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THE ARF–KERVAIRE INVARIANT OF FRAMED MANIFOLDS AS AN OBSTRUCTION TO EMBEDDABILITY

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ABSTRACT. We prove that no 14-connected (resp. 30-connected) stably parallelizable manifold N^{30} (resp. N^{62}) of dimension 30 (resp. 62) with the Arf-Kervaire invariant 1 can be smoothly embedded into $\mathbb{R}^{36}$ (resp. $\mathbb{R}^{83}$).

1. Closed stably parallelizable manifolds with a nontrivial Arf–Kervaire invariant

Let us consider a closed stably framed n–dimensional manifold. Such a manifold is presented by the pair (N^n, Ξ), where N^n is a closed manifold of the dimension $\dim(N) = n$, and Ξ is an isomorphism of bundles $\Xi : \nu(N) = \mathbb{R}^k \times N^n$, where $\nu(N)$ is the k–dimensional normal bundle of N^n, $k > n+1$. A stably parallelizable manifold is a stably framed manifold with the forgotten stable framing.

Suppose that $n = 2^\ell - 2 = 4l + 2$ and that N^n is $2l$–connected. Then N^n is diffeomorphic to the connected sum of manifolds of the following three types (see Kreck (2000) for the proof and further references):

- closed manifold Σ^n, homotopically equivalent to the standard n–dimensional sphere;
- product of two standard spheres $S^{2l+1} \times S^{2l+1}$;
- standard Arf-Kervaire manifold (constructed later).

A connected sum of two third type manifolds is diffeomorphic to the standard sum of some number of first and second type manifolds. The dimension $\dim(H_{2l+1}(N^n;\mathbb{Z}/2))$ is always even and equals to $2p$, where p is the number of the summands of second and third type. Following the statement of the Hill, Hopkins and Ravenel Theorem, third type manifold can be constructed only for $\ell = 5,6$ and eventually 7 (see Hill *et al.* 2016). The above statement was proved by Akhmetiev (2022, p. 3), except for possibly finite number of exceptional cases (there is an error in the proof of Proposition 2: the mapping of regular cobordism classes on p.34 is not a homomorphism).

Definition 1.1. A $2l$–connected manifold N^n, $n = 4l+2$, is said to have a nontrivial Arf–Kervaire invariant if $n = 2^\ell - 2$, $\ell = 5,6$ or 7, and N^n is diffeomorphic to the connected sum of a third type manifold and some number of first and second type manifolds.

Standard Arf–Kervaire manifold. The third type manifold is based on manifold M_0^{4l+2}, $4l+2=n$. This manifold was constructed by "plumbing" in (Browder 1972, Theorem V.2.11). For more methods and techniques see, *e.g.*, Cavicchioli *et al.* (2002, 2016). Manifold M_0^{4l+2} is well-defined for all nonnegative l, but the condition $\partial M_0^{4l+2} = S^{4l+1}$ is fulfilled only for $4l+2 = 2,6,14,30,62$ and eventually 126. For $4l+2 \neq 2,6,14,30,62$ and eventually $\neq 126$, the boundary ∂M_0^{4l+2} is *PL*–homeomorphic but not diffeomorphic to the standard $(4l+1)$–dimensional sphere. In these exceptional dimensions, manifold N^{4l+2} is defined as M_0^{4l+2} with the standard $(4l+2)$-dimensional disk glued along the boundary ∂M_0^{4l+2}. In the case $n=2,6,14$, a second type manifold is obtained. In the case $n=30,62$ and eventually 126, the obtained manifold is of the third type. A simplified proof of existence of M_0^{4l+2} for $n=62$ was given by Lin (2001). Jones and Rees (1978) remark that the manifold N^n of the third type is *PL*–embeddable into $\mathbb{R}^{n+2}$.

Our main result is formulated in the following theorem.

Theorem 1.2. *(1) Let N^{30} (resp. N^{62}) be an arbitrary closed* 14*-connected (resp.* 30*-connected) stably parallelizable manifold with a nontrivial Arf–Kervaire invariant. Then the product $N^{30}\times I$ (resp. $N^{62}\times I$) with the interval $I=[0,1]$ is not smoothly embeddable into the Euclidean space $\mathbb{R}^{46}$ (resp. $\mathbb{R}^{94}$), provided that the corresponding embedding is equipped by a nondegenerate normal field of* 9*–frames (resp.* 10*–frames) on the complement of the Cartesian product of the interval I and a point $N^{30}\times I\setminus\{pt\}\times I$ (resp. on $N^{62}\times I\setminus\{pt\}\times I$).*

(2) No stably parallelizable manifold N^{30} (resp. N^{62}) is smoothly embeddable into the Euclidean space $\mathbb{R}^{36}$ (resp. $\mathbb{R}^{83}$).

Remark 1.3. Obviously, Assertion 2 of Theorem 1.2 follows from Assertion 1. Indeed, the composition $N^{30}\subset\mathbb{R}^{36}\subset\mathbb{R}^{46}$ provides the embedding $N^{30}\times I\times D^9\subset\mathbb{R}^{46}$, where D^9 is the standard disk. The restriction of this embedding to the submanifold $N^{30}\times I\subset N^{30}\times I\times D^9$ ensures the condition of stable parallelizability in Assertion 2. Analogously, the composition $N^{62}\subset\mathbb{R}^{83}\subset\mathbb{R}^{94}$ provides the embedding $N^{62}\times I\times D^{10}\subset\mathbb{R}^{94}$, and the restriction of this embedding to the submanifold $N^{62}\times I\subset N^{62}\times I\times D^{10}$ ensures the condition of stable parallelizability in Assertion 2. Nevertheless, we give an independent proof of Assertion 2 since this proof is simpler than the proof of Assertion 1.

Remark 1.4. Eccles (1979, Corollary 1.2) constructed a stably parallelizable 30-dimensional (resp. 62-dimensional) manifold N^{30} (resp. N^{62}) with the Arf–Kervaire invariant 1, which is embeddable in $\mathbb{R}^{46}$.

Remark 1.5. Other geometric applications of the Arf–Kervaire invariant, related to the problem of embeddability, can be found in (Randall 1999).

Remark 1.6. A generalization of the Kervaire invariant 1 problem and applications was given by Akhmetiev (2016).

Remark 1.7. A preliminary version of this paper was posted on the arXiv (Akhmetiev *et al.* 2010).

2. Cobordism group of immersions and the Arf–Kervaire invariant of stably framed manifolds

Let us denote the cobordism group of immersions of oriented n-manifolds in the codimension 1 by $Imm^{fr}(n,1)$. The class of the regular cobordism of immersions $f : N^n \looparrowright \mathbb{R}^{n+1}$ represents an element of this cobordism group. The set of these elements is equipped with an equivalence relation of cobordance. It follows by the Pontryagin-Thom construction (Pontryagin 1955), in the form proposed by Wells (1966), that the group $Imm^{fr}(n,1)$ is isomorphic to the stable n-homotopy group of spheres. First, we describe the Pontryagin-Thom construction. A stably framed manifold is a pair (N^n,Ξ) where N^n is a smooth manifold and Ξ is a trivialization of the normal bundle ν_N. Namely, N^n is diffeomorphic to a submanifold M in $\mathbb{R}^{n+k}$. Then the normal bundle ν_N is isomorphic to the trivial normal bundle ν_M and Ξ is the chosen trivialization. The word "stably" means that $k >> n$ (in fact $k \geq n+2$ suffices).

It is convenient to introduce the direct limit $k \to +\infty$. The Pontryagin-Thom construction (Pontryagin 1955, Ch. 6), gives the map $F : S^{n+k} \to S^k$ as a composition of the standard projection $S^{n+k} \to ME(k)(N^n)$ and the standard map $ME(k)(N^n) \to S^k$. Here, $ME(k)(N^n)$ (or $M(N^n)$) denotes the Thom space of the trivial k–dimensional bundle. This space is homeomorphic to the k–fold suspension of $N^n_+ = N^n \cup \{x\}$, where x is a point. The base point $pt \in S^k$ is a regular value of the map F and the preimage of a small neighbourhood of that point defines the framed manifold (N^n,Ξ), corresponding to the subspace of zero section of $M(N^n)$. The homotopy class $[F] \in \Pi_n$ is well-defined. Moreover, if $F' : S^{n+k} \to S^n$ is another map, which is homotopic to F, and the base point pt is also a regular value of F', then it can be constructed a framed manifold (N',Ξ') analogously, with a framed cobordism (W,Ξ_W), connecting (N,Ξ) and (N',Ξ'). Therefore mapping $[F] \mapsto [(N^n,\Xi)]$ defines an isomorphism between the stable homotopical group of spheres and the cobordism group of stably framed manifolds. By the Smale-Hirsch theorem (Hirsch 1961), a stably framed manifold (N^n,Ξ) defines an immersion $f : N^n \looparrowright \mathbb{R}^n$. Immersion f is not defined uniquely; if f' is another immersion, corresponding to (N^n,Ξ), then f and f' are regularly cobordant. If (N'^n,Ξ') is cobordant to (N^n,Ξ), then the corresponding immersion $f' : N'^n \looparrowright \mathbb{R}^n$, is an element of the same regular cobordism class as the immersion f. Mapping $[(N^n,\Xi)] \mapsto [f]$, constructed by Hirsch and mapping $[F] \mapsto [(N^n,\Xi)]$, constructed by Pontryagin, define the isomorphism between the cobordism group of immersions $Imm^{fr}(n,1)$, and the stable homotopy group of spheres Π_n, constructed by Wells.

Consider the case $n = 4l+2$.

Definition of the $\mathbb{Z}/2$-quadratic form of an immersion and its Arf-invariant. Let $f : N^{4l+2} \looparrowright \mathbb{R}^{4l+3}$ be the immersion, representing an element in the group $Imm^{fr}(4l+2,1)$. Homology group $H_{2l+1}(N^{4k+2};\mathbb{Z}/2)$ is denoted shortly by H. By the Poincaré duality, the bilinear nondegenerate form $b : H \times H \to \mathbb{Z}/2$ is well-defined. Take the map $S^{4l+2+k} \to M(N^{4l+2}_+)$, defined by the Pontryagin–Thom construction. Obviously, the map fulfills the conditions of Browder (1969, Theorem 1.4), hence a quadratic form $q : H \to \mathbb{Z}/2$, associated with the form b, can be defined. Define the Arf-invariant $Arf(H,q)$ as the equivalence class of q in the Witt group of quadratic forms (Browder 1972, Sect. 4). It turns out that $Arf(H,q)$ is an invariant of the regular cobordism class of immersion f. It is said to be the Arf–Kervaire invariant of f. Hence, by the Wells theorem, this invariant defines a homomorphism of groups $\Theta : Imm^{fr}(4l+2,1) \longrightarrow \mathbb{Z}/2$, (see Browder (1969, Sect. 6) and Kervaire and Milnor (1963, Sect. 8)).

The form q can be constructed in a different way. Take a cycle $x \in H$. By the Thom theorem, there exist a (possibly nonoriented) manifold X^{2l+1} and a map $i_X : X^{2l+1} \to N^{4l+2}$ such that $i_{X,*}([X]) = x$, where $[X]$ is the fundamental class of manifold X. Because of general position, it can be assumed without loss of generality, that map $f \circ i_X : X^{2l+1} \to N^{4l+2}$ is an immersion with only finitely many transversal self-intersections. Denote the self-intersection points of the immersion i_X by $\{x_1,\dots,x_s\}$. For each point x_i there exists a neighbourhood consisting of two $2l+1$–disks intersecting in x_i. Perform a surgery to obtain a manifold $Y^{2l+1} \subset N^{4l+2}$ such that also $i_{Y,*}([Y]) = x$, where $i_Y : Y^{2l+1} \subset N^{4l+2}$ is the inclusion. To this end, remove both disks and glue their boundaries by a 1-handle. The idea of such a surgery in the case $l = 0$ is known from Pontryagin (1955, Ch. 15, Theorem 22). Take the immersion $f : N^{4l+2} \looparrowright \mathbb{R}^n$ and consider the map $j_Y = f \circ i_Y : Y^{2l+1} \subset \mathbb{R}^n$. By the general position argument, j_Y is also an embedding and is equipped with a cross-section ξ_Y of the normal bundle ν_{j_Y}. This cross-section is defined by the oriented normal bundle of the immersion f along the cycle. The linking number of the framed embedding (i_Y,ξ_Y) is denoted by $lk(i_Y,\xi_Y)$ (it is defined as the linking number between $i_Y(Y)$ and its copy along ξ_Y). Define

$$q(x) = lk(i_Y,\xi_Y) \ (mod\ 2). \tag{1}$$

Lemma 2.1. *Function $q : H \to \mathbb{Z}/2$, given by $q(x) = lk(i_Y,\xi_Y)(mod\ 2)$, is well defined. It coincides with the Browder function, constructed by Browder (1969, Lemma 1.2).*

Corollary 2.2. *Function $q(x) = lk(i_Y,\xi_Y)(mod\ 2)$ is the quadratic form, associated to bilinear form $b : H \times H \to \mathbb{Z}/2$.*

Proof of Corollary 2.2. The proof can be found in (Browder 1969, Theorem 1.4). □

Proof of Lemma 2.1. Consider a stably framed cobordism (W^{4l+3},Ξ_W), connecting given pairs (N^{4l+2},Ξ) and (N_1^{4l+2},Ξ_1) of stably framed manifolds. First suppose that the following conditions are satisfied:

(1) manifold N_1^{4l+2} is $2l$–connected;

(2) cobordism W consists of i–handles, $1 \le i \le 2l$.

The construction of cobordism W is based on spherical surgery as described by Kervaire and Milnor (1963, Sect. 5). Novikov (1964, Sect. 1) developed spherical surgery for a more general situation. It follows from condition 2 that mapping $H_{2l+1}(N^{4l+2};\mathbb{Z}/2) \to H_{2l+1}(W^{4l+3};\mathbb{Z}/2)$, induced by the inclusion $N^{4l+2} \subset W^{4l+3}$, is an isomorphism while mapping $H_{2l+1}(N_1^{4l+2};\mathbb{Z}/2) \to H_{2l+1}(W^{4l+3};\mathbb{Z}/2)$, induced by the inclusion $N_1^{4l+2} \subset W^{4l+3}$, is an epimorphism. Let q' be the function, constructed by Browder. Under condition 1, the function q' has the following geometric meaning (Browder 1969, the last paragraph of the proof of Theorem 3.2 and the corresponding reference). By the Hurewicz theorem, an element $x \in H_{2l+1}(N_1^{4l+2};\mathbb{Z}/2)$ can be realized as a map of spheres: $\varphi : S^{2l+1} \to N_1^{4l+2}$. Furthermore, φ can be realized in its homotopy class by an embedding $\varphi_0 : S^{2l+1} \subset N_1^{4l+2}$. Consider the embedding $I_{N_1} \circ \varphi_0 : S^{2l+1} \to N_1^{4l+2} \subset \mathbb{R}^{4l+2+k}$, where I_{N_1} is the inclusion which parametrizes the manifold N_1^{4l+2}. The embedding $I_{N_1} \circ \varphi_0$ is equipped by the normal vector field of k–frames. Then by the Hirsch theorem, the immersion $I_{N_1} \circ \varphi_0$ is regularly homotopic to the immersion into standard space $\mathbb{R}^{4l+2} \subset \mathbb{R}^{4l+2+k}$. Hence the framing vectors are parallel complements to the subspace of coordinate axes. This immersion is denoted by $\bar{\varphi}_0 : S^{2l+1} \looparrowright \mathbb{R}^{4l+2}$. The stable Hopf invariant of the immersion $\bar{\varphi}_0$ is defined as the number of transversal self-intersection points. This number is denoted by $q'(x)$ of the embedding φ_0. It depends neither on the choice of the embedding φ_0 in the homotopy class of φ, nor on the choice of the map φ realizing the homology class x. Apply the Hirsch theorem to the embedding $i_Y : S^{2l+1} \subset \mathbb{R}^{4l+3}$, equipped with the cross-section ξ_Y, to construct the immersion $i'_Y : S^{2l+1} \looparrowright \mathbb{R}^{4l+2}$. The value of $lk(i_Y, \xi_Y)$ in the right part of formula (1) coincides with the parity of number of self-intersection points of the immersion i'_Y. This proves that under condition 1 the function q, defined in (1), coincides with the function q', constructed by Browder.

Now we prove Lemma **2.1** in the general case. Let us consider the cobordism W under condition 2. Take an arbitrary element $x \in H_{2l+1}(N^{4l+2};\mathbb{Z}/2)$ and an element $x_1 \in H_{2l+1}(N_1^{4l+2};\mathbb{Z}/2)$ so that the homological class $x + x_1$ is trivial in $H_{2l+1}(W^{4l+3};\mathbb{Z}/2)$. It follows by (Browder, 1971; Lemma 3.1), that $q'(x) = q'(x_1)$. We have proved that $q'(x_1) = q(x_1)$. Let us prove the following equality

$$q(x) = q(x_1). \tag{2}$$

Let the homology class x be equal to the image of the fundamental class under the embedding $i_Y : Y^{2l+1} \subset N^{4l+2}$ and let the homology class x_1 be equal to the image of the fundamental class under the embedding $i_{Y_1} : Y_1^{2l+1} \subset N_1^{4l+2}$. Suppose that the mapping of polyhedron $i_Z : Z^{2l+2} \to W$, which realizes the singular boundary of homology classes x, x_1, is represented by the submanifolds Y^{2l+1} and Y_1^{2l+1}. It is well known that the polyhedron W can be chosen to be a manifold in the complement of some subpolyhedron of codimension 2. Consider the singular points and the self-intersection curve of the polyhedron $i_Z(Z^{2l+2})$. The self-intersection curve of the polyhedron $i_Z(Z^{2l+2})$ lies outside the considered codimension 2 subpolyhedron of the polyhedron Z^{2l+2}. The boundary of self-intersection curve is the set of critical points of the map i_Z, and the number of these points is even. Modify the polyhedron Z^{2l+2} on its regular part and modify the map i_Z by surgery in 1-handles in such a way that the map i_Z has no critical points.

Consider the immersion $f : N^{4l+2} \looparrowright \mathbb{R}^{4l+3} \times \{0\}$, the immersion $f_1 : N_1^{4l+2} \looparrowright \mathbb{R}^{4l+3} \times \{1\}$ and the immersion $F : W^{4l+3} \looparrowright \mathbb{R}^{4l+3} \times [0,1]$, such that its restriction on the upper and the lower components of boundary coincides with the immersions f and f_1, respectively. Consider the pairs of embeddings and corresponding normal sections $(i_Y : Y \subset \mathbb{R}^{4l+3} \times \{0\}; \xi_Y)$, $(i_{Y_1} : Y_1 \subset \mathbb{R}^{4l+3} \times \{1\}; \xi_{Y_1})$. Take the pair of embedding and the corresponding normal section $(i_Z : Z^{2l+2} \looparrowright \mathbb{R}^{4l+3} \times [0,1], \xi_Z)$, such that its restriction to both components of boundary coincides with the pairs (i_Y, ξ_Y), $(i_{Y_1}; \xi_{Y_1})$, respectively. Obviously, the self-linking numbers of boundary embeddings with given normal sections are equal modulo 2. Move $i_Y(Y)$ along ξ_Y; the obtained manifold is denoted by $(i_Y(Y))'$. Analogously, denote by $(i_{Y_1}(Y_1))'$ the manifold obtained from $i_{Y_1}(Y_1)$ by sliding along ξ_{Y_1} and by $(i_Z(Z))'$ the manifold obtained from $i_Z(Z)$ by sliding along ξ_Z. The self-linking number of framed embedding (i_{Y_1}, ξ_{Y_1}) is defined as the parity of number of points of self-intersection of the manifold $(i_{Y_1}(Y_1))'$ with the manifold $i_{Y_1}(Y_1)$ by homotoping $(i_{Y_1}(Y_1))'$ to infinity. The self-linking number of (i_{Y_1},ξ_{Y_1}) is defined similarly. Both self-linking numbers are congruent modulo 2 since $(i_Z(Z))'$ intersects $i_Z(Z)$ in an even number of points and when homotoping $(i_Z(Z))'$ to infinity, the intersection of $(i_Z(Z))'(t)$ and $(i_Z(Z))$ is a collection of curves lying completely in the regular part of polyhedra Z' and Z. Therefore, the boundary of this 1-manifold consists of an even number of points and these points are intersection points of two families of the boundary polyhedra. Formula (2) and Lemma **2.1** are thus proved. $\square$

The cobordism group of stably skew-framed immersions. Let (φ,Ψ_L) be a pair consisting of a $(2l+1)$-dimensional closed manifold $\varphi : L^{2l+1} \looparrowright \mathbb{R}^{4l+2}$ and of a skew-framing Ψ_L of the normal bundle ν_φ, i.e., an isomorphism $\Psi_L : \nu_\varphi = (2l+1)\kappa$, where κ is the orientation line bundle L^{2l+1}. It means that $w_1(\kappa) = w_1(L^{2l+1})$. The cobordism relation of pairs is the standard one. The set of all such pairs forms an abelian group $Imm^{sf}(2l+1,2l+1)$ with respect to the operation of disjoint union. The Pontryagin-Thom construction in the form of Wells can be applied to this cobordism group. It induces the following isomorphism

$$Imm^{sf}(2l+1,2l+1) \equiv \Pi_{4l+2}(P_{2l+1}),$$

where $P_{2l+1} = \mathbb{RP}^\infty/\mathbb{RP}^{2l}$ is the skew projective space and $\Pi_{4l+2}(P_{2l+1}) = \lim_{t\to+\infty}\pi_{4l+2+t}(\Sigma^t P_{2l+1})$ is the stable homotopy group of the $4l+2$-dimensional space P_{2l+1} (Akhmetiev and Eccles 2007).

The connecting homomorphism δ. Define the homomorphism

$$\delta : Imm^{fr}(4l+2) \to Imm^{sf}(2l+1),$$

which is called the connecting homomorphism. It is a modification of the transfer homomorphism of Kahn-Priddy (Eccles 1981). Let the immersion $f : N^{4l+2} \looparrowright \mathbb{R}^{4k+3}$ represent an element in the group $Imm^{fr}(4l+2,1)$. Construct a skew-framed immersion (φ,Ψ_L), where $\varphi : L^{2l+1} \looparrowright \mathbb{R}^{4l+2}$. Consider the immersion $I\circ f$, where $I : \mathbb{R}^{4k+3} \subset \mathbb{R}^{6l+3}$ denotes the standard embedding. The immersion $I\circ f$ is equipped with the standard framing. Let $g : N^{4l+2} \looparrowright \mathbb{R}^{6l+3}$ be an immersion, obtained from f by a small deformation which ensures general position. Hence the immersion g self-intersects transversally. The double point manifold of immersion g is denoted by L^{2l+1}. Let $h : L^{2l+1} \looparrowright \mathbb{R}^{6l+3}$ be the parametrizing immersion of L^{2l+1}. The normal bundle ν_h of immersion h is naturally isomorphic to the bundle $l\varepsilon \oplus l\kappa$, where κ is the line bundle over L^{2l+1}, which is associated to the canonical 2-fold covering of a double point manifold. By the Hirsch theorem, there exists an immersion h_1 regularly homotopic to $h$having its image in the subspace $\mathbb{R}^{4l+2} \subset \mathbb{R}^{6l+3}$. The regular homotopy between the immersions h and h_1 can be extended to the regular homotopy of normal bundles, hence the direct summands of normal bundle are parallel to the complementary coordinate axes of the subspace $\mathbb{R}^{4l+2} \subset \mathbb{R}^{6l+3}$. The immersion $h_1 : L^{2l+1} \looparrowright \mathbb{R}^{4l+2}$ is equipped with a skew-framing of the normal bundle, defined by the bundle isomorphism $\Psi_L : k\kappa \equiv \nu_{h_1}$. Starting from the immersion f, we have constructed the skew-framed immersion (h_1,Ψ_L). Define the element $\delta([f]) \in Imm^{sf}(2l+1,2l+1)$ to be the regular skew-framed cobordism class $[(h_1,\Psi_L)]$.

The Browder-Eccles invariant. An alternative definition of the Arf-Kervaire invariant of framed immersions was given by Eccles (1981). Such a definition uses the characteristic numbers of double point manifolds and is based on a theorem by Browder (1969). In his theorem, the Arf-Kervaire invariant is constructed by means of the Adams spectral sequence. The following simplest version of the definition was given by Akhmetiev and Eccles (1999).

Definition 2.3. Define the Browder-Eccles invariant $\bar{\Theta}(f)$ of a framed immersion f by the formula

$$\bar{\Theta}(f, \Xi_N) = h \circ \delta(f, \Xi_N) \pmod 2,$$

where $h \circ \delta(f, \Xi_N) = h(I \circ f)$ is the number of self-intersection points of the immersion $I \circ f$.

3. Cobordism groups of stably skew-framed immersions

In this section we define new variants of cobordism groups, namely the cobordism groups of stably framed immersions (stably skew-framed immersions, respectively), i.e., the immersions which are not framed in their image–Euclidean space but are framed only in ambiental Euclidean spaces of sufficiently big dimensions. Such a framing (skew-framing, respectively) is said to be the stable framing (stable skew-framing, respectively). Cobordism groups of stably framed and stably skew-framed immersions generalize *intermediate cobordism groups*, as introduced by Eccles (1979). The Arf-Kervaire and the Browder-Eccles invariants can be generalized to the invariants defined on the cobordism group of stably framed immersions. The new invariants are called the *twisted Arf-Kervaire invariant* and the *twisted Browder-Eccles invariant*, respectively. The definition of the twisted Arf-Kervaire invariant is closely connected to the definition of the *Arf-changeable invariant* of framed immersions in the sense of Jones and Rees (1978).

Definition of stably framed cobordism groups $Imm^{stfr}(4l+2, 2l+1)$. Let (f, Ξ_N) be a pair, where $f : N^{4l+2} \looparrowright \mathbb{R}^{6l+3}$ is an immersion in the codimension $2l+1$, Ξ_N be a stable framing of the manifold N^{4l+2}, i.e., a framing of the normal bundle of the composition $I \circ f : N^{4l+2} \looparrowright \mathbb{R}^{6l+3} \subset \mathbb{R}^r$, $r \geq 8l+6$. The set of pairs described above is equipped by an equivalence relation, which is given by the standard relation of cobordism. Up to the cobordism relation the set of pairs generates an Abelian group whose operation is determined by the disjoint union. This group is denoted by $Imm^{stfr}(4l+2, 2l+1)$.

Definition of stably skew-framed cobordism groups $Imm^{stsf}(2l+1, 2l+1)$. Let (φ, Ψ) be a pair, where $\varphi : L^{2l+1} \looparrowright \mathbb{R}^{4l+2}$ is an immersion in codimension $2l+1$, Ψ_L a stable skew-framing of the manifold L^{2l+1} in codimension $2l+1$, i.e., a skew-framing of the normal bundle of the composition $I \circ f : L^{2l+1} \looparrowright \mathbb{R}^{4l+2} \subset \mathbb{R}^r$, $r \geq 4l+4$ with the bundle $(2l+1)\kappa \oplus (r-2l-1)\varepsilon$, where ε is a trivial line bundle on L^{2l+1}, κ is a given line bundle over L^{2l+1}, which coincides with the oriented line bundle over L^{2l+1}, since $w_1(L^{2l+1}) = w_1(\nu_\varphi) = w_1((2l+1)\kappa) = w_1(\kappa)$. The set of pairs described above is equipped by an equivalence relation, which is defined by the standard regular cobordism. The set of equivalence classes generates an Abelian group whose operation is determined by the disjoint union. This group is denoted by $Imm^{stsf}(2l+1, 2l+1)$.

Homomorphisms $A : Imm^{fr}(4l+2,1) \longrightarrow Imm^{st\,fr}(4l+2,2l+1)$**,** $B : Imm^{st\,fr}(4l+2,2l+1) \to Imm^{fr}(4l+2,1)$**.** An arbitrary immersion $f : N^{4l+2} \looparrowright \mathbb{R}^{4l+3}$ determines the immersion $(I \circ f, \Xi_N)$ in codimension $2l+1$, $I : \mathbb{R}^{4l+3} \subset \mathbb{R}^{6l+3}$, which is stably framed in the ambient space $\mathbb{R}^{6l+3} \subset \mathbb{R}^n$. The homomorphism A is defined. An arbitrary stably framed immersion $(f : N^{4l+2} \looparrowright \mathbb{R}^{6l+3}, \Psi_N)$ obviously induces the immersion into the space $\mathbb{R}^r$, $r > 8l+6$. The Hirsch theorem, applied to this immersion of codimension $r-6l-3$, ensures the existence of a framed immersion (φ, Ξ_N), where $\varphi : N^{4l+2} \looparrowright \mathbb{R}^{4l+3}$. Now define $B([(f,\Psi_N)]) = [(\varphi,\Xi_N)] \in Imm^{fr}(4l+2,1)$. Obviously, by construction $B \circ A = Id : Imm^{fr}(4l+2,1) \to Imm^{fr}(4l+2,1)$.

Twisted Arf-Kervaire invariant $\Theta^{st} : Imm^{st\,fr}(4l+2,2l+1) \longrightarrow \mathbb{Z}/2$**.** We generalize the Arf-Kervaire homomorphism $\Theta : Imm^{fr}(4l+1,1) \to \mathbb{Z}/2$ and define a homomorphism $\Theta^{st} : Imm^{st\,fr}(4l+2,2l+1) \to \mathbb{Z}/2$ provided that $2l+1 \neq 1,3,7$, called *the twisted Arf-Kervaire invariant* such that the following diagram commutes:

$$\begin{array}{ccc} Imm^{fr}(4l+2,1) & \xrightarrow{A} & Imm^{st\,fr}(4l+2,2l+1) \\ \searrow \Theta & & \swarrow \Theta^{st} \\ & \mathbb{Z}/2 & \end{array} \qquad (*)$$

□

Auxiliary homomorphism $\pi : H_{2l+1}(N^{4l+2};\mathbb{Z}/2) \to \mathbb{Z}/2$**,** $2l+1 \neq 1,3,7$**.** It is known that for $2l+1 \neq 1,3,7$ there exist exactly two stably trivial S^{2l+1}-dimensional vector SO–bundles - this fact was applied in the proof of Kervaire and Milnor (1963, Lemma 8.3). One of these bundles is trivial; we denote it by $E(2l+1)$. The other bundle is nontrivial, and coincides with the tangent bundle $T(S^{2l+1})$ over the sphere S^{2l+1}. We need a generalization of this fact to the case of a $(2l+1)$–dimensional stably trivial bundle over an arbitrary closed $(4l+2)$–dimensional manifold, possibly nonoriented. Let M^{2l+1} be a closed, possibly nonoriented manifold, ξ a $SO(2l+1)$–bundle over M^{2l+1} such that it is trivial as a stable SO–bundle. Let M_1^{2l+1} and ξ be another manifold and a $SO(2l+1)$–bundle as above, respectively. Let W^{2l+2} be a $(2l+2)$–dimensional polyhedron, such that it is a manifold in the exterior of the codimension 2 skeleton. Let the polyhedron W realize a homology between the fundamental classes of the manifolds M^{2l+1} and M_1^{2l+1}, i.e. $\partial W^{2l+2} = M^{2l+1} \cup M_1^{2l+1}$. In addition, suppose that there exists a stably trivial $SO(2l+1)$–bundle Ξ over W^{2l+2} such that the restrictions of Ξ on M^{2l+1} and on M_1^{2l+1} coincide with the bundles ξ and ξ_1, respectively.

Lemma 3.1. *For an arbitrary above described pair* (M^{2l+1},ξ) *there exists an obstruction* $c(M,\xi) \in \mathbb{Z}/2$ *to the trivialization of the bundle* ξ*. Moreover,* $c(M^{2l+1},\xi) = c(M_1^{2l+1},\xi_1)$.

Corollary 3.2. *Let* $f : M^{2l+1} \to S^{2l+1}$ *be a map of a closed manifold to the standard sphere such that* $\deg(f) = 1 \pmod 2$*. Let* ξ *be a stably trivial* $SO(2l+1)$*–bundle over* S^{2l+1} *such that* $c(S^{2l+1},\xi) = 1$*. Then* $f^*(\xi)$ *is a stably trivial* $SO(2l+1)$*–bundle over* M^{2l+1} *and* $c(M^{2l+1}, f^*(\xi)) = 1 \pmod 2$.

Proof of Corollary 3.2. The corollary follows from the fact that there exists a homology between $f_*([M])$ and $[S]$, where $[M]$ and $[S]$ are the $(2l+1)$-dimensional fundamental classes of M^{2l+1} and S^{2l+1}. □

Proof of Lemma 3.1. Let (M^{2l+1}, ξ) be the pair described in the preamble of the lemma. Denote by $M^{[2l]} \subset M^{2l+1}$ the complement of the highest $(2l+1)$–dimensional cell in the skeleton of a cellular decomposition of M^{2l+1}. By the dimensionality argument, the restriction of ξ on $M^{[2l]}$ is a trivial bundle. Hence there exists a map $f : (M^{2l+1}, M^{[2l]}) \to (S^{2l+1}, pt)$, such that $f^*(\psi) = \xi$, where (S^{2l+1}, ψ) is a bundle over S^{2l+1}, satisfying the conditions of Lemma **3.1**. In the case when $M^{2l+1} = S^{2l+1}$, Lemma **3.1** is true since in the proof of Browder (1969, Theorem 3.2) the obstruction (S^{2l+1}, ξ) is constructed by the corresponding functional cohomological operation. Define $(M^{2l+1}, \xi) = (S^{2l+1}, \psi)$. Let (M_1^{2l+1}, ξ_1) be the second pair described above and (W^{2l+2}, Ξ) the homology, connecting (M^{2l+1}, ξ) and (M_1^{2l+1}, ξ_1). If c equals to zero for both pairs, Lemma **3.1** is proved.

Suppose that for at least one pair – say (M^{2l+1}, ξ) – the value of the obstruction (M^{2l+1}, ξ) is 1. Consider the handle (cell) decomposition of the cobordism (W^{2l+2}, M^{2l+1}). The index of handles can be restricted to $\leq 2l$ (the dimension of handles to $2l+1$) as in the case when W^{2l+2} is a smooth manifold. Indeed, the handle (cell) decomposition of (W^{2l+2}, M^{2l+1}) can be chosen so that all $(2l+2)$–dimensional cells retract to the $(2l+1)$–skeleton of the polyhedron W^{2l+2} without the $(2l+1)$–dimensional cells of the upper base M_1^{2l+1}. Define the map $F : (W^{2l+2}, M^{2l+1}) \to (S^{2l+1} \times I, S^{2l+1} \times \{0\})$ such that

$$F^* \pi^*(\psi) = \Xi, \tag{3}$$

where $\pi : S^{2l+1} \times I \to S^{2l+1}$ is a projection onto the lower base. The map F can be extended to the $(2l)$–dimensional handles uniquely up to homotopy. The map F can be extended also to the $(2l+1)$–dimensional handles, but possibly nonuniquely. By assumption, $c(S^{2l+1}, \psi) = 1$. Hence the extension of the map F to the $(2l+1)$–dimensional handles can be realized in a way that the condition (3) is satisfied. Now, on the upper base of the cobordism we have $f_1^*(\psi) = \xi_1$, therefore by definition, $c(M_1^{2l+1}, \xi_1) = 1$. Lemma **3.1** is thus proved. □

Let $(f : N^{4l+2} \looparrowright \mathbb{R}^{6l+3}, \Xi_N)$ be a pair defining an element of the group $Imm^{stfr}(4l+2, 2l+1)$. Consider an arbitrary cycle $x \in H = H_{2l+1}(N^{4l+2}; \mathbb{Z}/2)$. It is represented by an embedding $i_Y : Y^{2l+1} \to N^{4l+2}$. Denote shortly by ξ the bundle $i_Y^*(\nu_f)$, where ν_f is the normal bundle of an immersion f. Since ν_f is a stably trivial bundle (because the manifold N^{4l+2} is stably framed by Ξ), for the pair $i_Y^*(\nu_f)$ the obstruction (Y^{2l+1}, ξ) is defined, provided that $2l+1 \neq 1, 3, 7$. Define the mapping

$$\pi : H_{2l+1}(N^{4l+2}; \mathbb{Z}/2) \to \mathbb{Z}/2 \tag{4}$$

given by the formula $\pi(x) = c(Y^{2l+1}, \xi)$, $y \in H = H_{2l+1}(N^{4l+2}; \mathbb{Z}/2)$. Lemma **3.1** implies that the value of $\pi(x)$ does not depend on the choice of the manifold Y^{2l+1} and on the choice of the embedding l_x, which realizes the given cycle x. It can be easily verified that the mapping in (4) is a homomorphism.

Definition 3.3. Let $q : H \to \mathbb{Z}/2$, $H = H_{2l+1}(N^{4l+2}; \mathbb{Z}/2)$, be the quadratic form defined in (1) for a stably framed manifold (N^{4l+2}, Ξ). For $2l+1 \neq 1, 3, 7$ define the twisted quadratic form q^{tw} by the formula $q^{tw} = q + \pi : H \to \mathbb{Z}/2$. The Arf invariant of this twisted quadratic form defines a homomorphism $\Theta^{st} : Imm^{stfr}(4l+2, 2l+1) \longrightarrow \mathbb{Z}/2$, which is said to be the twisted Arf-Kervaire invariant.

Twisted Browder-Eccles invariant. Define the invariant $\bar{\Theta}^{st} : Imm^{stfr}(4l+2, 2l+1) \to \mathbb{Z}/2$, which is said to be the twisted Browder-Eccles invariant, starting by the construction of the homomorphism

$$\delta^{st} : Imm^{stfr}(4l+2, 2l+1) \to Imm^{stfs}(2l+1, 2l+1). \tag{5}$$

Suppose that an element of the group $Imm^{stfr}(4l+2, 2l+1)$ is represented by the pair $(f : N^{4l+2} \looparrowright \mathbb{R}^{6l+3}, \Xi)$. The double point manifold of immersion f is denoted by L^{2l+1}. This manifold is equipped by the parametrizing immersion $\varphi' : L^{2l+1} \looparrowright \mathbb{R}^{6l+3}$. The corresponding normal bundle $\nu_{\varphi'}$ admits (for a sufficiently big natural k) a stable isomorphism $\Psi' : \nu_{\varphi'} \oplus k\varepsilon \oplus k\kappa = (2l+1+k)\varepsilon \oplus (2l+1+k)\kappa$. The stable isomorphism Ψ' defines a stable isomorphism $\Psi_L : \nu_L \oplus k\varepsilon = (2l+1)\kappa \oplus k\varepsilon$, where by ν_L is denoted the $(2l+1)$–dimensional normal bundle over L. By the Smale-Hirsch construction an immersion $\varphi : L^{2l+1} \looparrowright \mathbb{R}^{4l+2}$ and a stably skew-framing Ψ_L of the normal bundle of this immersion are defined. Define $\delta^{st}([(f, \Xi_f)])$ to be the element of the group $Imm^{stsf}(2l+1, 2l+1)$ corresponding to the pair (φ, Ψ_L). Consider the homomorphism $Imm^{stsf}(2l+1, 2l+1) \xrightarrow{h} Imm^{\mathbf{D}_4}(0, 4l+2) = \mathbb{Z}/2$, defined as the parity of the number of double points of stably skew-framed immersions (this invariant is called the stably Hopf invariant). The Browder-Eccles invariant $\bar{\Theta}^{st}$ is defined as the composition $Imm^{stfr}(4l+2, 2l+1) \xrightarrow{h\circ\delta^{st}} Imm^{\mathbf{D}_4}(0, 4l+2) = \mathbb{Z}/2$.

Subgroup $Imm^{stfr}(4l+2, 2l+1)^* \subset Imm^{stfr}(4l+2, 2l+1)$**.** Define an auxiliary subgroup $Imm^{stfr}(4l+2, 2l+1)^* \subset Imm^{stfr}(4l+2, 2l+1)$ as the complete preimage of the group $\delta^{-1}(Imm^{sf}(2l+1, 2l+1) \subset Imm^{stsf}(2l+1, 2l+1))$ by the homomorphism (5), with the following additional assumption: there exists a representative $[(f, \Psi_f)] \in Imm^{stfr}(4k+2, 2k+1)^*$, $f : M^{4k+2} \looparrowright \mathbb{R}^{6n+3}$, such that M^{4k+2} is the boundary of a stably parallelizable manifold W^{4l+3}, $\partial(W^{4l+3}) = M^{4k+2}$. It is convinient to introduce $Imm^{stfr}(4l+2, 2l+1)^*$, which is an equivalent geometrical definition of an element from the preimage of δ. This definition is valid for all natural $l \geq 4$, but not for $l = 0, 1, 3$. The pair $(f : N^{4l+2} \looparrowright \mathbb{R}^{6l+3}, \Xi_N)$ represents an element $x \in Imm^{stfr}(4l+2, 2l+1)^*$ if the following is true. Take the pair $(\varphi : L^{2l+1} \looparrowright \mathbb{R}^{4l+2}, \Psi_L)$ representing the element $\delta^{st}(x)$; here L^{2l+1} is the double point manifold of the immersion f. Consider the canonical covering $\bar{L}^{2l+1} \to L^{2l+1}$ of the double point manifold L^{2l+1} (more details can be found in Adams 1962). Let $\bar{g} : \bar{L}^{2l+1} \looparrowright N^{4l+2}$ be the parametrizing immersion, $[\bar{L}] \in H = H_{2l+1}(N^{4l+2}; \mathbb{Z}/2)$ be the cycle obtained as the image of the fundamental class of $\bar{L}^{2l+1}$ by the immersion $\bar{g}$. Consider the value $\pi([\bar{L}])$, where $\pi : H \to \mathbb{Z}/2$ was defined in (4). Without losing generality we may assume that the stably framed immersion (f, Ξ) is chosen from the regular cobordism class so that the manifold $\bar{L}^{2l+1}$ is connected. This goal can be achieved through the 1-handles surgery on the double point manifold of immersion f. The techniques of such a surgery was invented by Haefliger (1962). Then the condition $x \in Imm^{stsf}(4l+2, 2l+1)^*$ is equivalent to $\pi([\bar{L}^{2l+1}]) = 0$. The last condition is equivalent to the fact that the pull-back $\bar{g}^*\nu_f$ over $\bar{L}^{2l+1}$ of the normal bundle ν_f is not only stably trivial but it is also trivial (in the case $l = 0, 1, 3$

the normal bundle ν_f is always trivial). It means that the immersion $\varphi : L^{2l+1} \looparrowright \mathbb{R}^{4l+2}$ is not only stably skew-framed but it is also skew-framed. This is the equivalent geometrical definition of the subgroup $Imm^{stfr}(4l+2, 2l+1)^*$. For convenience, all homomorphisms which have been constructed, include into the common commutative diagram:

$$\begin{array}{ccccc}
Imm^{fr}(4l+2,1) & \xrightarrow{B} & Imm^{stfr}(4l+2,2l+1)^* & \subset & Imm^{stfr}(4l+2,2l+1) \\
\downarrow \delta & & \downarrow \delta^{st} & & \downarrow \delta^{st} \\
Imm^{sf}(2l+1,2l+1) & = & Imm^{sf}(2l+1,2l+1) & \subset & Imm^{stsf}(2l+1,2l+1) \\
\downarrow h & & & & \downarrow h \\
Z/2 = Imm^{\mathbf{D}_4}(0,4l+2) & & = & & Imm^{\mathbf{D}_4}(0,4l+2)
\end{array}$$

4. Arf-Kervaire and Browder-Eccles (twisted) homomorphisms coincide

The following lemma is necessary for our proof of Theorem **1.2**.

Lemma 4.1. *The twisted Arf-Kervaire homomorphism $\Theta^{st} : Imm^{stfr}(4l+2, 2l+1) \to \mathbb{Z}/2$ coincides with the twisted Browder-Eccles homomorphism $\bar{\Theta}^{st} : Imm^{stfr}(4l+2, 2l+1) \to \mathbb{Z}/2$, on the subgroup $Imm^{stfr}(4l+2, 2l+1)^* \subset Imm^{stfr}(4l+2, 2l+1)$.*

We shall derive Lemma **4.1** from the following lemma.

Lemma 4.2. *(a) The homomorphism Θ^{st} and $\bar{\Theta}^{st}$ coincide on the subgroup $Im(B) \subset Imm^{stfr}(4l+2, 2l+1)^* \subset Imm^{stfr}(4l+2, 2l+1)$.*

(b) For an arbitrary element $x \in Imm^{fr}(4k+2, 1)$, there exists a representative $f : M^{4l+2} \looparrowright \mathbb{R}^{4l+3}$, $[f] = x$, such that M^{4l+2} is the boundary of a stably parallelizable manifold $W^{4l+3}, \partial(W^{4l+3}) = M^{4l+2}$.

Proof of Lemma 4.2. (a) This was proved by Eccles (1981), reformulated in the required form in (Akhmetiev and Eccles, 1999).

(b) This is a well-known observation by Nigel Ray, concerning Khan-Priddy transfer (Eccles 1979, ref. 7). □

Proof of Lemma 4.1. Let $(f : N^{4l+2} \looparrowright \mathbb{R}^{6l+3}, \Psi_N)$ be a stably framed immersion. Let $\eta : W^{4l+3} \to \mathbb{R}^{6l+3} \times \mathbb{R}^1_+$ be a generic mapping of a stably parallelized manifold (W^{4l+3}, Ψ_W) defining the boundary of stably framed manifold (N^{4l+2}, Ψ) but in general, not the immersion f itself. The dimension of the critical point manifold Σ of η equals to $2l+1$; this is less than half of dimension of the manifold–preimage W^{4l+3}. The critical points of η are of type $\Sigma^{1,0}$. By the Moren theorem (Arnold *et al.* 2012, Ch. 1, Par. 9, Sect. 6, case $k = 2$) the critical point manifold has the normal form called the extended Whitney umbrella. The formulae describing the singularities of Whitney umbrella $\mathbb{R}^s \to \mathbb{R}^{2s-1}$ were given by Pontryagin (1955, Ch. 1, Par. 4). The notion "extended" means the inclusion of the standard singularity of umbrella into the identical polyparametrical collection of maps. One may assume without loss of generality, after a corresponding repair of the singularity of map η, that the critical point manifold Σ satisfies the following properties.

(1) Σ is connected with connected canonical double covering $\bar{\Sigma}$.

(2) $\eta(\Sigma)$ belongs to the hyperspace $\mathbb{R}^{6l+3} \times \{1\}$ and the double point manifold K^{2l+2} of map η with the boundary $\partial K^{2l+2} = L^{2l+1} \cup \Sigma^{2l+1}$ is regular in a small neighborhood of the boundary with respect to the height function on $\mathbb{R}^1_+$ in such way that the subspace $\mathbb{R}^{6l+3} \times \{1\}$ is higher than the manifold (i.e. K^{2l+2} immerses into the subspace $[+\varepsilon, 1-\varepsilon] \times \mathbb{R}^{6l+3}$ outside its regular neighborhood.

Let $\eta_{1-\varepsilon} : N^{4l+2}_{1-\varepsilon} \looparrowright \mathbb{R}^{6l+3} \times \{1-\varepsilon\}$ be the immersion defined as the restriction of η on the complete preimage of the hyperspace $\mathbb{R}^{6l+3} \times \{1-\varepsilon\}$. Let $L^{2l+1}_{1-\varepsilon}$ be the component of the double point manifold $\eta_{1-\varepsilon}(N^{4l+2}_{1-\varepsilon})$ in the neighborhood of the critical point boundary Σ^{2l+1} of K^{2l+2}. From the assumption $\pi(\xi) = 0$ we may deduce that the normal bundle $\nu_{L_{1-\varepsilon}}$ of the manifold $L^{2l+1}_{1-\varepsilon}$ is decomposed into the direct sum of the trivial bundle $\nu_\varepsilon = (2l+1)\varepsilon$ and a nontrivial bundle $\nu_\kappa = \nu_\varepsilon \otimes \kappa$, where κ is the orientation line bundle over $L^{2l+1}_{1-\varepsilon}$. Since the canonical covering is connected, κ is nontrivial.

Construction of the stably framed immersion. Let us construct the stably framed immersion

$$(\xi_0 : N^{4l+2}_0 \looparrowright \mathbb{R}^{6l+3}, \Psi_0) \tag{6}$$

such that the double point manifold L^{2l+1}_0 (equipped with a skew-framing Ξ_0 of the normal bundle) coincides with an arbitrary given skew-framed immersion. Let us start the construction by the description of standard immersion $g_0 : S^{2l+1} \looparrowright \mathbb{R}^{4l+2}$ with the self-intersection points at the origin of the coordinate system $\mathbb{R}^{2l+1}_1 \oplus \mathbb{R}^{2l+1}_2 = \mathbb{R}^{4l+2}$. Let $\mathbb{R}^{2l+1}_{diag}$, $\mathbb{R}^{2l+1}_{antidiag}$ be two coordinate subspaces defined by means of the sum and the difference of the base vectors in the standard coordinate spaces $\mathbb{R}^{2l+1}_1, \mathbb{R}^{2l+1}_2$. Consider two standard unit disks $D^{2l+1}_1 \subset \mathbb{R}^{2l+1}_1$, $D^{2l+1}_2 \subset \mathbb{R}^{2l+1}_2$. Take a manifold C diffeomorphic to the cylinder $S^{2l} \times I$ defined as the collection of all the segments such that each connects a pair of points in ∂D^{2l+1}_1 and ∂D^{2l+1}_2 with equal coordinates. The union of two disks $D^{2l+1}_1 \cup D^{2l+1}_2$ with C (after the identification of corresponding components of the boundary) is the image of the standard sphere S^{2l+1} by a PL-immersion g_0 into $\mathbb{R}^{4l+2}$, with one self-intersection point at the origin. After the smoothing of corners along ∂C we obtain the smooth immersion of sphere under construction.

Let us describe the manifold N^{4l+2}_0, the stable framing Ξ_{N_0} over this manifold and the immersion $f_0 : N^{4l+2}_0 \looparrowright \mathbb{R}^{6l+3}$, $[(f_0, \Xi_{N_0})] \in Imm^{stsf}(4l+2, 2l+1)$. Take the embedding $\eta_0 : L^{2l+1}_0 \subset \mathbb{R}^{6l+3}$ with the normal bundle $\nu_{L_0} = \nu_1 \oplus \nu_1 \otimes \kappa$, where ν_1 is a trivial $(2l+1)$–dimensional bundle (with the prescribed trivialization) and κ is the orientation line bundle over L^{2l+1}_0. Consider the $(2l+2)$–dimensional bundle $\nu_1 \otimes \kappa \oplus \varepsilon$ over L^{2l+1}_0 and define the manifold N^{4l+2}_0 as the boundary $S(\nu_1 \otimes \kappa \oplus \varepsilon)$ of the disk bundle of this vector bundle. The locally trivial fibration $p : N^{4l+2}_0 \to L^{2l+1}_0$ is well-defined. Because $\nu_1 \otimes \kappa \oplus \varepsilon$ is the normal bundle of L^{2l+1}_0, the manifold N^{4l+2}_0 admits an embedding in codimension 1. This embedding determines the framing Ψ_{N_0} over N^{4l+2}_0 such that the constructed framed manifold (N^{4l+2}_0, Ψ_{N_0}) is bounding. Let us define the immersion $f_0 : N^{4l+2}_0 \looparrowright \mathbb{R}^{6l+3}$. Take the normal bundle $\nu_1 \otimes \kappa \oplus \nu_1$ of the immersion η_0 and consider the collection of the standard immersed spheres $g_0(S^{2l+1})$ constructed above in each fiber of η_0. The pair (f_0, Ψ_0) is the stably framed immersion under construction.

Calculation of invariants of the constructed stably framed immersion. This section is devoted to the calculation of the twisted Arf-Kervaire and the twisted Browder-Eccles invariants for the stably framed immersion (6). The group $H = H_{2l+1}(N_0^{4l+2};\mathbb{Z}/2)$ is generated by two elements. The first generator $x \in H$ is represented by a spherical fiber of the fibration $p : N_0^{4l+2} \to L_0^{2l+1}$. The fibration p has a standard section p^{-1} constructed by the trivial direct summand in the bundle $\nu_1 \otimes \kappa \oplus \varepsilon$. The image of the fundamental class of the base L_0^{2l+1} induced by p^{-1}, represents the second generator $y \in H$. Let us prove under the assumption $2l+1 \neq 1,3,7$ that the homomorphism $\pi : H \to \mathbb{Z}/2$ is defined by $\pi(x) = 1, \pi(y) = 0$. The condition $\pi(x) = 1$ holds since for an arbitrary immersion of a sphere $f_0 : S^{2l+1} \looparrowright \mathbb{R}^{4l+2}$ with one self-intersection point the corresponding normal $2l+1$–dimensional bundle is nontrivial. Let us prove that the following holds

$$\pi(y) = 0. \tag{7}$$

The cycle $y \in H$ is represented by the image of the fundamental class induced by the map of section $p^{-1}(L_0^{2l+1}) \to N_0^{4l+2}$ of the fibration p. The collection of the bases in the fibers of the subbundle $\nu_1 \subset \nu_{L_0}$ defines the trivialization of the normal bundle of the immersion f_0 over the submanifold $p^{-1}(L_0^{2l+1}) \subset N_0^{4l+2}$. This proves (7). The twisted Arf-Kervaire invariant of a stably framed immersion (ξ_0,Ψ_0) is equal to $q(y)$, i.e., coincides with the twisted Browder-Eccles invariant. This gives the required computations.

Let us finish the proof of Lemma **4.1**. Consider a stably framed immersion $(f_{1-\varepsilon},\Psi_{N_{1-\varepsilon}})$ with the skew-framed double point manifold $(L_{1-\varepsilon}^{2l+1},\Xi_{L_{1-\varepsilon}})$. The stably framed immersion $(\eta_{1-\varepsilon},\Psi_{N_{1-\varepsilon}})$ is regularly cobordant to the immersion (f,Ψ_N) from the beginning of the proof of Lemma **4.1**. Obviously, by considering the normal form of Whitney umbrella, $[(f_{1-\varepsilon},\Psi_{1-\varepsilon})] \in Imm^{st\,fr}(4l+2,2l+1)^* \subset Imm^{st\,fr}(4l+2,2l+1)$ and the stably framed immersion $(f_{1-\varepsilon},\Psi_{N_{1-\varepsilon}})$ is in fact, framed and the stably skew-framed immersion $(L_{1-\varepsilon}^{2l+1},\Xi_{L_{1-\varepsilon}})$ is in fact, skew-framed. The normal bundle $\bar{\nu}$ of the submanifold $\bar{L}_{1-\varepsilon}^{2l+1} \subset N_{1-\varepsilon}^{4l+2}$ is a stably trivial bundle. Let us prove that this bundle is trivial. Indeed, the fibers of $\bar{\nu}$ over antipodal pair of points of the projection $p : \bar{L}_{1-\varepsilon}^{2l+1} \to L_{1-\varepsilon}^{2l+1}$ are cannonicaly isomorphic (by the normal form arguments) and the bundle $\bar{\nu}$ is the pull-back of a stable-trivial bundle ν over $L_{1-\varepsilon}^{2l+1}$. This is true since the obstruction π, given by the formula (4), is trivial. By this fact, the normal bundle over $L_{1-\varepsilon}^{2l+1}$ splits into a framed subbundle ν_0 (which in general, does not coincide with ν) and a skew-framed subbundle $\nu_0 \otimes \kappa_p$, like at the beginning of the construction of the stably framed immersion $(\xi_0 : N_0^{4l+2} \looparrowright \mathbb{R}^{6l+3},\Psi_0)$ from formula (6).

Let us apply the construction (6), where the standard stably framed immersion (f_0,Ψ_{N_0}) is obtained so that its self-intersection points are opposite to the points of the skew-framed immersion $(L_{1-\varepsilon}^{2l+1},\Xi_{L_{1-\varepsilon}})$. Note that the disjoint union $(\eta_{1-\varepsilon},\Psi_{N_{1-\varepsilon}}) \cup (f_0,\Psi_{N_0})$ is a stably framed boundary. Note that the skew-framings of $\delta^{st}(\xi_0,\Psi_0)$ can be distingushed from the skew-framing of the immersion, which is determined by the boundary W^{4k+3}. The two skew-framings admit a common Browder-Eccles invariant since they are well-defined as skew-framings of a common immersion into R^{4l+2}. Therefore for the stably framed immersion $(\xi_{1-\varepsilon},\Psi_{1-\varepsilon}) \cup (\xi_0,\Psi_0)$, both invariants are trivial. On other hand, by the calculations, the twisted Arf-Kervaire invariant and the twisted Browder-Eccles invariant of (f_0,Ψ_{N_0}) coincide. Hence for the stably framed immersion $(\eta_{1-\varepsilon},\Xi_{N_{1-\varepsilon}})$, both invariants coincide and moreover, for the stably framed immersion $(f, \Xi_N) \in Imm^{stfr}(4l+2, 2l+1)^*$, both invariants coincide. Lemma **4.1** is thus proved. $\square$

Proof of Theorem 1.2. Let $4l+2 = 30$ or $4l+2 = 62$. Consider a closed framed $2l$–connected manifold (N^{4l+2}, Ψ) with the Arf-Kervaire invariant 1. Let us assume that for $l = 7$ ($l = 15$) there exists an embedding $J : N^{4l+2} \subset \mathbb{R}^{6l+3}$ and that the normal $(2k+2)$–bundle $\nu_{\bar{J}}$ of this embedding is equipped with 9 (resp. 10) linearly independent sections. The restriction of the normal bundle ν_J on an arbitrary embedded sphere $i_{S^{2l+1}} : S^{2l+1} \to N^{4l+2}$ is the trivial $(2l+1)$–dimensional, i.e., 15–dimensional (resp. 31–dimensional) bundle (the bundle $i^*_{S^{2l+1}}(\nu_J)$ is stably dimensional and trivial), equipped with 9 (resp. 10) linearly independent sections. There is only one stably trivial but nontrivial 15–dimensional (resp. 31–dimensional) bundle over S^{15} (resp. S^{31}); this bundle is the tangent bundle $T(S^{2l+1})$ (Kervaire and Milnor 1963, p. 534). The bundle $i^*_{S^{2l+1}}(\nu_J)$ over S^{15} (S^{31}) is trivial if and only if $c(S^{15}, i^*_{S^{15}}(\nu_J)) = 0$ (resp. $c(S^{31}, i^*_{S^{31}}(\nu_J) = 0)$. By the Adams theorem (Novikov 1964, Ch. 3, Par. 8, p. 106, the reference on the Adams result), the tangent bundle $T(S^{15})$ (resp. $T(S^{31})$) admits no more than 8 (resp. 9) linearly independent sections. By our assumption the bundle $i^*_{S^{2l+1}}(\nu_J)$ admits 9 (resp. 10) linearly independent sections. Therefore $i^*_{S^{2l+1}}(\nu_J)$ is a trivial bundle. Hence the auxiliary homomorphism $\pi : H \to \mathbb{Z}/2$ is trivial and the Arf-Kervaire invariant of the stably framed manifold (N^{4l+2}, Ξ) is equal to the twisted Arf-Kervaire invariant of the pair (J, Ξ), so both are equal to 1. On the other hand, by Lemmas **4.1** and **4.2**, the twisted Arf-Kervaire coincides with the twisted Browder-Eccles invariant.

Let us assume that for $l = 7$ ($l = 15$) there exists an embedding $\bar{J} : N^{4l+2} \times I \subset \mathbb{R}^{6l+4}$ and that the normal $(2k+2)$–bundle $\nu_{\bar{J}}$ of this embedding is equipped with 9 (resp. 10) linearly independent sections over the complement to the base segment $pt \times I \in N^{4l+2} \times I$. The restriction of the normal bundle $\bar{\nu}(\bar{J})$ over an arbitrary embedded sphere $i_{S^{2l+1}} : S^{2l+1} \subset N^{4l+2}$ is the trivial $(2l+1)$–bundle since it is equipped with 9 (resp. 10) linearly independent sections. By the Rourke-Sanderson compression theorem (Rourke and Sanderson 2001), we may assume, after an appropriate isotopy of the framed embedding $\bar{J}$, that the collection of segments I is vertically up with respect to the axis of projection. After the projection we obtain an immersion $J : N^{4l+2} \looparrowright \mathbb{R}^{6l+3}$, which is framed at least outside a neighborhood of a point. The double point manifold L^{2l+1} of the immersion J is stably skew-framed, hence in fact it is framed. Let us denote this framing by Ξ.

The framed manifold (L^{2l+1}, Ξ) determines an element of the group $Imm^{stsf}(2l+1, 2l+1)$ lying in the image of the homomorphism $\Pi_{2l+1} = Imm^{fr}(2l+1, 2l+1) \to Imm^{stsf}(2l+1, 2l+1)$. Therefore the twisted Browder-Eccles invariant of the stably framed immersion (J, Ψ_N) is equal to the stable Hopf invariant of the framed manifold (L^{2l+1}, Ξ_L). By the Toda theorem for $2l+1 = 15$ and by the Adams theorem for $2l+1 = 31$, the Hopf invariant is equal to 0 (Mosher and Tangora 1968, Sect. 18). On the other hand, by Lemma **4.1**, the twisted Browder-Eccles invariant of (J, Ψ_N) is equal to the twisted Arf-Kervaire invariant of (J, Ψ_N). The latter is equal to the Arf-Kervaire invariant of the framed manifold (N, Ψ_N) because the auxiliary homomorphism $\pi : H \to \mathbb{Z}/2$ for the stably framed immersion (J, Ψ_N) is trivial. Therefore the twisted Arf-Kervaire invariant is equal to 1. This contradiction shows that if the manifold $N^{4l+2} \times I$ embeds in $\mathbb{R}^{6l+4}$, then the collection of linearly independent sections of the normal bundle does not exist. Theorem **1.2** is thus proved. $\square$

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