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On well-splitting posets

Dušan Repovš[1,2,3] · **Lyubomyr Zdomskyy**[4]

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Abstract
We introduce a class of proper posets which is preserved under countable support iterations, includes ω^ω-bounding, Cohen, Miller, and Mathias posets associated to filters with the Hurewicz covering properties, and has the property that the ground model reals remain splitting and unbounded in corresponding extensions. Our results may be considered as a possible path towards solving variations of the famous Roitman problem.

Keywords Splitting · Bounding · Miller forcing · Filter · Hurewicz space · Mad family · Roitman problem

Mathematics Subject Classification Primary 03E35 · 03E17; Secondary 54D20

1 Introduction

The famous Roitman problem asks whether it is consistent that $\mathfrak{d} = \omega_1 < \mathfrak{a}$. Here, $\mathfrak{d}$ is the minimal cardinality of a subfamily of ω^ω which is *dominating* with respect to the preorder relation $\le^*$ on ω^ω, where $a \le^* b$ for $a, b \in \omega^\omega$ means that $a(n) \le b(n)$ for all but finitely many n, and $\mathfrak{a}$ is the minimal cardinality of an infinite *mad subfamily*

✉ Lyubomyr Zdomskyy
lzdomsky@gmail.com
http://www.logic.univie.ac.at/~lzdomsky/

Dušan Repovš
dusan.repovs@guest.arnes.si
https://repovs.splet.arnes.si/

[1] Faculty of Education, University of Ljubljana, 1000 Ljubljana, Slovenia

[2] Faculty of Mathematics and Physics, University of Ljubljana, 1000 Ljubljana, Slovenia

[3] Institute of Mathematics, Physics and Mechanics, 1000 Ljubljana, Slovenia

[4] Institut für Mathematik, Kurt Gödel Research Center, Universität Wien, Kolingasse 14-16, 1090 Wien, Austria

$\mathcal{A}$ of $[\omega]^\omega$, i.e., an infinite subfamily whose distinct elements have finite intersection and which is maximal with respect to this property.

Without the restriction $\mathfrak{d} = \omega_1$, the consistency of $\mathfrak{d} < \mathfrak{a}$ has been established in a breakthrough work of Shelah [14]. Regarding the original Roitmann problem, even its weaker version stated in [4] remains open: Is it consistent that $\mathfrak{s} = \mathfrak{b} = \omega_1 < \mathfrak{a}$? Here, $\mathfrak{s}$ is the minimal cardinality of a *splitting* family, i.e., a family $\mathcal{S} \subset [\omega]^\omega$ such that for every $X \in [\omega]^\omega$ there exists $S \in \mathcal{S}$ for which both $S \cap X$ and $X \setminus S$ are infinite, and $\mathfrak{b}$ is the minimal cardinality of a subfamily of ω^ω which is *unbounded* with respect to $\leq^*$. It is well-known that $\max\{\mathfrak{b}, \mathfrak{s}\} \leq \mathfrak{d}$ and the strict inequality holds, e.g., in the Cohen model (see [2, 15] for more information on these and many other cardinal characteristics of the continuum).

In this paper we isolate the class of *well-splitting* posets (see the next section for the definition) with properties described in the abstract, aiming at the solution of the aforementioned weak version of Roitman's problem. This class includes among others, Mathias posets associated to filters on ω with the Hurewicz covering property. This motivates the following

Question 1.1 (CH) *Can every mad family be destroyed by a well-splitting poset? In particular, given a mad family $\mathcal{A}$, is there a well-splitting poset $\mathbb{P}$ such that in $V^{\mathbb{P}}$, $\{\omega \setminus A : A \in \mathcal{A}\}$ can be enlarged to a Hurewicz filter, or more generally to a filter, whose Mathias forcing is well-splitting?*

By our Theorem 2.8, proved in the next section, the affirmative answer to Question 1.1 would allow one to construct a model of $\mathfrak{b} = \mathfrak{s} = \omega_1 < \mathfrak{a} = \omega_2$.

Recall from [8] that a topological space X is said to have the *Hurewicz covering property* (or is simply called a *Hurewicz space*) if for every sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of open covers of X there exists a sequence $\langle \mathcal{V}_n : n \in \omega \rangle$ such that each $\mathcal{V}_n$ is a finite subfamily of $\mathcal{U}_n$ and the collection $\{\cup \mathcal{V}_n : n \in \omega\}$ is a γ-cover of X, i.e., the set $\{n \in \omega : x \notin \cup \mathcal{V}_n\}$ is finite for each $x \in X$. It is clear that σ-compact spaces are Hurewicz, but by [9, Theorem 5.1] there also exist non-σ-compact sets of reals having the Hurewicz property. We consider each filter on ω with the subspace topology inherited from $\mathcal{P}(\omega)$, the latter being a topological copy of the Cantor space 2^ω via characteristic functions. As it was proved in [7], the Mathias forcing associated to a filter $\mathcal{F}$ is almost ω^ω-bounding in terminology of [13] if and only if $\mathcal{F}$ is Hurewicz. It is worth mentioning here that in general, almost ω^ω-bounding posets can make ground model reals non-splitting, see, e.g., [13, Lemma 1.14], so by our Lemma 2.1, almost ω^ω-bounding posets do not have to be well-splitting.

Building on the proof of [3, Theorem 3.1], it was established in [16] that under CH, for every mad family $\mathcal{A}$, the collection $\{\omega \setminus A : A \in \mathcal{A}\}$ can be enlarged to an ultrafilter $\mathcal{F}$ with a certain covering property which is weaker (but similar) to the Hurewicz one, and whose Mathias forcing does not produce any new real dominating the given ground model unbounded subset. The construction in [16] cannot be directly used to answer Question 1.1 since by Lemma 2.1, the Mathias forcing for ultrafilters cannot be well-splitting because it adds an unsplit real. However, it is natural to ask how far can one weaken the Hurewicz property of a filter so that its Mathias forcing is still well-splitting.

Question 1.2 *Let $\mathcal{F}$ be a filter on ω whose Mathias forcing is well-splitting. Is then $\mathcal{F}$ Hurewicz? In other words, are well-splitting and almost ω^ω-bounding equivalent for such posets?*

2 Well-splitting posets

Throughout this section we denote by E_0 and E_1 the sets of all even and odd natural numbers, respectively. A strictly increasing function $f \in \omega^\omega$ is said to *well-split* a set M if the sets $\{n \in E_j : |[f(n), f(n+1)) \cap Y| \geq 2\}$ are infinite for all $Y \in M \cap [\omega]^\omega$ and $j \in 2$. A simple diagonalization argument shows that for every countable M there is a function well-splitting M.

We shall say that a poset $\mathbb{P}$ is *well-splitting* if the following is satisfied: Whenever $\mathbb{P} \in M$, where M is a countable elementary submodel of $H(\theta)$ for any sufficiently large θ, $p \in M \cap P$, and f well-splits M, then there is some $q \leq p$ which is $(M, \mathbb{P})$-generic and such that q forces f to well-split $M[\Gamma]$, where Γ is the canonical name for $\mathbb{P}$-generic filter.

Lemma 2.1 *Supose that $\mathbb{P}$ is well-splitting and G is $\mathbb{P}$-generic. Then $V \cap [\omega]^\omega$ is splitting and $V \cap \omega^\omega$ is unbounded in $V[G]$.*

Proof To see that $\omega^\omega \cap V$ is unbounded, let us fix a $\mathbb{P}$-name $\dot{h}$ for an element of $\omega^{\uparrow\omega}$ (the family of all strictly increasing functions in ω^ω), $p \in \mathbb{P}$, and pick a countable elementary submodel M of $H(\theta)$ such that $\mathbb{P}, \dot{h}, p \in M$. Suppose that f well-splits M and $q \leq p$ is any $(M, \mathbb{P})$-generic condition which forces f to well-split $M[\Gamma]$. Let $\dot{h}_1 \in M$ be a $\mathbb{P}$-name for the following function: $\dot{h}_1(0) = 1$, $\dot{h}_1(n+1) = \dot{h}(\dot{h}_1(n)) + 1$ for all $n \in \omega$. It follows from the above that q forces the set

$$\dot{I} := \{n \in E_0 : |[f(n), f(n+1)) \cap \mathrm{range}(\dot{h}_1)| \geq 2\}$$

to be infinite. Let $G \ni q$ be $\mathbb{P}$-generic and set $I = \dot{I}^G$, $h = \dot{h}^G$, and $h_1 = \dot{h}_1^G$. Let us note that $|[f(\min I), f(\min I + 1)) \cap \mathrm{range}(h_1)| \geq 2$ yields

$$f(\min I + 1) \geq f(\min I) + 2 \geq \min I + 2 = (\min I + 1) + 1,$$

and therefore $f(i) \geq i + 1$ for every $i > \min I$, because f is strictly increasing.

In $V[G]$, for every $i \in I \setminus \{\min I\}$ we can find $n_i \in \omega$ such that $h_1(n_i), h_1(n_i+1) \in [f(i), f(i+1))$. Thus[1]

$$h_1(n_i + 1) = h(h_1(n_i)) + 1 < f(i+1) \leq f(h_1(n_i)),$$

i.e.,

$$\big\{h_1(n_i) : i \in I \setminus \{\min I\}\big\} \subset \{k : h(k) < f(k)\},$$

[1] The second inequality follows from $h_1(n_i) \geq f(i) \geq i + 1$.

and hence h does not dominate f. Summarizing the above, we conclude that for any $p \in \mathbb{P}$ and $\mathbb{P}$-name $\dot{h}$ for some element of $\omega^{\uparrow\omega}$, there is a stronger condition q and $f \in \omega^{\uparrow\omega} \cap V$ such that q forces the set $\{k : \dot{h}(k) < f(k)\}$ to be infinite. This means precisely that $\omega^{\uparrow\omega} \cap V$ is unbounded in $V[G]$ for any $\mathbb{P}$-generic filter G.

To prove that $[\omega]^\omega \cap V$ is splitting, let us fix a $\mathbb{P}$-name $\dot{Y}$ for some element of $[\omega]^\omega$, $p \in \mathbb{P}$, and pick a countable elementary submodel M of $H(\theta)$ such that $\mathbb{P}, \dot{Y}, p \in M$. Suppose that f well-splits M and $q \leq p$ is any $(M, \mathbb{P})$-generic condition which forces f to well-split $M[\Gamma]$. Then q forces the sets

$$\dot{I}_j := \bigcup_{n \in E_j} [f(n), f(n+1)) \cap \dot{Y}$$

to be infinite for all $j \in 2$. Since the sets $\bigcup_{n \in E_j} [f(n), f(n+1))$, $j \in 2$, are disjoint, infinite, and are both in V, this completes our proof. □

The converse of Lemma 2.1 does not hold. To get the corresponding counterexample we shall use the standard Mathias forcing $\mathcal{M}_\mathcal{F}$ associated to a filter $\mathcal{F}$ on ω defined as follows: $\mathcal{M}_\mathcal{F}$ consists of pairs $\langle s, F\rangle$ such that $s \in [\omega]^{<\omega}$, $F \in \mathcal{F}$, and $\max s < \min F$. A condition $\langle s, F\rangle$ is said to be *stronger* than $\langle t, G\rangle$ if $F \subset G$, s is an end-extension of t, and $s \setminus t \subset G$. $\mathcal{M}_\mathcal{F}$ is easily seen to introduce a generic subset $X \in [\omega]^\omega$ such that $X \subset^* F$ for all $F \in \mathcal{F}$, thus turning the ground model subsets of ω into an unsplitting family if $\mathcal{F}$ is an ultrafilter.

Each filter $\mathcal{F}$ on ω gives rise to the filter $(\mathcal{F})^{<\omega}$ on $\mathsf{fin} = [\omega]^{<\omega} \setminus \{\emptyset\}$ generated by $\{[F]^{<\omega} \setminus \{\emptyset\} : F \in \mathcal{F}\}$ as a base. Standardly, by $((\mathcal{F})^{<\omega})^+$ we denote the family of all subsets X of fin such that $X \cap Y \neq \emptyset$ for all $Y \in (\mathcal{F})^{<\omega}$.

Example 2.2 There exists a ccc non-well-splitting poset which preserves ground model reals as a splitting and unbounded family. Indeed, let $\mathbb{C}_{\omega_1}$ be the poset obtained by adding ω_1-many Cohen reals by an iteration with at most countable supports and G a $\mathbb{C}_{\omega_1}$-generic. Let $\mathcal{U} \in V$ be an ultrafilter on ω and $\mathcal{U}' = \{X \in \mathcal{P}(\omega)^{V[G]} : \exists U \in \mathcal{U}(U \subset X)\}$ be the filter in $V[G]$ generated by $\mathcal{U}$ as its base. We claim that in $V[G]$ the poset $\mathbb{Q} := \mathcal{M}_{\mathcal{U}'}$ preserves ground model reals (i.e., $([\omega]^\omega)^{V[G]}$) as a splitting and unbounded family in $V[G * H]$, where H is $\mathbb{Q}$-generic over $V[G]$. Indeed, by a rather standard argument similar to that in the proof of [12, Theorem 11], one can check that $(\mathcal{U}')^{<\omega}$ is $+$-Ramsey in $V[G]$ in the sense of [10], i.e., for every sequence $\langle X_n : n \in \omega\rangle$ in $((\mathcal{U}')^{<\omega})^+$ there is a selector $\langle a_n \in X_n : n \in \omega\rangle$ such that $\{a_n : n \in \omega\} \in ((\mathcal{U}')^{<\omega})^+$. By [6, Prop. 1 and Th. 19], we get that $(\omega^\omega)^{V[G]}$ is non-meager in $(\omega^\omega)^{V[G*H]}$. This implies that $(\omega^\omega)^{V[G]}$ is unbounded in $V[G * H]$, and $([\omega]^\omega)^{V[G]}$ is a splitting family in $V[G * H]$. Indeed, if there existed $b \in (\omega^\omega)^{V[G*H]}$ bounding $(\omega^\omega)^{V[G]}$, then $(\omega^\omega)^{V[G]} \subset \bigcup_{n\in\omega} K_n$, where $K_n = \{x \in \omega^\omega : \forall m \geq n (x(m) \leq b(m))\}$, and each K_n is a closed and nowhere dense subset of ω^ω. Similarly, if $B \in ([\omega]^\omega)^{V[G*H]}$ were unsplit by $([\omega]^\omega)^{V[G]}$, then

$$([\omega]^\omega)^{V[G]} \subset \bigcup_{n\in\omega} L_n \cup \bigcup_{n\in\omega} M_n,$$

where $L_n = \{X \subset \omega : X \cap B \subset n\}$ and $M_n = \{X \subset \omega : B \backslash n \subset X\}$, and each L_n and M_n is easily seen to be closed and nowhere dense subset of $\mathcal{P}(\omega)$.

On the other hand, $\mathbb{C}_{\omega_1} * \mathbb{Q}$ adds a pseudointersection to $\mathcal{U}$ and hence $([\omega]^\omega)^V$ is not a splitting family in $V[G * H]$. Since $\mathbb{C}_{\omega_1}$ is well-splitting by Lemma 2.3 combined with Corollary 2.6, we conclude that $\mathbb{Q}$ is not well-splitting in $V[G]$: If it were well-splitting in $V[G]$, then $\mathbb{C}_{\omega_1} * \mathbb{Q}$ would be well-splitting in V by Lemma 2.3, and hence $([\omega]^\omega)^V$ would be a splitting family in $V[G * H]$ by Lemma 2.1. □

It is clear that each well-splitting poset is proper and an iteration of finitely many well-splitting posets is again well-splitting. Next, we shall establish that being well-splitting is also preserved by countable support iterations. The proof of the following lemma is similar to that of [1, Lemma 2.8], with some additional control on the sequence $\langle \dot{p}_i : i \in \omega \rangle$.

Let us make a couple of standard conventions regarding our notation. Whenever $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \delta \rangle$ is an iterated forcing construction, we denote by $\mathbb{P}_{[\alpha_0,\alpha_1)}$ a $\mathbb{P}_{\alpha_0}$-name for the quotient poset $\mathbb{P}_{\alpha_1}/\mathbb{P}_{\alpha_0}$, viewed naturally as an iteration over the ordinals $\xi \in \alpha_1 \backslash \alpha_0$. For a $\mathbb{P}_{\alpha_0}$-generic G and a $\mathbb{P}_{\alpha_1}$-name τ, where $\alpha_0 \le \alpha_1$, we denote by τ^G the $\mathbb{P}^G_{[\alpha_0,\alpha_1)}$-name in $V[G]$ obtained from τ by partially interpreting it with G. This allows us to speak about, e.g., $\mathbb{P}^G_{[\alpha_1,\alpha_2)}$ for $\alpha_0 \le \alpha_1 \le \alpha_2 \le \delta$ and a $\mathbb{P}_{\alpha_0}$-generic filter G. For a poset $\mathbb{P}$ we shall denote the standard $\mathbb{P}$-name for $\mathbb{P}$-generic filter by $\Gamma_{\mathbb{P}}$. We shall write Γ_α instead of $\Gamma_{\mathbb{P}_\alpha}$ whenever we work with an iterated forcing construction which will be clear from the context. Also, $\Gamma_{[\alpha_0,\alpha_1)}$ is a $\mathbb{P}_{\alpha_1}$-name whose interpretation with respect to a $\mathbb{P}_{\alpha_0}$-generic filter G is $\Gamma_{\mathbb{P}^G_{[\alpha_0,\alpha_1)}}$, which is an element of $V[G]$.

Lemma 2.3 *If $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \delta \rangle \in M$ is a countable support iteration of well-splitting (hence proper) posets, then $\mathbb{P}$ is also well-splitting.*

Proof The proof is by induction on δ. The successor case is clear. So assume that δ is limit, $p \in \mathbb{P}_\delta$, and $M \ni \mathbb{P}_\delta, p$ is a countable elementary submodel of $H(\theta)$ for a sufficiently large θ. Pick an increasing sequence $\langle \delta_i : i \in \omega \rangle$ cofinal in $\delta \cap M$, with $\delta_i \in M$ for all $i \in \omega$. Let also $\{D_i : i \in \omega\}$ and $\{\dot{Y}_i : i \in \omega\}$ be an enumeration of all open dense subsets of $\mathbb{P}_\delta$ and all $\mathbb{P}_\delta$-names for some infinite subset of ω which are elements of M, respectively. We can assume without loss of generality, that for every $\mathbb{P}_\delta$-name $\dot{Y} \in M$ for an element of $[\omega]^\omega$ the set $\{i \in \omega : \dot{Y} = \dot{Y}_i\}$ is infinite. Suppose that f well-splits M. By induction on $i \in \omega$ we will define a condition $q_i \in \mathbb{P}_{\delta_i}$ and $\mathbb{P}_{\delta_i}$-names $\dot{p}_i, \dot{n}^0_i, \dot{n}^1_i$ such that

(i) $\dot{p}_i$ is a name for an element of $\mathbb{P}_\delta$, $q_0 \Vdash_{\delta_0} \dot{p}_0 \le \check{p}$, and $q_{i+1} \Vdash_{\delta_{i+1}} \dot{p}_{i+1} \le \dot{p}_i$;
(ii) $q_{i+1} \restriction \delta_i = q_i$;
(iii) q_i is $(M, \mathbb{P}_{\delta_i})$-generic;
(iv) $\dot{n}^0_i, \dot{n}^1_i$ are $\mathbb{P}_{\delta_i}$-names for natural numbers bigger than i; and
(v) q_i forces over $\mathbb{P}_{\delta_i}$ that "$\dot{p}_i \restriction \delta_i \in \Gamma_{\delta_i}$, $\dot{p}_i \in D_i \cap M$, and $\dot{p}_i$ forces over $\mathbb{P}_\delta$ that $\dot{n}^j_i \in E_j$ and $|[f(\dot{n}^j_i), f(\dot{n}^j_i + 1)) \cap \dot{Y}_i| \ge 2$ for all $j \in 2$".

Suppose now that we have constructed objects as above and set $q = \bigcup_{i \in \omega} q_i$. Since $q_i = q \restriction \delta_i$ forces over $\mathbb{P}_{\delta_i}$ that $\dot{p}_i \restriction \delta_i \in \Gamma_{\delta_i}$ and $q_{i+1} \Vdash_{\delta_{i+1}} \dot{p}_{i+1} \le \dot{p}_i$ for all i, a standard argument yields that q is $(M, \mathbb{P}_\delta)$-generic and $q \Vdash_\delta \dot{p}_i \in \Gamma_\delta$ for all $i \in \omega$, see, e.g., the proof of [1, Lemma 2.8] for details. Then q forces that $\tau_0 := \{n \in E_0 : |[f(n), f(n+1)) \cap \dot{Y}| \geq 2\}$ and $\tau_1 := \{n \in E_1 : |[f(n), f(n+1)) \cap \dot{Y}| \geq 2\}$ are infinite for any $\mathbb{P}_\delta$-name $\dot{Y}$ for an infinite subset of ω: Given $\mathbb{P}_\delta$-generic $G \ni q$, note that $p_i := \dot{p}_i^G \in G$ for all i. Now (v) implies $n_i^j \in \tau_j^G$ for all $j \in 2$ and $i \in \omega$ such that $\dot{Y} = \dot{Y}_i$, where $n_i^j = (\dot{n}_i^j)^G$.

Returning now to the inductive construction, assume that $q_i \in \mathbb{P}_{\delta_i}$, $\mathbb{P}_{\delta_i}$-names $\dot{p}_i, \dot{n}_i^0, \dot{n}_i^1$ satisfying (i)-(v) have already been constructed. Let G_{δ_i} be $\mathbb{P}_{\delta_i}$-generic containing q_i and $p_i = \dot{p}_i^{G_{\delta_i}} \in \mathbb{P}_\delta \cap M$. By (v) we know that $p_i \restriction \delta_i \in G_{\delta_i}$. In $V[G_{\delta_i}]$ let $p_i' \in M \cap D_{i+1}$ be such that $p_i' \leq p_i$ and $p_i' \restriction \delta_i \in G_{\delta_i}$. By the maximality principle we get a $\mathbb{P}_{\delta_i}$-name $\dot{p}_i'$ for a condition in $\mathbb{P}_\delta$ such that $q_i \Vdash_{\delta_i}$ "$\dot{p}_i' \leq \dot{p}_i$, $\dot{p}_i' \in M \cap D_{i+1}$, and $\dot{p}_i' \restriction \delta_i \in \Gamma_{\delta_i}$".

Given a $\mathbb{P}_{\delta_{i+1}}$-generic filter R, construct in $V[R]$ a decreasing sequence $\langle r_m : m \in \omega\rangle \in M[R]$ of conditions in $\mathbb{P}^R_{[\delta_{i+1},\delta)}$ below $(\dot{p}_i' \restriction [\delta_{i+1}, \delta))^R$ such that for some $a_m \in [\omega]^m$ we have $r_m \Vdash_{\mathbb{P}^R_{[\delta_{i+1},\delta)}}$ "a_m is the set of the first m elements of $\dot{Y}_{i+1}$". By the maximality principle we get a sequence $\langle \rho_m : m \in \omega\rangle \in M$ of $\mathbb{P}_{\delta_{i+1}}$-names for elements of $\mathbb{P}_{[\delta_{i+1},\delta)}$ such that

$$\Vdash_{\delta_{i+1}} \Big[\rho_{m+1} \leq \rho_m \wedge \exists \nu_m \in [\omega]^m \; (\rho_m \Vdash_{\mathbb{P}_{[\delta_{i+1},\delta)}} \text{"}\nu_m \text{ is the set of the first } m \text{ many elements of } \dot{Y}_{i+1}\text{ ")}\Big].$$

In the notation used above, let $\dot{Z}$ be a $\mathbb{P}_{\delta_{i+1}}$-name for $\bigcup_{m\in\omega} \nu_m$ and note that $\dot{Z}$ is a $\mathbb{P}_{\delta_{i+1}}$-name for an infinite subset of ω.

Let again G_{δ_i} be $\mathbb{P}_{\delta_i}$-generic containing q_i and $p_i' = (\dot{p}_i')^{G_{\delta_i}} \in \mathbb{P}_\delta \cap M \cap D_{i+1}$. It also follows from the above that $p_i' \restriction \delta_i \in G_{\delta_i}$. For a while we shall be working in $V[G_{\delta_i}]$. Since by our inductive assumption $\mathbb{P}^{G_{\delta_i}}_{[\delta_i,\delta_{i+1})}$ is well-splitting in $V[G_{\delta_i}]$, there exists a $(M[G_{\delta_i}], \mathbb{P}^{G_{\delta_i}}_{[\delta_i,\delta_{i+1})})$-generic condition $\pi \leq p_i' \restriction [\delta_i, \delta_{i+1})^{G_{\delta_i}}$ such that

$$\pi \Vdash_{\mathbb{P}^{G_{\delta_i}}_{[\delta_i,\delta_{i+1})}} \tau_j := \{n \in E_j : |[f(n), f(n+1)) \cap \dot{Z}^{G_{\delta_i}}| \geq 2\}$$

is infinite for all $j \in 2$. Let H be $\mathbb{P}^{G_{\delta_i}}_{[\delta_i,\delta_{i+1})}$-generic over $V[G_{\delta_i}]$ containing π, and $n^j_{i+1} \in \tau_j^H \setminus (i+2)$, where $j \in 2$. In $V[G_{\delta_i} * H]$ pick $m \in \omega$ such that

$$r_m := \rho_m^{G_{\delta_i}*H} \Vdash_{\mathbb{P}^{G_{\delta_i}*H}_{[\delta_{i+1},\delta)}} \dot{Z}^{G_{\delta_i}*H} \cap f(\max_{j\in 2}(n^j_{i+1}) + 1) = \dot{Y}^{G_{\delta_i}*H}_{i+1} \cap f(\max_{j\in 2}(n^j_{i+1}) + 1).$$

In $M[G_{\delta_i}]$ pick a condition $s \in M[G] \cap H$ below $p_i' \restriction [\delta_i, \delta_{i+1})^{G_{\delta_i}}$, forcing the above properties of n^j_{i+1}, τ_j, and ρ_m, where $j \in 2$. By the maximality principle we obtain $\mathbb{P}^{G_{\delta_i}}_{[\delta_i,\delta_{i+1})}$-names $\dot{s}$ and ρ in $M[G_{\delta_i}]$ for some elements of $\mathbb{P}^{G_{\delta_i}}_{[\delta_i,\delta_{i+1})}$ and $\mathbb{P}^{G_{\delta_i}}_{[\delta_{i+1},\delta)}$, and names $\dot{n}^j_{i+1}$ for natural numbers such that

$$\pi \Vdash_{\mathbb{P}_{[\delta_i,\delta_{i+1})}{}^{G_{\delta_i}}} \dot{s} \in M[G_{\delta_i}] \cap \Gamma^{G_{\delta_i}}_{[\delta_i,\delta_{i+1})} \wedge \dot{s} \leq \dot{p}' \upharpoonright [\delta_i, \delta_{i+1})^{G_{\delta_i}} \wedge \dot{s} \Vdash_{\mathbb{P}_{[\delta_i,\delta_{i+1})}{}^{G_{\delta_i}}}$$
$$\rho \leq \dot{p}' \upharpoonright [\delta_{i+1}, \delta)^{G_{\delta_i}} \wedge \rho \Vdash_{\mathbb{P}_{[\delta_{i+1},\delta)}{}^{G_{\delta_i}}} \quad (1)$$
$$\forall j \in 2\, |[f(\dot{n}^j_{i+1}), f(\dot{n}^j_{i+1}+1)) \cap \dot{Y}_{i+1}| \geq 2.$$

Using the maximality principle again, we can find $\mathbb{P}_{\delta_i}$-names for the objects appearing in Eq. (1) such that q_i forces this equation. We shall use the same notation for these names. It remains to set $q_{i+1} = q_i\hat{}\pi$ and $\dot{p}_{i+1} = \dot{p}'_i \upharpoonright \delta_i\hat{}\dot{s}\hat{}\rho$ and note that they together with the names $\dot{n}^j_{i+1}$, $j \in 2$, satisfy (i)-(v) for $i+1$. □

By a *Miller tree* we understand a subtree T of $\omega^{<\omega}$ consisting of increasing finite sequences such that the following conditions are satisfied:

- Every $t \in T$ has an extension $s \in T$ which is *splitting* in T , i.e., there are more than one immediate successors of s in T;
- If s is splitting in T, then it has infinitely many successors in T.

The *Miller forcing* is the collection $\mathbb{M}$ of all Miller trees ordered by inclusion, i.e., smaller trees carry more information about the generic. This poset was introduced in [11]. For a Miller tree T we shall denote the set of all splitting nodes of T by $\mathrm{Split}(T)$. $\mathrm{Split}(T)$ may be written in the form $\bigcup_{i\in\omega} \mathrm{Split}_i(T)$, where

$$\mathrm{Split}_i(T) = \{t \in \mathrm{Split}(T) \ : \ |\{s \in \mathrm{Split}(T) : s \subsetneq t\}| = i\}.$$

If $T_0, T_1 \in \mathbb{M}$, then $T_1 \leq_i T_0$ means $T_1 \leq T_0$ and $\mathrm{Split}_i(T_1) = \mathrm{Split}_i(T_0)$. It is easy to check that for any sequence $\langle T_i : i \in \omega\rangle \in \mathbb{M}^\omega$, if $T_{i+1} \leq_i T_i$ for all i, then $\bigcap_{i\in\omega} T_i \in \mathbb{M}$.

For a node t in a Miller tree T we denote by T_t the set $\{s \in T : s$ is compatible with $t\}$. It is clear that T_t is also a Miller tree.

Lemma 2.4 *The Miller forcing* $\mathbb{M}$ *is well-splitting.*

Proof Let N be an elementary submodel of $H(\theta)$ and $T \in \mathbb{M} \cap N$. Let $\{\dot{Y}_i : i \in \omega\}$ be an enumeration of all $\mathbb{M}$-names for infinite subsets of ω which are elements of N, in which every such name appears infinitely often. Let also $\{D_i : i \in \omega\}$ be an enumeration of all open dense subsets of $\mathbb{M}$ which belong to N. Suppose that $f \in \omega^\omega$ well-splits N. We shall inductively construct a sequence $\langle T_i : i \in \omega\rangle$ such that $T_{i+1} \leq_i T_i$ and $T_\infty = \bigcap_{i\in\omega} T_i$ is as required. Set $T_0 = T$ and suppose that T_i has already been constructed. Moreover, we shall assume that $(T_i)_t \in N$ for all $t \in \mathrm{Split}_i(T_i)$. Let $\{t_j : j \in \omega\}$ be a bijective enumeration of $\mathrm{Split}_i(T_i)$. For every j and $k \in \omega$ such that $t_j\hat{}k \in T_i$ fix a decreasing sequence $\langle S^{i,j,k}_n : n \in \omega\rangle \in N$ of elements of D_i below $(T_i)_{t_j\hat{}k}$ such that each $S^{i,j,k}_n$ decides some $a^{i,j,k}_n \in [\omega]^n$ to be the set of the first n many elements of $\dot{Y}_i$. Thus $Y^{i,j,k} := \bigcup_{n\in\omega} a^{i,j,k}_n \in N \cap [\omega]^\omega$, and hence there are $E_p \ni m^{i,j,k}_{n,p} \geq i$ such that

$$|[f(m^{i,j,k}_{n,p}), f(m^{i,j,k}_{n,p}+1)) \cap Y^{i,j,k}_n| \geq 2$$

for all $p \in 2$. Let $n(i, j, k)$ be such that

$$Y^{i,j,k} \cap \max_{p\in 2} f(m^{i,j,k}_{n(i,j,k),p} + 1) \subset a^{i,j,k}_{n(i,j,k)}$$

and set

$$T_{i+1} = \bigcup\{S^{i,j,k}_{n(i,j,k)} : j \in \omega, t_j\hat{\ }k \in T_i\}.$$

This completes our inductive construction of the fusion sequence $\langle T_i : i \in \omega\rangle$. We claim that T_∞ is as required. First of all, T_∞ is $(N, \mathbb{M})$-generic because the collection $\bigcup\{S^{i,j,k}_{n(i,j,k)} : j \in \omega, t_j\hat{\ }k \in T_i\}$ is a subset of D_i and predense below T_{i+1} (and hence also below T_∞). Now fix a $\mathbb{M}$-name $\dot{Y} \in N$ for an element of $[\omega]^\omega$ and suppose to the contrary, that there exist $i \in \omega$, $p \in 2$, and $R \le T_\infty$ that forces $|[f(m), f(m+1)) \cap \dot{Y}| \le 1$ for all $E_p \ni m \ge i$. Enlarging i, if necessary, we may assume that $\dot{Y} = \dot{Y}_i$. Passing to a stronger condition, if necessary, we may assume that $R \le (T_i)_{t_j\hat{\ }k}$ for some $i, j \in \omega$ and k such that $t_j\hat{\ }k \in T_i$. But then $R \le S^{i,j,k}_{n(i,j,k)}$, and the latter condition forces

$$|[f(m^{i,j,k}_{n,p}), f(m^{i,j,k}_{n,p} + 1)) \cap Y^{i,j,k}_n| = |[f(m^{i,j,k}_{n,p}), f(m^{i,j,k}_{n,p} + 1)) \cap \dot{Y}| \ge 2,$$

which leads to a contradiction since $m^{i,j,k}_{n,p}$ has been chosen to be above i. This contradiction completes our proof. □

In the proof of the next lemma we shall work with clopen subsets of $\mathcal{P}(\omega)$ of the form $\uparrow s = \{X \subset \omega : s \subset X\}$, where $s \in [\omega]^{<\omega}$.

Lemma 2.5 *Suppose that $\mathcal{F}$ is a Hurewicz filter. Then $\mathcal{M}_\mathcal{F}$ is well-splitting.*

Proof Suppose that f well-splits $M \prec H(\theta)$, and $\mathcal{F} \in M$. We shall prove that any $\langle s_0, F_0\rangle \in \mathcal{M}_\mathcal{F} \cap M$ forces that f well-splits $M[\Gamma]$. This suffices because all conditions in $\mathcal{M}_\mathcal{F}$ are $(M, \mathcal{M}_\mathcal{F})$-generic. Suppose, contrary to our claim, that there exists $\langle s_1, F_1\rangle \le \langle s_0, F_0\rangle$ such that

$$\begin{aligned}\langle s_1, F_1\rangle \Vdash \exists\sigma\exists j\exists n_0 \big(&\sigma \in M \cap [\omega]^\omega \wedge j \in 2 \wedge n_0 \in \omega \wedge \\ &\wedge \forall n \in E_j\setminus n_0\, (|[f(n), f(n+1)) \cap \sigma| \le 1)\big).\end{aligned}$$

Replacing $\langle s_1, F_1\rangle$ with a stronger condition, if necessary, we may fix $j \in 2$, $n_0 \in \omega$, and a $\mathcal{M}_\mathcal{F}$-name $\dot{Y} \in M$ for an infinite subsets of ω such that

$$\langle s_1, F_1\rangle \Vdash \forall n \in E_j\setminus n_0\, (|[f(n), f(n+1)) \cap \dot{Y}| \le 1).$$

Let $\dot{g} \in M$ be a name for a function such that $\dot{g}(n)$ is forced to be the nth element of $\dot{Y}$. For every $m \in \omega$ let $\mathcal{S}_m$ be the set of all $s \in [F_0\setminus(\max s_1 + 1)]^{<\omega}$ such that there exist $F_s \in \mathcal{F}$ with the property that $\langle s_1 \cup s, F_s\rangle$ forces $\dot{g}(m+1)$ to be equal to some $l_{s,m} \in \omega$. It is clear that for every $F \in \mathcal{F}$ there exists $s \in \mathcal{S}_m$ such that $s \subset F$. In other words, $\mathcal{U}_m := \{\uparrow s : s \in \mathcal{S}_m\}$ is an open cover of $\mathcal{F}$. Since $\mathcal{F}$ is Hurewicz, there exists for every m, a finite $\mathcal{V}_m \subset \mathcal{U}_m$ such that $\left\{\bigcup \mathcal{V}_m : m \in \omega\right\}$ is a γ-cover of $\mathcal{F}$. Let $\mathcal{T}_m \in [\mathcal{S}_m]^{<\omega}$ be such that $\mathcal{V}_m = \{\uparrow s : s \in \mathcal{T}_m\}$ and $h(m) = \max\{l_{s,m} : s \in \mathcal{T}_m\} + 1$. By elementarity, we can in addition assume that $\langle \mathcal{U}_m, \mathcal{V}_m, \mathcal{S}_m, \mathcal{T}_m : m \in \omega\rangle \in M$ as well as $h \in M$.

Set $h'(0) = h(0)$ and $h'(m+1) = h(h'(m))$ for all $m \in \omega$. Let m_0 be such that for every $m \geq m_0$ there exists $s \in \mathcal{S}_m \cap \mathcal{P}(F_1)$. Set $n_1 = \max\{n_0, h'(m_0)\}$. Since f well-splits M, the set $I_j := \{n \in E_j : |[f(n), f(n+1)) \cap \mathrm{range}(h')| \geq 2\}$ is infinite. In particular, it contains some $n_2 > n_1$. Thus there exists $m \in \omega$ such that

$$f(n_2) \leq h'(m) < h'(m+1) = h(h'(m)) < f(n_2).$$

Now, f is strictly increasing, hence by the definition of n_1, we have that $m > m_0$, and therefore there exist $s \in \mathcal{S}_{h'(m)} \cap \mathcal{P}(F_1)$. Thus there exists $F_s \in \mathcal{F}$ such that

$$\langle s_1 \cup s, F_s\rangle \Vdash \dot{g}(h'(m)+1) = l_{s,h'(m)} < h(h'(m)) < f(n_2).$$

Also, $\langle s_1 \cup s, F_s\rangle \Vdash \dot{g}(h'(m)) \geq h'(m) \geq f(n_2)$. It follows from the above that $\langle s_1 \cup s, F_s\rangle$ forces that $[f(n_2), f(n_2)+1)$ contains at least two elements of $\dot{Y}$, namely the $h'(m)$-th and $h'(m)+1$-st. On the other hand, $\langle s_1 \cup s, F_s\rangle$ is compatible with $\langle s_1, F_1\rangle$ because $s \subset F_1$ and $\max s_1 < \min s$, $n_2 > n_0$, $n_2 \in E_j$, and $\langle s_1, F_1\rangle$ forces $|[f(n), f(n+1)) \cap \dot{Y}| \leq 1$ for all $n \in E_j \setminus n_0$. In this way two compatible conditions $\langle s_1, F_1\rangle$ and $\langle s_1 \cup s, F\rangle$ force contradictory facts, which is impossible. This completes our proof. ◻

Let us mention that there is another property of posets $\mathcal{M}_{\mathcal{F}}$ for Hurewicz filters $\mathcal{F}$ which is preserved by *finite* support iterations and which guarantees that the ground model reals remain splitting and unbounded, see [5, Prop. 84].

Corollary 2.6 *The Cohen forcing is well-splitting.*

Proof The Cohen forcing is isomorphic to any countable atomless poset, in particular to $\mathcal{M}_{\mathfrak{Fr}}$, where $\mathfrak{Fr}$ is the Fréchet filter consisting of all cofinite subsets of ω. It remains to note that $\mathfrak{Fr}$ is Hurewicz. ◻

Recall that a poset $\mathbb{P}$ is *ω^ω-bounding* if $\omega^\omega \cap V$ is dominating in $V^{\mathbb{P}}$.

Lemma 2.7 *Every proper ω^ω-bounding poset $\mathbb{P}$ is well-splitting.*

Proof Let us fix a $\mathbb{P}$-name $\dot{Y}$ for an element of $[\omega]^\omega$, $p \in \mathbb{P}$, and pick a countable elementary submodel M of $H(\theta)$ such that $\mathbb{P}, \dot{Y}, p \in M$. Suppose that f well-splits M and $q \leq p$ is any $(M, \mathbb{P})$-generic condition. Let $\dot{g} \in M$ be a name for the function in $\omega^{\uparrow\omega}$ which is the increasing enumeration of $\dot{Y}$. Since $\mathbb{P}$ is ω^ω-bounding and q is $(M, \mathbb{P})$-generic, there exist $k_0 \in \omega$ and $h \in M \cap \omega^{\uparrow\omega}$ such that $q \Vdash$ "$\dot{g}(k) < h(k)$ for all $k \geq k_0$". Let $h_1 \in M$ be the following function: $h_1(0) = 0$, $h_1(n+1) = h(h(h_1(n))) + 1$ for all $n \in \omega$. Let G be $\mathbb{P}$-generic containing q and Y, g be the evaluations of $\dot{Y}$, $\dot{g}$ with respect to G, respectively. It follows from the above that the set

$$I := \{i \in E_0 : |[f(i), f(i+1)) \cap \operatorname{range}(h_1)| \geq 2\}$$

is infinite. For every $i \in I$ we can find $n_i \in \omega$ such that $h_1(n_i), h_1(n_i+1) \in [f(i), f(i+1))$. Thus if $i \geq k_0$ then we have

$$\begin{aligned} f(i) \leq h_1(n_i) \leq g(h_1(n_i)) < h(h_1(n_i)) \leq g(h(h_1(n_i))) \\ < h(h(h_1(n_i))) = h_1(n_i+1) < f(i), \end{aligned}$$

and hence $|[f(i), f(i+1)) \cap Y| \geq 2$ because $g(h_1(n_i)), g(h(h_1(n_i)))$ belong to the latter intersection. Therefore in $V[G]$ we have $I \subset \{i \in E_0 : |[f(i), f(i+1)) \cap Y| \geq 2\}$. Since $G \ni q$ was chosen arbitrarily, we can conclude that q forces the set

$$\{n \in E_0 : |[f(n), f(n+1)) \cap \dot{Y}| \geq 2\}$$

to be infinite. Analogously, we can conclude that q forces also the set

$$\{n \in E_1 : |[f(n), f(n+1)) \cap \dot{Y}| \geq 2\}$$

to be infinite, which completes our proof. □

Summarizing the results proved in this section, we get the main result of this paper.

Theorem 2.8 *The class of all well-splitting posets preserves ground model reals splitting and unbounded, is closed under countable support iterations, and includes ω^ω-bounding, Cohen, Miller, and Mathias forcing associated to filters with the Hurewicz covering properties.*

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