int_\Omega |v_n|^{1-\delta(x)}\,\mathrm{d}x\right], \\ &\le \lim_{n\to\infty} E_\lambda(tv_n) \le \lim_{n\to\infty} E_\lambda(v_n) = \inf_{v\in\mathcal{N}_\lambda^-} E_\lambda(v), \end{aligned}$$

which contradicts with the fact that $tv_0 \in \mathcal{N}_\lambda^-$. Hence, $v_n \to v_0$ strongly in $W_0^{1,p(x)}(\Omega)$ and $E_\lambda(v_0) = \inf_{v\in\mathcal{N}_\lambda^-} E_\lambda(v)$. This completes the proof of Theorem 5.1. ■

6. Proof of Theorem 1.1

Proof: By Theorem 4.1 and Theorem 5.1, for all $\lambda \in (0,\lambda_0)$, there exist $u_0 \in \mathcal{N}_\lambda^+$, $v_0 \in \mathcal{N}_\lambda^-$ such that

$$E_\lambda(u_0) = \inf_{u\in\mathcal{N}_\lambda^+} E_\lambda(u)$$

and

$$E_\lambda(v_0) = \inf_{v\in\mathcal{N}_\lambda^-} E_\lambda(v).$$

On the other hand, since $E_\lambda(u_0) = E_\lambda(|u_0|)$ and $|u_0| \in \mathcal{N}_\lambda^+$ and in the same way, $E_\lambda(v_0) = E_\lambda(|v_0|)$ and $|v_0| \in \mathcal{N}_\lambda^-$, we suppose that $u_0, v_0 \ge 0$. Using Lemma 3.2, u_0, v_0 are critical points of E_λ on $W_0^{1,p(x)}(\Omega)$ and thus weak solutions of (P_λ).

Finally, by the Harnack inequality and by Zhang–Liu [26], we obtain that u_0, v_0 are nonnegative solutions of (P_λ).

It remains to prove that the solutions we obtained for Theorem 4.1 and Theorem 5.1 are distinct. Indeed, since $\mathcal{N}_\lambda^- \cap \mathcal{N}_\lambda^+ = \emptyset$, it follows that u_0 and v_0 are different. This completes the proof of Theorem 1.1. ■

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References

[1] Acerbi E, Mingione G. Regularity results for a class of functionals with nonstandard growth. Arch Ration Mech Anal. 2001;156:121–140.
[2] Diening L. Theoretical and numerical results for electrorheological fluids [Ph.D. thesis]. Freiburg: University of Freiburg; 2002.
[3] Rădulescu VD, Repovš DD. Partial differential equations with variable exponents: variational methods and qualitative analysis. Boca Raton (FL): Chapman and Hall/CRC, Taylor & Francis Group; 2015.
[4] Zhang Q. Existence and asymptotic behavior of positive solutions to $p(x)$-Laplacian equations with singular nonlinearities. J Inequal Appl. 2007;70: 305–316, Art. ID 19349, 9 pp.
[5] Saoudi K. Existence and non-existence of solution for a singular nonlinear Dirichlet problem involving the $p(x)$-Laplace operator. J Adv Math Stud. 2016;9(2):292–303.
[6] Fan XL. Solutions for $p(x)$-Laplacian Dirichlet problems with singular coefficients. J Math Anal Appl. 2005;312:464–477.
[7] Saoudi K, Ghanmi A. A multiplicity results for a singular problem involving the fractional p-Laplacian operator. Complex Var Elliptic Equ. 2016;61(9):1199–1216.
[8] Saoudi K, Kratou M, Al Sadhan S. Multiplicity results for the $p(x)$-Laplacian equation with singular nonlinearities and nonlinear Neumann boundary condition. Int J Differ Equ. 2016;2016, Article ID 3149482, 14 pp.
[9] Saoudi K, Ghanmi A. A multiplicity results for a singular equation involving the $p(x)$-Laplace operator. Complex Var Elliptic Equ. 2017;62(5):695–725.
[10] Coclite MM, Palmieri G. On a singular nonlinear Dirichlet problem. Comm Partial Differ Equ. 1989;14:1315–1327.
[11] Crandall MG, Rabinowitz PH, Tartar L. On a Dirichlet problem with a singular nonlinearity. Comm Partial Differ Equ. 1977;2:193–222.
[12] Ghergu M, Rădulescu V. Singular elliptic problems: bifurcation and asymptotic analysis. Oxford: Oxford University Press; 2008. (Oxford lecture series in mathematics and its applications; 37).
[13] Giacomoni J, Saoudi K. Multiplicity of positive solutions for a singular and critical problem. Nonlinear Anal. 2009;71(9):4060–4077.
[14] Giacomoni J, Schindler I, Takáč P. Sobolev versus Hölder local minimizers and global multiplicity for a singular and quasilinear equation. Ann Sc Norm Super Pisa Cl Sci V. 2007;6:117–158.
[15] Saoudi K. Existence and non-existence for a singular problem with variables potentials. Electron J Differ Equ. 2017;2017(291):1–9.
[16] Saoudi K, Kratou M. Existence of multiple solutions for a singular and quasilinear equation. Complex Var Elliptic Equ. 2015;60(7):893–925.
[17] Mashiyev RA, Ogras S, Yucedag Z, Avci M. The Nehari manifold approach for Dirichlet problem involving the $p(x)$-Laplacian equation. J Korean Math Soc. 2010;47(4):845–860.
[18] Saiedinezhad S, Ghaemi MB. The fibering map approach to a quasilinear degenerate $p(x)$-Laplacian equation. Bull Iranian Math Soc. 2015;41:1477–1492.
[19] Saoudi K. A singular elliptic system involving the p(x)-Laplacian and generalized Lebesgue-Sobolev spaces. Int J Math. 2019;30(116):1950064.
[20] Papageorgiou NS, Rădulescu VD, Repovš DD. Nonlinear analysis – theory and methods. Cham: Springer; 2019. (Springer monographs in mathematics).
[21] Fan XL, Zhao D. On spaces $L^p(x)(\Omega)$ and $W^m, p(x)(\Omega)$. J Math Anal Appl. 2001;263: 424–446.

[22] Fan XL, Shen JS, Zhao D. Sobolev embedding theorems for spaces $W^k, p(x)(\Omega)$. J Math Anal Appl. 2001;262:749–760.
[23] Kováčik O, Rákosnik J. On spaces $L^p(x)$ and $W^k, p(x)$. Czechoslovak Math J. 1991;41(116): 592–618.
[24] Zhang QH. A strong maximum principle for differential equations with nonstandard $p(x)$-growth conditions. J Math Anal Appl. 2005;312(1):24–32.
[25] Drabek P, Pohozaev SI. Positive solutions for the p-Laplacian: application of the fibering method. Proc Royal Soc Edinb Sect A. 1997;127:703–726.
[26] Zhang X, Liu X. The local boundedness and Harnack inequality of $p(x)$-Laplace equation. J Math Anal Appl. 2007;332:209–218.