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On existence of PI-exponent of algebras with involution

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We study polynomial identities of algebras with involution of nonassociative algebras over a field of characteristic zero. We prove that the growth of the sequence of $*$-codimensions of a finite-dimensional algebra is exponentially bounded. We construct a series of finite-dimensional algebras with fractional $*$-PI-exponent. We also construct a family of infinite-dimensional algebras C_α such that $\exp^*(C_\alpha)$ does not exist.

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1. Introduction

Let A be an algebra over a field Φ of characteristic zero. One of the modern approaches to the study of polynomial identities of A is to investigate their numerical invariants. The most important numerical characteristic of identities of A is the sequence $\{c_n(A)\}$ of codimensions and its asymptotic behavior. For a wide class of algebras, the growth of the sequence $\{c_n(A)\}$ is exponentially bounded. This class includes associative PI-algebras [1,2], finite-dimensional algebras of arbitrary signature [3,4], affine Kac-Moody algebras [5], infinite-dimensional simple Lie algebras of Cartan type [6], Virasoro algebra, Novikov algebras [7], and many others.

In the case of exponential upper bound, the corresponding sequence of roots $\{\sqrt[n]{c_n(A)}\}$ is bounded and its lower and upper limits

$$\underline{\exp}(A) = \liminf_{n\to\infty} \sqrt[n]{c_n(A)}, \quad \overline{\exp}(A) = \limsup_{n\to\infty} \sqrt[n]{c_n(A)}$$

are called the *lower* and the *upper* PI-*exponent* of A, respectively. In the case when $\underline{\exp}(A) = \overline{\exp}(A)$, the ordinary limit

$$\exp(A) = \lim_{n\to\infty} \sqrt[n]{c_n(A)}$$

is called the (*ordinary*) PI-*exponent* of A.

In the late 1980's, S. Amitsur conjectured that the PI-exponent of any associative PI-algebra exists and is a nonnegative integer. Amitsur's conjecture was confirmed in [8]. It was also proved for finite-dimensional Lie algebras [9], Jordan algebras [10], and some others. The class of algebras for which Amitsur's conjecture was partially confirmed is much wider. Namely, the existence (but not the integrality, in general) was proved in a series of papers.

For example, it was shown in [11] that the PI-exponent exists for any finite-dimensional simple algebra. The question about existence of PI-exponents is one of the main problems of numerical theory of polynomial identities. Until now, only two results about algebras without PI-exponent have been proved. An example of a two-step left-nilpotent algebra without PI-exponent was constructed in [12]. Analogous result for unitary algebras was obtained in [13].

If an algebra A is equipped with an additional structure (like an involution or a group grading), then one may consider identities with involution, graded identities, etc. Recall that in the associative case, the celebrated theorem of Amitsur [14] states that if A is an algebra with involution $* : A \to A$, satisfying a $*$-polynomial identity, then A satisfies an ordinary (non-involution) polynomial identity. As a consequence, the sequence of $*$-codimensions $\{c_n^*(A)\}$ is exponentially bounded. In [15,16] the existence and integrality of $\exp^*(A)$ was proved for any associative PI-algebra with involution.

In the present paper we shall show that the class of algebras with exponentially bounded $*$-codimension sequence is sufficiently large. In particular, it contains all finite-dimensional algebras.

Theorem A *(see Theorem 3.1 in Section 3). Let A be a finite-dimensional algebra with involution $*: A \to A$ and $d = \dim A$. Then $*$-codimensions of A satisfy the following inequality*

$$c_n^*(A) \le d^{n+1}.$$

Nevertheless, as it will be shown, the results of [15,16] cannot be generalized to the general nonassociative case. We shall construct a series of finite-dimensional algebras with fractional $*$-PI-exponent. For any integer $T \ge 2$ we shall construct an algebra A_T with the following property.

Theorem B *(see Theorem 4.1 in Section 4). The $*$-PI-exponent of algebra A_T exists and*

$$exp^*(A_T) = \frac{1}{\theta_T^{\theta_T}(1-\theta_T)^{1-\theta_T}},$$

where $\theta_T = \frac{1}{2T+1}$.

We shall also present a family of algebras C_α with involution $*$ which has an exponentially bounded sequence $\{c_n^*(C_\alpha)\}$ such that $\exp^*(C_\alpha)$ does not exist.

Theorem C *(see Theorem 5.1 in Section 5). For any real number $\alpha > 1$ there exists an algebra C_α such that*

$$\underline{exp}^*(C_\alpha) = 1, \quad \overline{exp}^*(C_\alpha) = \alpha.$$

The necessary background on numerical theory of polynomial identities can be found in [17].

2. Preliminaries

Let A be an algebra with involution $*: A \to A$ over a field Φ of char $\Phi = 0$. Recall that an element $a \in A$ is called *symmetric* if $a^* = a$, whereas an element $b \in A$ is called *skew-symmetric* if $b^* = -b$. Denote

$$A^+ = \{a \in A | a^* = a\}, \quad A^- = \{b \in A | b^* = -b\}.$$

Obviously, we have a vector space decomposition $A = A^+ \oplus A^-$. In order to study $*$-polynomial identities we need to introduce free objects in the following way.

Let $\Phi\{X,Y\}$ be a free (nonassociative) algebra over Φ with the set of free generators $X\cup Y,\ X=\{x_1,x_2,\dots\},Y=\{y_1,y_2,\dots\}$. A map $*: X\cup Y\to X\cup Y$ such that $x_i^*=x_i, y_i^*=-y_i, i=1,2,\dots$, can be naturally extended to an involution on $\Phi\{X,Y\}$. A polynomial $f=f(x_1,\dots,x_m,y_1,\dots,y_n)\in\Phi\{X,Y\}$ is said to be *a $*$-identity* of A if

$$f(a_1,\dots,a_m,b_1,\dots,b_n)=0,\ \text{for all } a_1,\dots a_m\in A^+,b_1,\dots,b_n\in A^-.$$

Denote by $Id^*(A)$ the set of all $*$-identities of A in $\Phi\{X,Y\}$. Then $Id^*(A)$ is an ideal of $\Phi\{X,Y\}$ and it is stable under involution $*$ and endomorphisms compatible with $*$.

Given $0\le k\le n$, denote the space of all multilinear polynomials in $\Phi\{X,Y\}$ in k symmetric variables $x_1,\dots,x_k$ and $n-k$ skew-symmetric variables $y_1,\dots,y_{n-k}$ by $P^*_{k,n-k}$. Denote also

$$P^*_n=P^*_{0,n}\oplus P^*_{1,n-1}\oplus\cdots\oplus P^*_{n,0}.$$

Clearly, the intersection $P^*_{k,n-k}\cap Id^*(A)$ is the subspace of all multilinear $*$-identities of A in k symmetric and $n-k$ skew-symmetric variables.

The following value

$$c^*_{k,n-k}(A)=\dim\frac{P^*_{k,n-k}}{P^*_{k,n-k}\cap Id^*(A)}$$

is called the *partial* $(k,n-k)$ $*$-*codimension* of A, whereas the value

$$c^*_n(A)=\sum_{k=0}^{n}\binom{n}{k}c^*_{k,n-k}(A)$$

is called the (*total*) $*$-*codimension* of A. We shall also use the following notations

$$P^*_{k,n-k}(A)=\frac{P^*_{k,n-k}}{P^*_{k,n-k}\cap Id^*(A)},\quad P^*_n(A)=\frac{P^*_n}{P^*_n\cap Id^*(A)}.$$

3. $*$-codimensions of finite-dimensional algebras

Let A be a finite-dimensional algebra with involution $*\colon A\to A$, where $\dim A=d$. Recall that A^+ and A^- are the subspaces of symmetric and skew-symmetric elements of A, respectively. In order to get an exponential upper bound for $c^*_n(A)$, we shall follow the approach of [3]. Choose a basis $a_1,\dots,a_p$ of A^+ and a basis $b_1,\dots,b_q$ of A^-. If $f(x_1,\dots,x_k,y_1,\dots,y_{n-k})\in P^*_{k,n-k}$ is a multilinear $*$-polynomial in k symmetric variables $x_1,\dots,x_k$ and $n-k$ skew-symmetric variables $y_1,\dots,y_{n-k}$, then f is a $*$-identity of A if and only if $\varphi(f)=0$, for all evaluations φ such that

$$\varphi(x_i)\in\{a_1,\dots,a_p\},\ 1\le i\le k,\quad \varphi(y_j)\in\{b_1,\dots,b_q\},\ 1\le j\le n-k. \tag{1}$$

Denote $N = \dim P^*_{k,n-k}$. Fix a basis $g_1, \dots, g_N$ of $P^*_{k,n-k}$ and write f as a linear combination $f = \alpha_1 g_1 + \cdots + \alpha_N g_N$. Then the value $\varphi(f)$ for φ of the type (1) can be written as

$$\varphi(f) = \lambda_1 a_1 + \cdots + \lambda_p a_p + \mu_1 b_1 + \cdots + b_q \mu_q,$$

where all $\lambda_1, \dots, \lambda_p, \mu_1, \dots, \mu_q$ are linear combinations of $\alpha_1, \dots, \alpha_N$. Hence $\varphi(f) = 0$ if and only if

$$\lambda_1 = \cdots = \lambda_p = \mu_1 = \cdots \mu_q = 0. \tag{2}$$

The total number of evaluations φ of type (1) is equal to $p^k q^{n-k}$. It follows that $f \equiv 0$ is a $*$-identity of A if and only if the N-tuple $(\alpha_1, \dots, \alpha_N)$ is the solution of system S of $p^k q^{n-k}(p+q)$ linear equations of type (2).

Denote by U the subspace of all solutions of system S in the space V of all N-tuples $(\alpha_1, \dots, \alpha_N)$. Then $\dim U = N - r$, where $r = \text{rank } S$ is the rank of S. Clearly,

$$r \le p^k q^{n-k}(p+q). \tag{3}$$

Since

$$c^*_{k,n-k}(A) = \mathrm{codim_V(U) = r},$$

it follows from (3) that

$$c^*_{k,n-k}(A) \le p^k q^{n-k}(p+q)$$

and

$$c^*_n(A) = \sum_{k=0}^{n} \binom{n}{k} c^*_{k,n-k}(A) \le (p+q) \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} = (p+q)^{n+1}.$$

Recall that $p + q = d = \dim A$. Hence we have proved the first main result of this paper.

Theorem 3.1. *Let A be a finite-dimensional algebra with involution $*\colon A \to A$ and $d =$ $\dim A$. Then $*$-codimensions of A satisfy the following inequality*

$$c^*_n(A) \le d^{n+1}. \quad \square$$

In the case of exponentially bounded sequence $\{c^*_n(A)\}$, the following natural question arises.

Question 3.1. *Does the $*$-PI-exponent*

$$\exp^*(A) = \lim_{n\to\infty} \sqrt[n]{c_n^*(A)}$$

exist and what are its possible values?

In Section 1 we mentioned that $c_n^*(A)$ exists and is a nonnegative integer for any associative $*$-PI-algebra A. The following hypotheses look very natural.

Conjecture 3.1. *For any finite-dimensional algebra A with involution $*$, its $*$-PI-exponent* $\exp^*(A)$ *exists.*

In the light of results of [18], we can assume that $*$-PI-exponent may take on all real values ≥ 1.

Conjecture 3.2. *For any real value $\alpha \geq 1$, there exists an algebra A_α with involution such that $*$-PI-exponent of A_α exists and* $\exp^*(A_\alpha) = \alpha$.

4. Algebras with fractional $*$-PI-exponent

In this section we shall discuss $*$-codimension growth of algebras A_T introduced in [19]. We shall prove the existence of $*$-PI-exponents of A_T and compute the precise value of $\exp^*(A_T)$. In Section 5 we shall use the properties of A_T for constructing several counterexamples.

Recall the structure of A_T. Given an integer $T \geq 2$, denote by A_T the algebra with basis $\{a, b, z_1, \ldots, z_{2T+1}\}$ and with multiplication

$$z_i a = a z_i = z_{i+1}, 1 \leq i \leq 2t, \; z_{2T+1} b = b z_{2T+1} = z_1,$$

where all remaining products are zero. Involution $* : A_T \to A_T$ is defined by

$$a^* = -a, b^* = b, z_i^* = (-1)^{i+1} z_i$$

and then

$$A^+ =< b, z_1, z_3, \ldots, z_{2T+1} >, A^- =< a, z_2, z_4, \ldots, z_{2T} > .$$

We shall need the following two results from [19].

Lemma 4.1. *([19, Lemma 3.7]) The $*$-codimensions of A_T satisfy the inequality $c_n^*(A_T) \leq n^3$, provided that $n \leq 2T$.*

Lemma 4.2. *([19, Corollary 3.8]) Let $f \equiv 0$ be a multilinear $*$-identity of A_T of degree $n \leq 2T$. Then f is also an identity of A_{T+1}.*

Note that algebras A_T are commutative and *metabelian*, i.e. they satisfy the following identity

$$(xy)(zt) \equiv 0.$$

Hence any product of elements $c_1, \ldots, c_n \in A$ can be written in the left-normed form. We shall omit brackets in the left-normed products, i.e. we shall write $c_1 c_2 \cdots c_n$ instead of $(\ldots(c_1c_2)\ldots)c_n$.

First, we shall find a lower bound for $*$-codimensions.

Lemma 4.3. *The following inequality holds for all $n \geq 2T+2$,*

$$c_n^*(A_T) \geq \frac{1}{n^2}\left(\frac{1}{\theta_T^{\theta_T}(1-\theta_T)^{1-\theta_T}}\right)^{n-2T-1}, \tag{4}$$

where

$$\theta_T = \frac{1}{2T+1}.$$

Proof. Write n in the form $n = (2T+1)k + t + 1$, where $0 \leq t \leq 2T$. Then the following product of n basis elements is nonzero

$$z_1 \underbrace{a^{2T}b \cdots a^{2T}b}_{k} a^t = z_{t+1} \neq 0.$$

Here, we use the notation xa^m for $x\underbrace{a\cdots a}_{m}$. Hence the polynomial

$$x_0 y_1 \cdots y_{2T} x_1 \cdots y_{2t(k-1)+1} \cdots y_{2Tk} x_k y_{2Tk+1} \cdots y_{2Tk+t}$$

is not an identity of A_T, that is,

$$P^*_{k+1,2Tk+t}(A_T) \neq 0, \quad c^*_{k+1,2Tk+t} \geq 1.$$

In particular,

$$c_n^*(A_T) \geq \binom{n}{k+1} \geq \binom{n_0}{k+1} \geq \binom{n_0}{k}, \tag{5}$$

where $n = 2Tk + k + t + 1, n_0 = 2Tk + k$.

Using the Stirling formula for factorials we get

$$\binom{(2T+1)k}{k} > \frac{1}{n^2}\frac{((2T+1)k)^{(2T+1)k}}{k^k(2Tk)^{2Tk}} \tag{6}$$

$$= \frac{1}{n^2}\left(\frac{1}{\left(\frac{1}{2T+1}\right)^{\frac{1}{2T+1}}\left(\frac{2T}{2T+1}\right)^{\frac{2T}{2T+1}}}\right)^{(2T+1)k} = \frac{1}{n^2}\left(\frac{1}{\theta_T^{\theta_T}(1-\theta_T)^{1-\theta_T}}\right)^{n_0}$$

$$\geq \frac{1}{n^2}\left(\frac{1}{\theta_T^{\theta_T}(1-\theta_T)^{1-\theta_T}}\right)^{n-2T-1},$$

where $\theta_T = \frac{1}{2T+1}$.

Finally, combining (5) and (6), we obtain the desired inequality (4). $\square$

Next, we shall find an upper bound for $c_n^*(A_T)$. First, we restrict the number of nonzero components $P_{k,n-k}^*(A_T)$ for a fixed n.

Lemma 4.4. *Given a positive integer n, there are at most three integers k, $0 \leq k \leq n$, such that $P_{k,n-k}^*(A_T) \neq 0$. Moreover, if $P_{k,n-k}^*(A_T) \neq 0$, then*

$$\frac{k-2}{n} \leq \frac{1}{2T+1}.$$

Proof. Clearly, all nonzero products of the basis elements of A_T are of the form

$$W = z_{2T+1-i}a^i b \underbrace{a^{2T}b\cdots a^{2T}b}_{p} a^j. \tag{7}$$

The number of symmetric factors k is equal to $p+1$ if i is odd, and $k = p+2$ if i is even. The total number of factors in W is equal to $n = (2T+1)p + i + j + 2$. Moreover, i and j in (7) satisfy inequalities $0 \leq i, j \leq 2T$. Hence

$$n - 4T - 2 \leq (2T+1)p \leq n - 2. \tag{8}$$

Clearly, there are at most two integers p satisfying (8). Since $k = p+1$ or $p+2$, at most 3 components $P_{k,n-k}^*(A_T)$ can be nonzero. Finally, according to (8), we have

$$\frac{k-2}{n} \leq \frac{p}{n} \leq \frac{n-2}{(2T+1)n} \leq \frac{1}{2T+1}. \quad \square$$

Lemma 4.5. *Let $n \leq 2T+2$. Then $c_{k,n-k}^*(A_T) \leq (2T+1)^3$.*

Proof. As it was mentioned earlier, all nonzero products of the basis elements of A_T are of the form

$$z_j a^p b \underbrace{a^{2T}b\cdots a^{2T}b}_{k} a^q, \quad 1 \leq j \leq 2T+1, \quad 0 \leq p, q \leq 2T.$$

Hence all nonzero modulo $Id^*(A_T)$ multilinear monomials are of the form

$$wy_{\sigma(1)}\cdots y_{\sigma(p)}x_{\tau(1)}y_{\sigma(p+1)}\cdots y_{\sigma(p+2T)}x_{\tau(2)}\cdots \tag{9}$$
$$y_{\sigma(2Tk-2T+p+1)}\cdots y_{\sigma(2Tk+p)}x_{\tau(k+1)}y_{\sigma(2Tk+p+1)}\cdots y_{\sigma(2Tk+p+q)},$$

where $\sigma \in S_{2Tk+p+q}, \tau \in S_{k+1}$, and w is either x_0 or y_0.

Moreover, any monomial (9) coincides (modulo $Id^*(A_T)$) with the special case (9) when $\sigma = 1, \tau = 1$. Hence, we have at most $(2T+1)^3$ linearly independent elements in $P^*_{k,n-k}(A_T)$, and so we are done. □

Lemma 4.6. *For all $n \geq 2T+2$, we have*

$$c_n^*(A_T) \leq 3(2T+1)^3 n^3 \left(\frac{1}{\theta_T^{\theta_T}(1-\theta_T)^{1-\theta_T}}\right)^n.$$

Proof. First we compute an upper bound for $c^*_{k,n-k}(A_T)$, provided that $P^*_{k,n-k}(A_T) \neq 0$. Note that

$$\binom{n}{k} \leq n^2 \binom{n}{k-2} \leq n^3 \frac{n^n}{m^m(n-m)^{n-m}},$$

by the Stirling formula, where $m = k-2$.

Since the function

$$\frac{1}{x^x(1-x)^{1-x}}$$

is nondecreasing on $(0, \frac{1}{2})$, we have by Lemma 4.4,

$$\binom{n}{k} \leq n^3 \left(\frac{1}{(m/n)^{m/n}(1-m/n)^{1-m/n}}\right)^n \leq n^3 \left(\frac{1}{\theta_T^{\theta_T}(1-\theta_T)^{1-\theta_T}}\right)^n. \tag{10}$$

Now relation (10), Lemma 4.4, and Lemma 4.5 imply

$$c_n^*(A_T) = \sum_{k=0}^{n} \binom{n}{k} c^*_{k,n-k}(A_T) \leq 3(2T+1)^3 n^3 \left(\frac{1}{\theta_T^{\theta_T}(1-\theta_T)^{1-\theta_T}}\right)^n. \quad \square$$

Finally, Lemma 4.3 and Lemma 4.6 imply the second main result of this paper.

Theorem 4.1. *The $*$-PI-exponent of algebra A_T exists and*

$$\exp^*(A_T) = \frac{1}{\theta_T^{\theta_T}(1-\theta_T)^{1-\theta_T}},$$

where $\theta_T = \frac{1}{2T+1}$. □

5. Algebras without $*$-PI-exponent

We modify construction of the algebra from Section 4. Denote by $\widetilde{A}_T$ an infinite-dimensional algebra with the basis

$$a, b_i, z_j^i, \quad 1 \le j \le 2T+1, \quad i = 1, 2, \ldots$$

and multiplication table

$$az_j^i = z_j^i a = z_{j+1}^i, 1 \le j \le 2T, \quad b_i z_{2T+1}^i = z_{2T+1}^i b_i = z_1^{i+1}.$$

Involution $* : \widetilde{A}_T \to \widetilde{A}_T$ is defined as follows

$$a^* = -a, \quad b_i^* = b_i, \quad (z_j^i)^* = (-1)^{j+1} z_j^i, \quad 1 \le j \le 2T+1, \quad i = 1, 2, \ldots \quad .$$

Lemma 5.1. *A multilinear polynomial $f \in P^*_{k,n-k}$ of degree $n \le 2T$ is a $*$-identity of $\widetilde{A}_T$ if and only if f is a $*$-identity of A_T.*

Proof. First, note that $P^*_{k,n-k}(A_T) = P^*_{k,n-k}(\widetilde{A}_T) = 0$, when $n \le 2T$ and $3 \le k \le n$.

Let $k = 0$. Then both A_T and $\widetilde{A}_T$ satisfy the following identity

$$y_{t+1} y_{\sigma(1)} \cdots y_{\sigma(t)} = y_{t+1} y_1 \cdots y_t,$$

for any $\sigma \in S_t$ and $t \le 2T-1$. Hence, modulo $Id^*(A_T)$ (and modulo $Id^*(\widetilde{A}_T)$), the polynomial f coincides with linear combination

$$f = \lambda_2 w_2 + \cdots + \lambda_n w_n, \quad \text{where } w_j = y_j y_1 \cdots y_{j-1} y_{j+1} \cdots y_n.$$

Let for example, $\lambda_n \ne 0$. Then $\varphi(f) \ne 0$ in A_T and $\widetilde{\varphi}(f) \ne 0$ in $\widetilde{A}_T$ for evaluations $\varphi, \widetilde{\varphi}$, where

$$\varphi(y_n) = z_1, \varphi(y_j) = a \text{ in } A_T, 2 \le j \le n-1, \ \widetilde{\varphi}(y_n) = z_1^1, \widetilde{\varphi}(y_j) = a \text{ in } \widetilde{A}_T, 2 \le j \le n-1.$$

Now let $k = 1$. Then all monomials $y_1 \cdots y_j x_1 y_{j+1} \cdots y_t$ are identities of A_T and $\widetilde{A}_T$ if $3 \le j \le t \le n-1$. Since

$$x_1 y_{\sigma(1)} \cdots y_{\sigma(n-1)} \equiv x_1 y_1 \cdots y_{n-1}, \quad \text{for all} \quad \sigma \in S_{n-1}$$

mod $Id^*(A_T)$ and mod $Id^*(\widetilde{A}_T)$, it follows that $f = \lambda x_1 y_1 \cdots y_{n-1}$, with $0 \ne \lambda \in \Phi$. Hence $f \notin Id^*(A_T)$ and $f \notin Id^*(\widetilde{A}_T)$.

Finally, let $k = 2$. Then modulo $Id^*(A_T)$ and modulo $Id^*(\widetilde{A}_T)$, any multilinear $*$-polynomial is a linear combination of monomials

$$w_p = x_1 y_1 \cdots y_p x_2 y_{p+1} \cdots y_{n-2} \quad \text{and} \quad v_q = x_2 y_1 \cdots y_q x_1 y_{q+1} \cdots y_{n-2},$$

where $0 \le p, q,\ p + q = n - 2$.

Suppose that

$$f = \sum_p \lambda_p w_p + \sum_q \mu_q v_q$$

and that at least one of the coefficients λ_p is nonzero. We may also assume that $\mu_0 = 0$ if $\lambda_0 \neq 0$. If all $\lambda_p = 0$ for p even and all $\mu_q = 0$ for q even, then $f \in Id^*(A_T) \cap Id^*(\widetilde{A}_T)$.

Denote

$$t = \max\{p | p \text{ even} \ \text{ and } \lambda_p \neq 0\}.$$

Then there exists odd j such that $j + t = 2T + 1$. Hence

$$\varphi(f) = \lambda_t z_j a^t b a^m = \lambda_t z_{t+1} \neq 0 \quad \text{in} \quad A_T$$

for the evaluation φ such that $\varphi(x_1) = z_j, \varphi(x_2) = b, \varphi(y_1) = \cdots = \varphi(y_{n-2}) = a$.

Similarly,

$$\widetilde{\varphi}(f) = \lambda_t z_{m+1}^2 \quad \text{in} \quad \widetilde{A}_T$$

if

$$\widetilde{\varphi}(x_1) = z_j^1, \widetilde{\varphi}(x_2) = b_1, \widetilde{\varphi}(y_1) = \cdots = \widetilde{\varphi}(y_{n-2}) = a.$$

It follows that

$$Id^*(A_T) \cap P_n^* = Id^*(\widetilde{A}_T) \cap P_n^*,$$

provided that $n \le 2T$. □

Remark 5.1. It follows from Lemma 4.1, Lemma 4.2, and Lemma 4.5, that $*$-codimensions of small degree of $\widetilde{A}_T$ are polynomially bounded,

$$c_n^*(\widetilde{A}_T) \le n^3 \ \text{ if } \ n \le 2T.$$

Also, any multilinear $*$-identitiy of $\widetilde{A}_T$ of degree $n \le 2T$ is an identity of all $\widetilde{A}_{T+1}, \widetilde{A}_{T+2}, \ldots$.

Unlike A_T, algebra $\widetilde{A}_T$ has an overexponential $*$-codimension growth.

Lemma 5.2. *Let $n \ge 4T + 3$. Then*

$$c_n^*(\widetilde{A}_T) > \left[\frac{n}{2T+1} - 1\right]!, \tag{11}$$

where $[t]$ *denotes the integer part of real number* $t > 0$.

Proof. Denote

$$w_\sigma = x_0 y_1 \cdots y_{2T} x_{\sigma(1)} y_{2T+1} \cdots y_{4T} x_{\sigma(2)} \cdots x_{\sigma(m)} y_{2mT+1} \cdots y_{2mT+j},$$

where $\sigma \in S_m, 0 \le j \le 2T$. Since

$$z_1^1 a^{2T} b_1 a^{2T} \cdots a^{2T} b_m a^j = z_{j+1}^{m+1} \neq 0,$$

while

$$z_1^1 a^{2T} b_{\sigma(1)} a^{2T} \cdots a^{2T} b_{\sigma(m)} a^j = 0,$$

for any $e \neq \sigma \in S_m$, all monomials w_σ of degree $n = (2T+1)m + j + 1$ are linearly independent modulo $Id^*(\widetilde{A}_T)$.

Hence

$$c_n^*(\widetilde{A}_T) \ge c_{m+1,n-m-1}^*(\widetilde{A}_T) \ge m! \quad . \tag{12}$$

Since

$$(2T+1)m = n - j - 1 \ge n - (2T+1),$$

we have

$$m \ge \frac{n}{2T+1} - 1$$

and (12) yields inequality (11). □

Now, let $\Phi[Z]$ be the polynomial ring over Φ and let $\Phi[Z]_0$ be its subring of polynomials with the zero constant term. Given an integer $N \ge 1$, denote by R_N the quotient

$$R_N = \frac{\Phi[Z]_0}{(Z)^{N+1}},$$

where $(Z)^{N+1}$ is the ideal of $\Phi[Z]_0$ generated by Z^{N+1}.

Denote $B(T,N) = \widetilde{A}_T \otimes R_N$. Then

$$P_{k,n-k}^*(B(T,N)) = P_{k,n-k}^*(\widetilde{A}_T), \text{ for all } 0 \le k \le n \le N, \tag{13}$$

whereas

$$P_{k,n-k}^*(B(T,N)) = 0, \text{ for all } n \ge N+1. \tag{14}$$

Given two infinite series of integers $T_1, T_2, \ldots$ and $N_1, N_2, \ldots$ such that

$$0 < T_1 < N_1 < \ldots < T_j < N_j < \ldots,$$

we define an algebra $C(T_1, T_2, \ldots, N_1, N_2, \ldots)$ as the direct sum

$$C(T_1, T_2, \ldots, N_1, N_2, \ldots) = B(T_1, N_1) \oplus B(T_2, N_2) \oplus \cdots \quad .$$

The next statement easily follows from Lemma 4.2, Lemma 5.1, and relations (13), (14).

Lemma 5.3. *Let $C = C(T_1, \cdots, N_1, \cdots)$. Then*

- $c_n^*(C) = c_n^*(\widetilde{A}_{T_1})$, *for all $n \le N_1$;*
- $c_n^*(C) = c_n^*(\widetilde{A}_{T_j})$, *for all $j \ge 2, N_{j-1} + 1 \le n \le T_j$;*
- $c_n^*(\widetilde{A}_{T_j}) \le c_n^*(C) \le c_n^*(\widetilde{A}_{T_j}) + c^*(\widetilde{A}_{T_{j+1}})$, *for all $j \ge 2, T_j < n \le N_j$.* □

Lemma 5.4. *Let $C = C(T_1, \ldots, N_1, \ldots)$. Then $c_n^*(C) \le 3nc_{n-1}^*(C)$.*

Proof. Fix $n \ge 3$ and $1 \le k \le n-1$. Denote by $f_1, \ldots, f_m$ a basis of $P_{k,n-k-1}^*$ modulo $Id^*(C)$, where $f_j, 1 \le j \le m$, are monomials in $x_1, \ldots, x_k, y_1, \ldots, y_{n-k-1}$ and $m = c_{k,n-k-1}^*$. Denote also by $g_1, \ldots, g_t$ a basis consisting of monomials in $x_1, \ldots, x_{k-1}$, $y_1, \ldots, y_{n-k}$ of $P_{k-1,n-k}^*$ modulo $Id^*(C)$, $t = c_{k-1,n-k}^*(C)$.

Then modulo $Id^*(C)$, the subspace $P_{k,n-k}^*$ coincides with the span of products

$$f_1^i y_i, \ldots, f_m^i y_i, g_1^j x_j, \ldots, g_t^j x_j, \;\; 1 \le i \le n-k, 1 \le j \le k,$$

where

$$\begin{aligned} f_p^i &= f_p(x_1, \ldots, x_k, y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_{n-k}), \\ g_q^j &= g_q(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_k, y_1, \ldots, y_{n-k}). \end{aligned}$$

Hence

$$c_{k,n-k}^*(C) \le n(c_{k-1,n-k}^*(C) + c_{k,n-k-1}^*(C)). \tag{15}$$

It follows from (15) and the next inequalities

$$\binom{n}{k} \le n\binom{n-1}{k}, \quad \binom{n}{k} \le n\binom{n-1}{k-1}$$

that

$$\binom{n}{k} c_{k,n-k}^*(C) \le n\left[\binom{n-1}{k-1} c_{k-1,n-k}^*(C) + \binom{n-1}{k} c_{k,n-k-1}^*(C)\right]. \tag{16}$$

Inequality (16) implies that

$$\sum_{k=1}^{n-1}\binom{n}{k}c^*_{k,n-k}(C)\le 2\sum_{j=0}^{n-1}\binom{n}{j-1}c^*_{j,n-j-1}(C)=2nc^*_{n-1}(C).$$

Finally, since $c^*_{0,n}=1$ and $c^*_{n,0}=1$ for $n\ge 3$, we have

$$c^*_n(C)\le 3nc^*_{n-1}(C).\quad\square$$

We are now ready to construct a family of examples of algebras with involution without $*$-PI-exponent. The following is the third main result of this paper.

Theorem 5.1. *For any real number $\alpha>1$, there exists an algebra C_α such that*

$$\underline{\exp}^*(C_\alpha)=1,\quad \overline{\exp}^*(C_\alpha)=\alpha.$$

Proof. Given $\alpha>1$, we construct an algebra C_α as $C(T_1,\dots,N_1,\dots)$ by the special choice of the sequences $T_1,T_2,\dots$ and $N_1,N_2,\dots$.

First, we fix T_1 such that $n^3<\alpha^n$, for all $n\ge T_1$. By Lemmas 4.1, 5.1 and 5.2, there exists N_1 such that

$$\begin{cases} c^*_n(\widetilde{A}_T)<\alpha^n \text{ if } n=N_1-1\\ c^*_n(\widetilde{A}_T)\ge\alpha^n \text{ if } n=N_1.\end{cases}$$

Then by Lemma 5.3 and Lemma 5.4,

$$\alpha^n\le c^*_n(C)\le 3n\alpha^n\quad\text{if}\quad n=N_1.$$

On the other hand, $c^*_{N_1+1}\le (N_1+1)^3$ by the choice of N_1. We now set $T_2=2N_1$.

Suppose that $T_1,N_1,\dots,T_{k-1},N_{k-1},T_k$ have already been chosen. Then as before, applying Lemmas 4.1, 5.1, 5.2 and 5.3, one can find N_k such that

$$\begin{cases} c^*_n(C)<\alpha^n \text{ if } n=N_k-1\\ c^*_n(C)\ge\alpha^n \text{ if } n=N_k.\end{cases}\tag{17}$$

Moreover,

$$\begin{cases} c^*_n(C)\le 3n\alpha^n\\ c^*_{n+1}(C)\le (n+1)^3\end{cases}\tag{18}$$

if $n=N_k$.

Denote by C_α the obtained algebra $C(T_1,\dots,N_1,\dots)$. Since $c^*_n(C_\alpha)\ne 0$ for all $n\ge 1$, relations (17), (18) give us the equations

$$\underline{\exp}^*(C_\alpha) = 1, \quad \overline{\exp}^*(C_\alpha) = \alpha$$

and we have thus completed the proof. □

Data availability

Data will be made available on request.

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