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Research Article

Mittag-Leffler Stability and Attractiveness of Pseudo Almost Periodic Solutions for Delayed Cellular Neural Networks

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We consider a class of nonautonomous cellular neural networks (CNNs) with mixed delays, to study the solutions of these systems which are type pseudo almost periodicity. Using general measure theory and the Mittag-Leffler function, we obtain the existence of unique solutions for cellular neural equations and investigate the Mittag-Leffler stability and attractiveness of pseudo almost periodic functions. We also present numerical examples to illustrate the application of our results.

1. Introduction

Due to the many applications of neural cell networks in various fields, these systems have been extensively studied. Image processing, robotics, optimization, etc. are among the fields used by these differential systems [1–4]. Due to the importance of network systems, stability analysis and synchronization control for these systems have always been considered by many researchers who have studied these systems with different tools. For example, we can mention [5–8], where Lyapunov functions have been used as a tool for these synchronization analyses.

We shall introduce a neural cellular system and investigate the solutions of this differential equation, which are of the ϕ-pseudo almost periodic type (for more details, see [9–11]). Assume that ϕ is a measure, η is a positive measurable function in $\mathbb{R}$, and ϕ_1 is singular Lebesgue measure. Here, measure ϕ is defined by $d\phi(y) = \eta(y)dy + d\phi_1(y)$.

The cellular neural system with mixed delay is described by

$$\begin{aligned} z_p'(y) = {} & -e_p(y)z_p(y) + \sum_{q=1}^{n} \bar{\vartheta}_{pq}(y)\bar{\Theta}_q(z_q(y)) + \sum_{q=1}^{n} \vartheta_{pq}(y)\Theta_q(z_q(y-\zeta_{pq})) \\ & + \sum_{q=1}^{n} h_{pq}(y)\int_0^{\infty} \psi_{pq}(r)\widehat{\Theta}_q(z_q(y-r))dr + L_p(y), \quad \text{for } y \in \mathbb{R}. \end{aligned} \tag{1}$$

This system with the initial value is expressed as follows:

$$\begin{aligned} z_p'(y) = {} & -e_p(y)z_p(y) + \sum_{q=1}^{n} \bar{\vartheta}_{pq}(y)\bar{\Theta}_q(z_q(y)) \\ & + \sum_{q=1}^{n} \vartheta_{pq}(y)\Theta_q(z_q(y-\zeta_{\ell j})) \\ & + \sum_{q=1}^{n} h_{pq}(y)\int_0^{\infty} \psi_{pq}(r)\widehat{\Theta}_q(z_q(y-r))dr + L_p(y), \quad \text{for } y \geq 0, \end{aligned} \tag{2}$$

$$z_p(y) = \xi_p(y), \quad \text{for } y \leq 0. \tag{3}$$

The parameters in this equation are as follows:

(i) $z_p(y)$ is the p-th neuron state

(ii) $e_p(y)$ represents the rate of decay,

(iii) Real functions $\bar{\Theta}_p, \Theta_p, \widehat{\Theta}_p$ are activation functions of the p-th neuron

(iv) $L_p(y)$ is the input

(v) ζ_{pq} are the delays that are constant

(vi) ψ_{pq} is the transmission delay kernel

Considering a special case of the stated measure, i.e., $d\phi(y) = \eta(y)d(y), \phi_1 = 0$, the ϕ-pseudo almost periodic solutions of the above system are of the weighted pseudo almost periodic functions type.

In the present paper, we shall derive some sufficient conditions for existence and uniqueness results for cellular neural equations [3, 12–14]. We first state the basic concepts and then obtain the unique solution for equation (1). In the sequel, we prove our main results, i.e., the Mittag-Leffler stability and attractiveness of ϕ-pseudo almost periodic solutions of equation (2), which improves upon and extends [11, 15–20].

We conclude the introduction by describing the structure of the paper. In Section 2, we collect the preliminary information. In Section 3, we present several examples of interesting measures. In Section 4, we prove our first main result (Theorem 19). In Section 5, we prove our second main result (Theorem 21). In Section 6, we prove our third main result (Theorem 23). In Section 7, we present some applications.

2. Preliminaries

We denote the space of all positive measures on Lebesgue ψ-field $\mathscr{A}$ with $\mathscr{N}$. If μ is a positive measure, then we have

(i) $\phi(\mathbb{R}) = +\infty$

(ii) $\phi([\ell, j]) < \infty$, for all $\ell, j \in \mathbb{R}(\ell \leq j)$

Considering $\mathscr{BC}(\mathbb{R}, \mathscr{Y})$ as the space of all continuous and bounded functions, as well as the supremum norm $\|g\|_\infty = \sup_{y\in\mathbb{R}}\|g(y)\|$, we have a Banach space.

Definition 1. The Mittag-Leffler function is defined by

$$E_\lambda(g) = \sum_{\varsigma=0}^{\infty} \frac{g^\varsigma}{\Gamma(\varsigma\lambda + 1)}, \tag{4}$$

where λ is a real number, $\lambda \leq 0$, and g is a complex variable.

The generalization of $E_\lambda(g)$ is defined as

$$E_{\lambda,\mu}(g) = \sum_{\varsigma=0}^{\infty} \frac{g^\varsigma}{\Gamma(\varsigma\lambda + \mu)}, \tag{5}$$

where $\lambda, \mu \in \mathbb{C}$, Re $(\lambda) > 0$, Re $(\mu) > 0$.

Definition 2. If $0 < \lambda \leq 1$ and ν is a complex number, then

$$\cosh_\lambda\left(\nu x^\lambda\right) = \sum_{\varsigma=0}^{\infty} \frac{\nu^{2\varsigma} x^{2\varsigma\lambda}}{\Gamma(2\varsigma\lambda + 1)} \tag{6}$$

is called the λ-order fractional hyperbolic cosine function and

$$\sinh_\lambda\left(\nu x^\lambda\right) = \sum_{\varsigma=0}^{\infty} \frac{\nu^{2\varsigma+1} x^{(2\varsigma+1)\lambda}}{\Gamma((2\varsigma + 1)\lambda + 1)} \tag{7}$$

is called the λ-order fractional hyperbolic sine function.

Proposition 3. *Assume that $0 < \lambda \leq 1$. Then,*

$$\begin{aligned} \cosh_\lambda\left(\nu x^\lambda\right) &= \frac{E_\lambda\left(\nu x^\lambda\right) + E_\lambda\left(-\nu x^\lambda\right)}{2}, \\ \sinh_\lambda\left(\nu x^\lambda\right) &= \frac{E_\lambda\left(\nu x^\lambda\right) - E_\lambda\left(-\nu x^\lambda\right)}{2}. \end{aligned} \tag{8}$$

Definition 4. A continuous function $g : \mathbb{R} \longrightarrow \mathscr{Y}$ is said to be almost periodic if $\|g(y + \zeta) - g(y)\| < \varpi$, for all $y \in \mathbb{R}, \varpi > 0$, $\zeta \in [\ell, j]$.

Definition 5. Let $\phi \in \mathscr{N}$. A bounded continuous function $g : \mathbb{R} \longrightarrow \mathscr{Y}$ is said to be ϕ-ergodic if

$$\lim_{u\longrightarrow+\infty} \frac{1}{\phi([-u, u])} \int_{-u}^{u} \|g(y)\| d\phi(y) = 0. \tag{9}$$

Definition 6. Suppose that $\phi \in \mathscr{N}$, k, and ω are almost periodic and ϕ-ergodic functions, respectively. Then, $g : \mathbb{R} \longrightarrow \mathscr{Y}$ is a ϕ-pseudo almost periodic function, provided that $g = k + \omega$.

We denote the space of all almost periodically functions by $\mathscr{AP}(\mathbb{R}, \mathscr{Y})$, the space of all ϕ-ergodic functions by $\mathscr{E}(\mathbb{R}, \mathscr{Y}, \phi)$, and the space of all ϕ-pseudo almost periodic functions by $\mathscr{PAP}(\mathbb{R}, \mathscr{Y}, \phi)$. All these spaces, equipped with the supremum norm, are Banach spaces. Also, we have $\mathscr{AP}(\mathbb{R}, \mathscr{Y}) \subset \mathscr{PAP}(\mathbb{R}, \mathscr{Y}, \phi) \subset \mathscr{BC}(\mathbb{R}, \mathscr{Y})$; for more details, see [9].

Definition 7. Let $z^*(y) = \{z_p^*(y)\}_{p=1}^{n}$ be a solution of equation (2), with initial value $\{z_p^*(y): y \leq 0\}$. Suppose that for every solution $z(y) = \{z_p(y)\}_{p=1}^n$ of equation (2) with initial value $\xi = \{\xi_p(y)\}$, there exist constants $\gamma > 0$ and $W_\xi > 1$ such that

$$\left|z_p(y) - z_y^*(y)\right| \le W_\xi \|\xi - z^*\|_1 \sum_{\varsigma=0}^{\infty} \frac{(-\gamma y)^\varsigma}{\Gamma(\varsigma\lambda + 1)}, \qquad (10)$$

for all $y > 0, p = 1, 2, 3, \cdots, n$, where

$$\|\xi - z^*\|_1 = \sup_{-\infty \le c \le 0} \max_{p=1,2,3,\cdots,n} \left|\xi_p(c) - z_p^*(c)\right|. \qquad (11)$$

Then, the property of Mittag-Leffler stability holds for z^*.

We can derive the Mittag-Leffler attractiveness from the Mittag-Leffler stability; for more details, see [21–27].

Definition 8. Let $z^*(y) = \{z_p^*(y)\}_{p=1}^n$ be a solution of equation (2), with initial value $\{z_p^*(y): y \le 0\}$. Suppose that there exists $\rho > 0$ such that

$$\lim_{y \longrightarrow +\infty} \sum_{\varsigma=0}^{\infty} \frac{(\varrho y)^\varsigma}{\Gamma(\varsigma\lambda + 1)} \|z(y) - z^*(y)\| = 0, \qquad (12)$$

for any solution $z(y) = \{z_p(y)\}_{p=1}^n$ of equation (2). Then, the property of Mittag-Leffler attractiveness holds for z^*.

If the Mittag-Leffler stability for any solution of equation (2) is established, then z depends on its initial value $\{z(y): -\infty < y \le 0\}$.

Definition 9. The convolution of functions v and x from $\mathbb{R}$ to $\mathbb{R}$, if any, is defined as follows:

$$(v \star x)(y) := \int_{-\infty}^{+\infty} v(r) x(y - r) dr, \qquad (13)$$

where $\phi \in \mathcal{N}$ and for $p, q = 1, 2, 3, \cdots, n, \bar{\vartheta}_{pq}, \vartheta_{pq}, h_{pq}, L_p \in \mathscr{PAP}(R, \mathbb{R}, \phi)$, and $e_p \in \mathscr{AP}(R, \mathbb{R})$.

Definition 10 (see [28]). Let $(G, \mathscr{A})$ be a Borel space. If ϕ and τ are measures on $(G, \mathscr{A})$, we say that ϕ and τ are mutually singular, if there exist disjoint sets R and D in $\mathscr{A}$ such that $G = R \cup D$ and $\tau(R) = \phi(D) = 0$.

Definition 11 (see [28]). Assume that ϕ and τ are measures on the Borel space $(G, \mathscr{A})$. We say that τ is absolutely continuous relative to ϕ, provided that $(\phi(R) = 0) \Rightarrow (\tau(R) = 0)$, for each $R \in \mathscr{A}$.

Following Lebesgue-Radon-Nikodym [28], we assume that $d\phi(y) = \eta(y)dy + d\phi_1$. We impose the following assumptions for every $p = 1, 2, 3, \cdots, n$:

($\mathbf{I}_1$) $\bar{\Theta}_q, \Theta_q, \Theta_q$ are globaly Lipschitzian with Lipschitz constants $\mathscr{T}_q^{\bar{\Theta}}, \mathscr{T}_q^{\Theta}$, and $\mathscr{T}_q^{\widehat{\Theta}}$, respectively

($\mathbf{I}_2$) $\psi_{pq} : \mathbb{R}^+ \longrightarrow \mathbb{R}$ is bounded and continuous

($\mathbf{I}_3$) There exists $n > 0$ such that

$$\left|\psi_{pq}(y)\right| \sum_{\varsigma=0}^{\infty} \frac{(ny)^\varsigma}{\Gamma(\varsigma\lambda + 1)} \qquad (14)$$

is integrable on $\mathbb{R}^+$

($\mathbf{I}_4$) For the bounded interval L and all $\zeta \in \mathbb{R}$, there exists $\varepsilon > 0$ such that $\phi(R + \zeta) \le \varepsilon\phi(R)$, when $R \in \mathscr{A}$ satisfies $R \cap L = \varnothing$

($\mathbf{I}_5$) There exist $\widehat{e}_p \in \mathscr{BC}(\mathbb{R}, [0, +\infty))$, $O_p > 0$, such that

$$\sum_{\varsigma=0}^{\infty} \frac{\left(-\int_b^y e_p(r)dr\right)^\varsigma}{\Gamma(\varsigma\lambda + 1)} \le O_p \sum_{\varsigma=0}^{\infty} \frac{\left(-\int_b^y \widehat{e}_p(r)dr\right)^\varsigma}{\Gamma(\varsigma\lambda + 1)}, \quad \text{for all } y, b \in \mathbb{R}, y \ge b,$$
$$\widehat{e}_p^+ = \sup_{y \in \mathbb{R}} \widehat{e}_p(y), \widehat{e}_p^- = \inf_{y \in R} \widehat{e}_p(y) > 0 \qquad (15)$$

($\mathbf{I}_6$) There exist $\widehat{e}_p \in \mathscr{BC}(\mathbb{R}, [0, +\infty))$, $O_p > 0$, $\kappa_p > 0$ and $\sigma_p > 0$ such that

$$\sup_{r \in \mathbb{R}} \left\{ -\widehat{e}_p(y) + O_p \left[\sigma_p^{-1} \sum_{q=1}^{n} \left(\left|\bar{\vartheta}_p(y)\right| \mathscr{T}_q^{\bar{\Theta}} + \left|\vartheta_{pq}(y)\right| \mathscr{T}_q^{\sigma} + \widehat{\mathscr{T}}_q^{\widehat{\Theta}} \left|h_p(y)\right| \int_0^{\infty} \left|\psi_p(r)\right| dr \right) \sigma_q \right] \right\} \ll -\kappa_p < 0 \qquad (16)$$

($\mathbf{I}_7$) There exist $\widehat{e}_p \in \mathscr{BC}(\mathbb{R}, [0, +\infty))$, $O_p > 0$, $\kappa_p > 0$ and $\sigma_p > 0$ such that

$$0 < \max_{p=1,2,3,\cdots,n} \left\{ \frac{\widehat{e}_p^+}{\widehat{e}_p^-} - \frac{\kappa_p}{\widehat{e}_p^+} \right\} < 1 \qquad (17)$$

($\mathbf{I}_8$) For all $\ell, j, i \in \mathbb{R}$ such that $0 \le \ell < j \le i$, there exist $\zeta_0 \ge 0$ and $\lambda_0 > 0$ such that

$$|\zeta| \ge \zeta_0 \Longrightarrow \phi((\ell + \zeta, j + \zeta)) \ge \lambda_0 \phi([\zeta, i + \zeta]) \qquad (18)$$

Hypothesis $(\mathbf{I}_4)$ implies hypothesis $(\mathbf{I}_8)$, whereas the converse is not true. Also, if hypothesis $(\mathbf{I}_8)$ holds, then $(\mathscr{PAP}(\mathbb{R}, \mathbb{R}^n, \phi), \|\cdot\|_\infty)$ is a Banach space. If hypothesis $(\mathbf{I}_4)$holds, then for any $g \in \mathscr{PAP}(\mathbb{R}, \mathbb{R}^n, \phi)$,$\zeta \in \mathbb{R}$, $g(y-\zeta) \in \mathscr{PAP}(\mathbb{R}, \mathbb{R}^n, \phi)$,$U \in \mathscr{E}(\mathbb{R}, \mathbb{R}, \phi)$, and $E \in L^1(\mathbb{R})$, we have $E \star U \in \mathscr{E}(\mathbb{R}, \mathbb{R}, \phi)$. The proofs can be found in [9].

Theorem 12 (see [29]). *For any integrable function* $g: \mathbb{R} \longrightarrow \mathbb{R}$*such that* $g \in \mathscr{AP}(\mathbb{R}, \mathbb{R})$*, we have* $g \star k \in \mathscr{AP}(\mathbb{R}, \mathbb{R})$.

Theorem 13 (see [30]). *For any* ζ *on the interval with positive length* l_ϱ *and any* $\varrho > 0$*, we have*

$$\|g(y-\zeta) - g(y)\| < \varrho, \|k(y-\zeta) - k(y)\| < \varrho, \tag{19}$$

where $g, k \in \mathscr{AP}(\mathbb{R}, \mathbb{R})$*, for all* $y \in \mathbb{R}$*. In particular,* $kg \in \mathscr{AP}(\mathbb{R}, \mathbb{R})$.

Remark 14. For a globally Lipschitzian mapping $\mathscr{Q}: \mathscr{Y} \longrightarrow \mathscr{Z}$such that $\mathscr{Y}$ and $\mathscr{Z}$ are Banach spaces and every almost periodic functions ω, we have $\mathscr{Q} \circ \omega \in \mathscr{AP}(\mathbb{R}, \mathscr{Y})$, which means that $\mathscr{Q} \circ \omega$ is an almost periodic function.

3. Examples of Measures Satisfying Hypotheses $(\mathbf{I}_4)$ and $(\mathbf{I}_8)$

Next, we shall introduce three examples of measures which satisfy hypotheses $(\mathbf{I}_4)$ and $(\mathbf{I}_8)$.

Example 15 (see [9]). We consider a measure $\phi \in \mathscr{N}$ which is not absolutely continuous and satisfies $(\mathbf{I}_8)$. This measure is defined as $d\phi(y) = dy + d\mathfrak{x}$, where dy is a measure of the Lebesgue type. Also, $\mathfrak{x}$ is the measure on $(\mathbb{R}, \mathscr{A})$, which in $\mathscr{A}$ is the ψ-field of the Lebesgue type. This measure is defined as follows:

$$\mathfrak{x}(R) = \begin{cases} \text{card } (\mathrm{R} \cap \mathbb{Z}), & \text{if } R \cap \mathbb{Z} \text{ is finite}, \\ \infty, & \text{if } R \cap \mathbb{Z} \text{ is infinite}. \end{cases} \tag{20}$$

Example 16. Consider the following measure:

$$d\phi_{\psi,\varkappa}(y) = \sum_{\varsigma=0}^{\infty} \frac{(\psi y)^\varsigma}{\Gamma(\varsigma\lambda+1)} dy + \varkappa \sum_{n=-\infty}^{\infty} \left(\sum_{\varsigma=0}^{\infty} \frac{(\psi n)^\varsigma}{\Gamma(\varsigma\lambda+1)} \right) \delta_n, \tag{21}$$

where $\psi \geq 0, \varkappa > 0$, and according to the integer n, δ_n is a Dirac measure (DM), and

$$\sum_{n=-\infty}^{\infty} \left(\sum_{\varsigma=0}^{\infty} \frac{(\psi n)^\varsigma}{\Gamma(\varsigma\lambda+1)} \right) \delta_n \tag{22}$$

is a generalized Dirac comb (GDC). When $\psi = 0$, this measure is called a Dirac comb (DC).

Since $[\zeta, b+\zeta] \subset [\zeta, [b] + 1 + b]$, we shall show that $(\mathbf{I}_8)$ is satisfied for $b > 0$, such that $b \geq j$:

$$\begin{aligned} \phi_{\psi,\varkappa}([\zeta, b+\zeta]) &= \int_\zeta^{i+\zeta} \sum_{\varsigma=0}^{\infty} \frac{(\psi y)^\varsigma}{\Gamma(\varsigma\lambda+1)} dy + \varkappa \sum_{n\in[\zeta, b+\zeta]} \left(\sum_{\varsigma=0}^{\infty} \frac{(\psi n)^\varsigma}{\Gamma(\varsigma\lambda+1)} \right) \\ &\leq \sum_{\varsigma=0}^{\infty} \frac{\psi^\varsigma \left((b+\zeta)^{\varsigma+1}/\varsigma + 1 - \zeta^{\varsigma+1}/\varsigma + 1 \right)}{\Gamma(\varsigma\lambda+1)} \\ &\quad + \varkappa \sum_{n=[\zeta]}^{b+\zeta} \left(\sum_{\varsigma=0}^{\infty} \frac{(\psi n)^\varsigma}{\Gamma(\varsigma\lambda+1)} \right) \\ &= \sum_{\varsigma=0}^{\infty} \frac{\psi^\varsigma \left((b+\zeta)^{\varsigma+1}/\varsigma + 1 - \zeta^{\varsigma+1}/\varsigma + 1 \right)}{\Gamma(\varsigma\lambda+1)} \\ &\quad + \varkappa \sum_{\varsigma=0}^{\infty} \frac{(\psi\zeta)^\varsigma}{\Gamma(\varsigma\lambda+1)} \sum_{z=0}^{b} \left(\sum_{\varsigma=0}^{\infty} \frac{(\psi z)^\varsigma}{\Gamma(\varsigma\lambda+1)} \right) \\ &= \sum_{\varsigma=0}^{\infty} \frac{(\psi)^\varsigma}{\Gamma(\varsigma\lambda+1)} \left[\left(\frac{(b+\zeta)^{\varsigma+1}}{\varsigma+1} - \frac{\zeta}{\zeta+1} \right) \right. \\ &\quad \left. + \varkappa(\zeta)^\varsigma \sum_{z=0}^{b} \left(\sum_{\varsigma=0}^{\infty} \frac{(\psi z)^\varsigma}{\Gamma(\varsigma\lambda+1)} \right) \right]. \end{aligned} \tag{23}$$

We also have that

$$\begin{aligned} \phi_{\psi,\varkappa}(\ell+\zeta, \jmath+\varsigma) &\geq \int_{\ell+\zeta}^{\jmath+\zeta} \sum_{\varsigma=0}^{\infty} \frac{(\psi y)^\varsigma}{\Gamma(\varsigma\lambda+1)} dy \\ &= \sum_{\varsigma=0}^{\infty} \frac{(\psi)^\varsigma}{\Gamma(\varsigma\lambda+1)} \left(\frac{(\jmath+\varsigma)^{\varsigma+1}}{\varsigma+1} - \frac{(\ell+\zeta)^{\varsigma+1}}{\varsigma+1} \right). \end{aligned} \tag{24}$$

The conclusion follows with $\zeta_0 = 0$ and

$$\varrho_0 = \frac{\sum_{\varsigma=0}^{\infty} ((\psi)/\Gamma(\varsigma\lambda+1)) \left(\left((\jmath+\zeta)^{\varsigma+1}/(\varsigma+1) \right) - \left((\ell+\zeta)^{\varsigma+1}/(\varsigma+1) \right) \right)}{\sum_{\varsigma=0}^{\infty} ((\psi)/\Gamma(\varsigma\lambda+1)) \left[\left(\left((b+\zeta)^{\varsigma+1}/(\zeta+1) \right) - (\zeta/(\zeta+1)) \right) + \varkappa(\zeta)^\varsigma \sum_{z=0}^{b} \left(\sum_{\varsigma=0}^{\infty} ((\psi z)^\varsigma/\Gamma(\varsigma\lambda+1)) e \right) \right]}. \tag{25}$$

This means that $(\mathscr{PAP}(\mathbb{R}, \mathbb{R}^n, \phi_{\psi,u}), \|.\|_\infty)$ is a Banach space.

The measure $\phi_{\psi,u}$ does not satisfy $(\mathbf{I}_4)$. In the sequel, we shall prove this. Let

$$\begin{aligned} D &= \cup_{n=-\infty}^{\infty}\left(n-\frac{1}{2}-\varrho_n, n-\frac{1}{2}+\varrho_n\right), \\ \varrho &= \frac{1}{\psi}\sinh^{-1}\left(\frac{1}{2^{|n|+1}\sum_{\varsigma=0}^{\infty}(\psi n)^{\varsigma}/\Gamma(\varsigma\lambda+1)}\right), \\ \zeta &= \frac{1}{2}. \end{aligned} \tag{26}$$

Then, $D+\zeta = \cup_{n=-\infty}^{\infty}(n-\rho_n, n+\rho_n)$ contains $\mathbb{Z}$ and

$$\begin{aligned} \phi_{\psi,\varkappa}(D+\zeta) &= \sum_{n=-\infty}^{\infty}\int_{n-\varrho_n}^{n+\varrho_n}\sum_{\varsigma=0}^{\infty}\frac{(\psi y)^{\varsigma}}{\Gamma(\varsigma\lambda+1)}dy + \sum_{n=-\infty}^{\infty}\sum_{\varsigma=0}^{\infty}\frac{(\psi n)^{\varsigma}}{\Gamma(\varsigma\lambda+1)} \\ &= \sum_{n=-\infty}^{\infty}\frac{1}{2^{|n|+1}} + \infty = \infty, \end{aligned} \tag{27}$$

provided that $D \cap \mathbb{Z} = \varnothing$. Now,

$$\begin{aligned} \phi_{\psi,u}(D) &= \sum_{n=-\infty}^{\infty}\int_{n-1/2+\rho_n}^{n+1/2+\rho_n}\sum_{\varsigma=0}^{\infty}\frac{(\psi y)^{\varsigma}}{\Gamma(\varsigma\lambda+1)}dy + 0 \\ &= \sum_{\varsigma=0}^{\infty}\frac{(\psi/2)^{\varsigma}}{\Gamma(\varsigma\lambda+1)}\sum_{n=-\infty}^{\infty}\frac{1}{2|n|^{+1}} < \infty. \end{aligned} \tag{28}$$

Therefore, if $R = D/L$, where L is a bounded interval, we obtain

$$\phi_{\psi,\varkappa}(R+\zeta) \geq \sum_{n\in\mathbb{Z}/L}\sum_{\varsigma=0}^{\infty}\frac{(\psi n)^{\varsigma}}{\Gamma(\varsigma\lambda+1)} = \infty, \tag{29}$$

provided that $\phi_{\psi,\varkappa}(R) \leq \phi_{\psi,\varkappa}(D) < \infty$.

Example 17. We consider the following measure for $\phi \in \mathscr{N}$, $\psi \geqslant 0, \varkappa > 0$,

$$d\phi_{\psi,\varkappa}(y) = \sum_{\varsigma=0}^{\infty}\frac{(\psi y)^{\varsigma}}{\Gamma(\varsigma\lambda+1)}dy + \varkappa\sum_{n=1}^{\infty}\frac{1}{n^2}\delta_{1/n}, \tag{30}$$

where $\delta_{1/n}$ is the Dirac measure at $1/n$ and satisfying $(\mathbf{I}_4)$.

For $\zeta \in \mathbb{R}$, let $L = (-1-|\zeta|, 1+\zeta]$. We can easily see that $R \cap L = \varnothing$, $R \cap [0,1] = \varnothing$ and also $(R+\zeta) \cap [0,1] = \varnothing$. Then,

$$\begin{aligned} \phi(R+\zeta) &= \int_{R+\zeta}\sum_{\varsigma=0}^{\infty}\frac{(\psi y)^{\varsigma}}{\Gamma(\varsigma\lambda+1)}dy + 0 \\ &= \sum_{\varsigma=0}^{\infty}\frac{(\psi)^{\varsigma}}{\Gamma(\varsigma\lambda+1)}\left(\int_{R+\zeta}y^{\varsigma}dy\right) \\ &= \sum_{\varsigma=0}^{\infty}\frac{(\psi)^{\varsigma}}{\Gamma(\varsigma\lambda+1)}\frac{(R+\zeta)^{\varsigma+1}}{\varsigma+1} \\ &= \sum_{\varsigma=0}^{\infty}\frac{(\psi)^{\varsigma}}{\Gamma(\varsigma\lambda+1)}\frac{(R+\zeta)^{\varsigma+1}}{(\varsigma+1)(R+\zeta)^{\varsigma+1}\zeta^{-\varsigma}R^{-(\varsigma+1)}} \\ &= \sum_{\varsigma=0}^{\infty}\frac{(\psi\zeta)^{\varsigma}}{\Gamma(\varsigma\lambda+1)}\frac{R^{\varsigma+1}}{\varsigma+1} = \phi(R). \end{aligned} \tag{31}$$

The conclusion now follows with $\vartheta = 1$.

4. On the Integral Solution of Equation (34)

Proposition 18. *Assume that* $(\mathbf{I}_1)$ *and* $(\mathbf{I}_2)$ *hold. If* $z_q \in \mathscr{BC}(\mathbb{R}, \mathbb{R})$, *then*

$$\begin{aligned} &\vartheta_{pq}(y)\Theta_q(z_q(y-\zeta_{pq})) \in \mathscr{BC}(\mathbb{R},\mathbb{R}), \bar{\vartheta}_{pq}(y)\bar{\Theta}_q(z_q(y)) \in \mathscr{BC}(\mathbb{R},\mathbb{R}), \\ &h_{pq}(y)\int_0^{\infty}\psi_{pq}(r)\widehat{\Theta}_q(z_q(y-r))dr \in \mathscr{BC}(\mathbb{R},\mathbb{R}), \end{aligned} \tag{32}$$

for $p, q = 1, 2, 3 \cdots, n$.

Assume further that $(\mathbf{I}_1), (\mathbf{I}_2)$, *and* $(\mathbf{I}_4)$ *hold. If* $z_q \in \mathscr{PAP}(\mathbb{R}, \mathbb{R}, \phi)$, *then*

$$\begin{aligned} &\vartheta_{pq}(y)\Theta_q(z_q(y-\zeta_{pq})) \in \mathscr{PAP}(\mathbb{R},\mathbb{R},\phi), \bar{\vartheta}_{pq}(y)\bar{\Theta}_q(z_q(y)) \in \mathscr{PAP}(\mathbb{R},\mathbb{R},\phi), \\ &h_{pq}(y)\int_0^{\infty}\psi_{pq}(r)\widehat{\Theta}_q(z_q(y-r))dr \in \mathscr{PAP}(\mathbb{R},\mathbb{R},\phi). \end{aligned} \tag{33}$$

Proof. This follows from [15] (Theorem 4.1).

In the sequel, we set $\bar{z}_p(y) = \sigma_y^{-1}z(y)$. Then, equation (1) is transformed into the following system:

$$\begin{aligned} \bar{z}_p(y) = &-e_p(y)\bar{z}_p(y) + \sigma_p^{-1}\sum_{q=1}^{n}\vartheta_p(y)\bar{\Theta}_q(\sigma\bar{z}_p(y)) \\ &+ \sigma_p^{-1}\sum_{q=1}^{n}\vartheta_{pq}(y)\Theta_q(\sigma\bar{z}_q(y-\zeta_{pq})) \\ &+ \sigma_p^{-1}\sum_{q=1}^{n}h_{pq}(y)\int_0^{\infty}\psi_{pq}(v)\widehat{\Theta}_q(\sigma_p\bar{z}_q(y-v))dv \\ &+ \sigma_p^{-1}L_p(y). \end{aligned} \tag{34}$$

Now we show that the integral solutions of equation (34) are mappings of $\mathscr{PAP}(\mathbb{R}, \mathbb{R}^n, \phi)$ to itself. □

Theorem 19. *Assuming that* $(\boldsymbol{I}_1), (\boldsymbol{I}_2)$, *and* $(\boldsymbol{I}_5)$ *hold, we define the nonlinear mapping* $\mathfrak{P}$ *on* $\mathscr{BC}(\mathbb{R}, \mathbb{R}^n)$ *for* $p = 1, 2, 3, \cdots, n$ *as follows:*

$$
\begin{aligned}
(\mathfrak{P}U)_p(y) = \int_0^y \sum_{\varsigma=0}^{\infty} \frac{\left(-\int_b^y e_p(r)dr\right)^{\varsigma}}{\Gamma(\varsigma\lambda+1)} \Bigg[& \sigma^{-1} \sum_{q=1}^{n} \bar{\vartheta}_{pq}(b) \bar{\Theta}_q(\sigma_q U_q(b)) \\
& + \sigma_p^{-1} \sum_{q=1}^{n} \vartheta_{pq}(b) \Theta_q(\sigma_q U_q(b-\zeta_q)) \\
& + \sigma_p^{-1} \sum_{p=1}^{n} h_{pq}(b) \int_0^{\infty} \psi_{pq}(v) \widehat{\Theta}_q(\sigma_q U_q(b-v)) dv \\
& + \sigma_p^{-1} L_p(b) \Bigg] db.
\end{aligned} \tag{35}
$$

If we assume condition $(\mathbf{I}_4)$ *along with the other three conditions, then* $\mathfrak{P} \in \mathscr{PAP}(\mathbb{R}, \mathbb{R}^n, \phi)$.

Proof. We have $\mathfrak{P}U \in \mathscr{BC}(\mathbb{R}, \mathbb{R}^n)$ (see [29]). According to Proposition 18, for $p = 1, 2, 3, \cdots, n$, there exist $\Delta_p \in \mathscr{AP}(\mathbb{R}, \mathbb{R})$ and $\Psi_p \in \mathscr{E}(\mathbb{R}, \mathbb{R}, \phi)$ such that

$$
\begin{aligned}
& \sigma_p^{-1} \sum_{q=1}^{n} \bar{\vartheta}_{pq}(y) \bar{\Theta}_q(\sigma_q U_q(y)) + \sigma_p^{-1} \sum_{q=1}^{n} \vartheta_{pq}(y) \Theta_q(\sigma_q U_q(y-\zeta_{pq})) \\
& + \sigma_p^{-1} \sum_{q=1}^{n} h_{pq}(y) \times \int_0^{\infty} \psi_{pq}(v) \widehat{\Theta}_q(\sigma_q U_q(y-v)) dv \\
& + \sigma_p^{-1} L_p(y) = \Delta_p + \Psi_p \in \mathscr{PAP}(\mathbb{R}, \mathbb{R}, \phi).
\end{aligned} \tag{36}
$$

(1) We claim that

$$
\int_0^y \sum_{\varsigma=0}^{\infty} \frac{\left(-\int_b^y e_p(r)dr\right)}{\Gamma(\varsigma\lambda+1)} \Delta_p(b)db \in \mathscr{AP}(\mathbb{R}, \mathbb{R}), p = 1, 2, 3, \cdots, n. \tag{37}
$$

In fact,

$$
\Lambda_p(y) =: \int_0^y \sum_{\varsigma=0}^{\infty} \frac{\left(-\int_b^y e_p(r)dr\right)^{\varsigma}}{\Gamma(\varsigma\lambda+1)} \Delta_p(b)db, p = 1, 2, 3, \cdots, n. \tag{38}
$$

According to Theorem 13, and since $\Delta_p, e_p \in \mathscr{AP}(\mathbb{R}, \mathbb{R})$, for every $\varpi > 0$, there exists a number such as ζ belonging to an interval of positive length l_ϖ such that $|\Delta_p(y+\zeta) - \Delta_p(y)| < \varpi$, and

$$
\left| \sum_{\varsigma=0}^{\infty} \frac{-\int_{b+\zeta}^{\alpha+\zeta} e_p(r)dr}{\Gamma(\varsigma\lambda+1)} e_p(y+\zeta) - \sum_{\varsigma=0}^{\infty} \frac{-\int_{\alpha}^{y} e_p(r)dr}{\Gamma(\varsigma\lambda+1)} e_p(y) \right| < \varpi, \tag{39}
$$

for all $y \in \mathbb{R}$ (see [30]). Then,

$$
\begin{aligned}
& |\Lambda_p(y+\zeta) - \Lambda_p(y)| \\
& = \left| \int_0^{y+\zeta} \sum_{\varsigma=0}^{\infty} \frac{\left(-\int_b^{y+\zeta} e_p(r)dr\right)^{\varsigma}}{\Gamma(\varsigma\lambda+1)} \Delta_p(b)db - \int_0^y \sum_{\varsigma=0}^{\infty} \frac{\left(-\int_b^y e_p(r)dr\right)^{\varsigma}}{\Gamma(\varsigma\lambda+1)} \Delta_p(b)db \right| \\
& = \left| \int_0^y \sum_{\varsigma=0}^{\infty} \frac{\left(-\int_{b+\zeta}^{y+\zeta} e_p(r)dr\right)^{\varsigma}}{\Gamma(\varsigma\lambda+1)} \Delta_p(b+\zeta)db - \int_0^y \sum_{\varsigma=0}^{\infty} \frac{\left(-\int_b^y e_p(r)dr\right)^{\varsigma}}{\Gamma(\varsigma\lambda+1)} \Delta_p(b)db \right| \\
& = \Bigg| \int_0^y \sum_{\varsigma=0}^{\infty} \frac{\left(-\int_{b+\zeta}^{y+\zeta} e_p(r)dr\right)^{\varsigma}}{\Gamma(\varsigma\lambda+1)} \Delta p(b+\zeta)db - \int_0^y \sum_{\varsigma=0}^{\infty} \frac{\left(-\int_b^y e_p(r)dr\right)^{\varsigma}}{\Gamma(\varsigma\lambda+1)} \Delta_p(b+\zeta)db \\
& \quad + \int_0^y \sum_{\varsigma=0}^{\infty} \frac{\left(-\int_b^y e_p(r)dr\right)^{\varsigma}}{\Gamma(\varsigma\lambda+1)} \Delta_p(b+\zeta)db - \int_0^y \sum_{\varsigma=0}^{\infty} \frac{\left(-\int_b^y e_p(r)dr\right)^{\varsigma}}{\Gamma(\varsigma\lambda+1)} \Delta p(b)db \Bigg| \\
& \leq \int_0^y \left| \sum_{\varsigma=0}^{\infty} \frac{\left(-\int_{b+\zeta}^{y+\zeta} e_p(r)dr\right)^{\varsigma}}{\Gamma(\varsigma\lambda+1)} - \sum_{\varsigma=0}^{\infty} \frac{\left(-\int_b^y e_p(r)dr\right)^{\varsigma}}{\Gamma(\varsigma\lambda+1)} \right| |\Delta_p(b+\zeta)| db \\
& \quad + \int_0^y \sum_{\varsigma=0}^{\infty} \frac{\left(-\int_b^y e_p(r)dr\right)^{\varsigma}}{\Gamma(\varsigma\lambda+1)} |\Delta_p(b+\zeta) - \Delta_p(r)| dr.
\end{aligned} \tag{40}
$$

Let

$$
\mathscr{J} = \left| \sum_{\varsigma=0}^{\infty} \frac{\left(-\int_{b+\varpi}^{y+\zeta} e_p(r)dr\right)^{\varsigma}}{\Gamma(\varsigma\lambda+1)} - \sum_{\varsigma=0}^{\infty} \frac{\left(-\int_b^y e_p(r)dr\right)^{\varsigma}}{\Gamma(\varsigma\lambda+1)} \right|. \tag{41}
$$

Since

$$
\left(\sum_{\varsigma=0}^{\infty} \frac{\left(-\int_b^y e_p(r)dr\right)^{\varsigma}}{\Gamma(\varsigma\lambda+1)} \right)'_y = \varsigma \left(-\int_b^y e_p(r)dr \right)^{\varsigma-1} (-e_p(r)), e_p \in \mathscr{AP}(\mathbb{R}, \mathbb{R}), \tag{42}
$$

using assumption $(\mathbf{I}_8)$, we obtain that

$$
\begin{aligned}
\mathscr{J} & = \left| \left[\sum_{\varsigma=0}^{\infty} \frac{\left(-\int_\alpha^y e_p(r)dr\right)^{\varsigma}}{\Gamma(\varsigma\lambda+1)} \sum_{\varsigma=0}^{\infty} \frac{\left(-\int_{b+\zeta}^{\alpha+\zeta} e_p(r)dr\right)^{\varsigma}}{\Gamma(\varsigma\lambda+1)} \right]_{\alpha+b}^{y} \right| \\
& = \Bigg| \Bigg[\int_b^y \sum_{\varsigma=0}^{\infty} \frac{\left(-\int_\alpha^y e_p(r)dr\right)^{\varsigma}}{\Gamma(\varsigma\lambda+1)} \left(\sum_{\varsigma=0}^{\infty} \frac{\left(-\int_{b+\zeta}^{\alpha+\zeta} e_p(r)dr\right)^{\varsigma}}{\Gamma(\varsigma\lambda+1)} \right)'_\alpha d\alpha \\
& \quad + \int_s^t \left(\sum_{\varsigma=0}^{\infty} \frac{\left(-\int_\alpha^y e_p(r)dr\right)^{\varsigma}}{\Gamma(\varsigma\lambda+1)} \right)'_\alpha \sum_{\varsigma=0}^{\infty} \frac{\left(-\int_{b+\zeta}^{\alpha+\zeta} e_p(r)dr\right)^{\varepsilon}}{\Gamma(\varsigma\lambda+1)} d\alpha \Bigg] \Bigg| \\
& = \Bigg| \Bigg[\int_b^y \sum_{\varsigma=0}^{\infty} \frac{\left(-\int_\alpha^{y e_p(r)dr}\right)^{\varsigma}}{\Gamma(\varsigma\lambda+1)} \left(\sum_{\varsigma=0}^{\infty} \frac{\varsigma\left(-\int_{b+\zeta}^{\alpha+\zeta} e_p(r)dr\right)^{\varsigma-1} (-e_p(\alpha+\zeta))}{\Gamma(\varsigma\lambda+1)} d\alpha \right) \\
& \quad + \int_b^y \sum_{\varsigma=0}^{\infty} \frac{\varsigma\left(-\int_\alpha^y e_p(r)dr\right)^{\varsigma-1} (-e_p(\alpha))}{\Gamma(\varsigma\lambda+1)} \sum_{\varsigma=0}^{\infty} \frac{-\int_{b+\zeta}^{\alpha+\zeta} e_p(r)dr}{\Gamma(\varsigma\lambda+1)} d\alpha \Bigg] \Bigg| \\
& = \Bigg| \int_b^y \sum_{\varsigma=0}^{\infty} \frac{\left(-\int_\alpha^y e_p(r)dr\right)^{\varsigma}}{\Gamma(\varsigma\lambda+1)} \sum_{\varsigma=0}^{\infty} \frac{\left(-\int_{b+\zeta}^{\alpha+\zeta} e_p(r)dr\right)^{\varsigma}}{\Gamma(\varsigma\lambda+1)} \\
& \quad \cdot \left(\sum_{\varsigma=0}^{\infty} \frac{-\int_{b+\zeta}^{\alpha+\zeta} e_p(r)dr}{\Gamma(\varsigma\lambda+1)} (e_p(\alpha+\zeta)) - \sum_{\varsigma=0}^{\infty} \frac{-\int_\alpha^y e_p(r)dr}{\Gamma(\varsigma\lambda+1)} (e_p(\alpha)) \right) d\alpha \Bigg| \\
& \leq \int_b^y \sum_{\varsigma=0}^{\infty} \frac{\left(-\int_\alpha^y e_p(r)dr\right)^{\varsigma}}{\Gamma(\varsigma\lambda+1)} \sum_{\varsigma=0}^{\infty} \frac{\left(-\int_{b+\zeta}^{\alpha+\zeta} e_p(r)dr\right)^{\varsigma}}{\Gamma(\varsigma\lambda+1)} \\
& \quad \cdot \left| \left(\sum_{\varsigma=0}^{\infty} \frac{-\int_{b+\zeta}^{\alpha+\zeta} e_p(r)dr}{\Gamma(\varsigma\lambda+1)} (e_p(\alpha+\zeta)) - \sum_{\varsigma=0}^{\infty} \frac{-\int_\alpha^y e_p(r)dr}{\Gamma(\varsigma\lambda+1)} (e_p(\alpha)) \right) \right| d\alpha \\
& \leq (O_p)^2 \in \int_b^y \sum_{\varsigma=0}^{\infty} \frac{\left(-\widehat{e}_p^{-(y-b)}\right)^{\varsigma}}{\Gamma(\varsigma\lambda+1)} d\alpha \leq O_p^2 \rho \sum_{\varsigma=0}^{\infty} \frac{\left(-\widehat{e}_p^{-(y-b)}\right)^{\varsigma}}{\Gamma(\varsigma\lambda+1)} (y-b).
\end{aligned} \tag{43}
$$

Then, Δ_p is a continuous and bounded function, given that $\Delta_p \in \mathscr{AP}(\mathbb{R}, \mathbb{R})$. Now, by putting equation (39) in equation (38), we have

$$\begin{aligned} &\left|\Lambda_p(y+\zeta) - \Lambda_p(y)\right| \\ &\leq O_p^2 \rho \int_0^y \sum_{\varsigma=0}^{\infty} \frac{\left(-\widehat{e_p}(y-b)\right)^{\varsigma}}{\Gamma(\varsigma\lambda+1)} (y-b)\left|\Delta_p(b+\zeta)\right| db + \rho \int_0^y \sum_{\varsigma=0}^{\infty} \frac{\left(-\int_b^y e_p(r)dr\right)^{\varsigma}}{\Gamma(\varsigma\lambda+1)} db \\ &\leq O_p^2 \rho \|\Delta_p\|_\infty \int_0^y \sum_{\varsigma=0}^{\infty} \frac{\left(-\widehat{e_p}(y-b)\right)^{\varsigma}}{\Gamma(\varsigma\lambda+1)} (y-b) db + O_p \rho \int_{-\infty}^y \sum_{\varsigma=0}^{\infty} \frac{\left(-\int_b^y \widehat{e}_p(r)dr\right)^{\varsigma}}{\Gamma(\varsigma\lambda+1)} db \\ &\leq O_p^2 \rho \|\Delta_p\|_\infty \int_0^y \sum_{\varsigma=0}^{\infty} \frac{\left(-\widehat{e_p}(y-b)\right)^{\varsigma}}{\Gamma(\varsigma\lambda+1)} (y-b) db + O_p \rho \int_0^y \sum_{\varsigma=0}^{\infty} \frac{\left(-\widehat{e_p}(y-b)\right)^{\varsigma}}{\Gamma(\varsigma\lambda+1)} d\lambda \\ &\leq (O_p)^2 \rho \|\Delta_p\|_\infty \sum_{\varsigma=0}^{\infty} \frac{\left(\widehat{e_p}\right)^{\varsigma} y^{\varsigma+2}}{\Gamma(\varsigma\lambda+1)(\varsigma+2)} + O_p \rho \sum_{\varsigma=0}^{\infty} \frac{\left(\widehat{e_p}\right)^{\varsigma} y^{\varsigma+1}}{\Gamma(\varsigma\lambda+1)(\varsigma+1)}. \end{aligned} \tag{44}$$

We obtain that

$$\int_0^y \sum_{\varsigma=0}^{\infty} \frac{\left(-\int_b^y e_p(r)dr\right)^{\varsigma}}{\Gamma(\varsigma\lambda+1)} \Delta_p(b) db \in \mathscr{AP}(\mathbb{R}, \mathbb{R}). \tag{45}$$

(2) Let us show that

$$\int_0^y \sum_{\varsigma=0}^{\infty} \frac{\left(-\int_b^y e_p(r)dr\right)^{\varsigma}}{\Gamma(\varsigma\lambda+1)} \Psi_p(b) db \in \mathscr{E}(\mathbb{R}, \mathbb{R}, \phi). \tag{46}$$

According to hypothesis $(\mathbf{I}_4)$, for $\Psi_p \in \mathscr{E}(\mathbb{R}, \mathbb{R}, \phi)$ and for $p = 1, \cdots, n$, we have

$$\begin{aligned} 0 &= \lim_{u\longrightarrow+\infty} \frac{1}{\phi((\quad |-u,u))} \int_{-u}^{u} \int_0^y \sum_{\varsigma=0}^{\infty} \frac{\left(-\widehat{e_p} r\right)^{\varsigma}}{\Gamma(\varsigma\lambda+1)} \left|\Psi_p(y-r)\right| dr d\phi(y) \\ &= \lim_{u\longrightarrow+\infty} \frac{1}{\phi(\quad |-uu)} \int_{-u}^{u} \int_0^y \sum_{\varsigma=0}^{\infty} \frac{\left(-\widehat{e_p}(y-b)\right)^{\varsigma}}{\Gamma(\varsigma\lambda+1)} \left|\Psi_p(b)\right| db d\phi(y). \end{aligned} \tag{47}$$

Then, by $(\mathbf{I}_5)$,

$$\begin{aligned} 0 &\leq \lim_{u\longrightarrow+\infty} \frac{1}{\phi(|-u,u))} \int_{-u}^{u} \int_0^y \sum_{\varsigma=0}^{\infty} \frac{\left(-\int_b^y e_p(\xi)d\xi\right)^{\varsigma}}{\Gamma(\varsigma\lambda+1)} \left|\Psi_p(b)\right| db d\phi(y) \\ &\leq O_p \lim_{u\longrightarrow+\infty} \frac{1}{\phi(|-u,u])} \int_{-u}^{u} \int_0^y \sum_{\varsigma=0}^{\infty} \frac{\left(-\widehat{e_p}(y-b)\right)^{\varsigma}}{\Gamma(\varsigma\lambda+1)} \left|\Psi_p(b)\right| db d\phi(y) = 0. \end{aligned} \tag{48}$$

Hence,

$$\int_0^y \sum_{\varsigma=0}^{\infty} \frac{\left(-\int_b^y e_p(r)dr\right)^{\varsigma}}{\Gamma(\varsigma\lambda+1)} \Psi_p(b) db \in \mathscr{E}(\mathbb{R}, \mathbb{R}, \phi), \tag{49}$$

for $p = 1, 2, 3, \cdots, n$. Combined with (37), we have

$$\begin{aligned} (\mathfrak{P}U)_p(y) &= \int_0^y \sum_{\varsigma=0}^{\infty} \frac{\left(-\int_b^y e_p(r)dr\right)^{\varsigma}}{\Gamma(\varsigma\lambda+1)} \Delta_p(b) db \\ &\quad + \int_0^y \sum_{\varsigma=0}^{\infty} \frac{\left(-\int_b^y e_p(r)dr\right)^{\varsigma}}{\Gamma(\varsigma\lambda+1)} \Psi_p(b) db \in \mathscr{PAP}(\mathbb{R}, \mathbb{R}, \phi). \end{aligned} \tag{50}$$

□

5. Existence and Uniqueness of ϕ-Pseudo Almost Periodic Solutions

Assuming that the solution of equation (1) is of the ϕ-pseudo almost periodic type, we shall prove the existence and uniqueness of these solutions.

Theorem 21. *(1) Given assumptions* $(\mathbf{I}_1), (\mathbf{I}_2)$, *and* $(\mathbf{I}_5)$, *there is a unique solution* $z^* \in \mathscr{BC}(\mathbb{R}, \mathbb{R}^n)$ *for equation (1).*

(2) Given assumptions $(\mathbf{I}_1), (\mathbf{I}_2), (\mathbf{I}_5), (\mathbf{I}_4), (\mathbf{I}_6)$, *and* $(\mathbf{I}_7)$, *we have* $z^* \in \mathscr{PAP}(\mathbb{R}, \mathbb{R}^n, \phi)$.

Proof. Let $U, G \in \mathscr{BC}(\mathbb{R}, \mathbb{R}^n)$ (resp., $U, G \in \mathscr{PAP}(\mathbb{R}, \mathbb{R}^n, \phi)$). Then, in view of Theorem 19, we have that $\mathfrak{P}U$, $\mathfrak{P}G \in \mathscr{BC}(\mathbb{R}, \mathbb{R}^n)$ (resp. $\mathfrak{P}U, \mathfrak{P}G \in \mathscr{PAP}(\mathbb{R}, \mathbb{R}^n, \phi)$).

Let $\mathscr{S} = |(\mathfrak{P}U)_p(y) - (\mathfrak{P}G)_p(y)|$. Using $(\mathbf{I}_1)$ and $(\mathbf{I}_5), (\mathbf{I}_6)$, we get that

$$\begin{aligned} \mathscr{S} &= \Bigg| \int_0^y \sum_{\varsigma=0}^{\infty} \frac{\left(-\int_b^y e_p(r)dr\right)^{\varsigma}}{\Gamma(\varsigma\lambda+1)} \sigma_p^{-1} \sum_{q=1}^{n} \Bigg(\bar{\vartheta}_{pq}(b) \left(\bar{\Theta}_q\left(\sigma_q U_q(b)\right)\right. \\ &\quad \left. - \bar{\Theta}_q\left(\sigma_q G_q(b)\right)\right) + \sigma_p^{-1} \sum_{q=1}^{n} \vartheta_{pq}(b) \left(\Theta_q\left(\sigma_q U_q\left(b-\zeta_{pq}\right)\right)\right. \\ &\quad \left. - \Theta_q\left(\sigma_q G_q\left(b-\zeta_{pq}\right)\right)\right) + \sigma_p^{-1} \sum_{q=1}^{n} h_{pq}(b) \int_0^{\infty} \psi_{pq}(v) \left(\widehat{\Theta}_q\left(\sigma_q U_q(b-v)\right)\right. \\ &\quad \left. - \widehat{\Theta}_q\left(\sigma_q U_q(b-v)\right)\right) dv \Bigg) db \Bigg| \\ &\leq O_p \sigma_p^{-1} \int_0^y \sum_{\varsigma=0}^{\infty} \frac{\left(-\int_b^y \widehat{e}_p(r)dr\right)^{\varsigma}}{\Gamma(\varsigma\lambda+1)} \sum_{q=1}^{n} \left|\bar{\vartheta}_{pq}(b)\right| \mathscr{T}_q^{\bar{\Theta}} \\ &\quad + \left|\vartheta_{pq}(b)\right| \mathscr{T}_q^{\Theta} + \mathscr{T}_q^{\widehat{\Theta}} \left|h_{pq}(s)\right| \left| \int_0^{\infty} \left|\psi_{pq}(r)\right| dr \right) \sigma_q db \Bigg) \|U-G\|_\infty \\ &\leq \int_0^y \sum_{\varsigma=0}^{\infty} \frac{\left(-\int_b^y \widehat{e}_p(r)dr\right)^{\varsigma}}{\Gamma(\varsigma\lambda+1)} \left[\widehat{e}_p(b) - \kappa_p\right] db \|U-G\|_\infty \\ &\leq \left[\widehat{e}_p^+ \int_0^y \sum_{\varsigma=0}^{\infty} \frac{\left(-\int_b^y \widehat{e}_p(r)dr\right)^{\varsigma}}{\Gamma(\varsigma\lambda+1)} db - \kappa_p \int_0^y \sum_{\varsigma=0}^{\infty} \frac{\left(-\int_b^y \widehat{e}_p(r)dr\right)^{\varsigma}}{\Gamma(\varsigma\lambda+1)} db \right] \|U-G\|_\infty \\ &\leq \max_{p=1,2,3,n} \left\{ \frac{\widehat{e}_p^+}{\widehat{e}_p^-} - \frac{\kappa_p}{\widehat{e}_p^+} \right\} \|U-G\|_\infty, \end{aligned} \tag{51}$$

so

$$\left| (\mathfrak{P}U)_p(y) - (\mathfrak{P}G)_p(y) \right| \leq \max_{p=1,2,3,\cdots,n} \left\{ \frac{\widehat{e}_p^+}{\widehat{e}_p^-} - \frac{\kappa_p}{\widehat{e}_p^+} \right\} \|U-G\|_\infty. \tag{52}$$

Invoking condition $(\mathbf{I}_7)$ and $0 < \max_{p=1,2,3,\cdots,n}\{(\widehat{e}_+^+/e_p^-) - (\kappa_1/e_p^+)\} < 1$, we conclude that $\mathfrak{P} \in \mathscr{BC}(\mathbb{R}, \mathbb{R}^n)$ is a contraction (since $\mathscr{PAP}(\mathbb{R}, \mathbb{R}^n, \mu)$ is a Banach space, according to condition $(\mathbf{I}_3)$ it is also a contraction on this space). Therefore, we conclude that $\bar{z}^* \in \mathscr{BC}(\mathbb{R}, \mathbb{R}^n)$ (or $\bar{z}^* \in \mathscr{PAP}(\mathbb{R}, \mathbb{R}^n, \phi)$) is a unique fixed point for $\mathfrak{P}$. Also, given (34), $z^* \in \mathscr{BC}(\mathbb{R}, \mathbb{R}^n)$ (or $z^* \in \mathscr{PAP}(\mathbb{R}, \mathbb{R}^n, \phi)$) is a unique solution of type ϕ-$\mathscr{PAP}$ for equation (1).

In the sequel, we shall investigate the Mittag-Leffler stability and the Mittag-Leffler attractiveness for the unique solution of equation (2), which is of type ϕ-$\mathscr{PAP}$. First, we state Lemma 22. □

Lemma 22. *Assuming conditions $(\boldsymbol{I}_3)$ and $(\boldsymbol{I}_6)$, we define $\mathfrak{Z} = \{\mathfrak{Z}_p\}_{p=1}^n : [0, m] \longrightarrow \mathbb{R}^n$ by $\mathfrak{Z}_p(w) = \sup_{y\in\mathbb{R}} \mathfrak{Z}_p(w, y)$, where*

$$\mathfrak{Z}_p(w, y) = w - \widehat{e}_p(y) + O_p\sigma_p^{-1} \sum_{q=1}^{n}\left(\left|\bar{\vartheta}_{pq}(y)\right|\mathscr{T}_q^{\bar{\Theta}} + \left|\vartheta_{pq}(y)\right|\mathscr{T}_q^{\Theta}\sum_{\varsigma=0}^{\infty}\frac{(w\zeta_{pq})^{\varsigma}}{\Gamma(\varsigma\lambda + 1)} + \mathscr{T}_q^{\widehat{\Theta}}\left|h_{pq}(y)\right|\int_0^{\infty}\left|\psi_{pq}(r)\right|\sum_{\varsigma=0}^{\infty}\frac{(wr)^{\varsigma}}{\Gamma(\varsigma\lambda + 1)}dr\right)\sigma_q. \tag{53}$$

Then $\mathfrak{Z}_p(\nu) < 0$ for all $0 < \nu_0 < m$ and $0 < \nu < \nu_0$.

Proof. According to condition $(\mathbf{I}_3)$, function $y \mapsto \mathfrak{Z}_p(w, y)$ is defined on the interval $[0, m]$. Then according to condition $(\mathbf{I}_6)$, we have

$$\mathfrak{Z}_p(0) = \sup_{y\in\mathbb{R}} \mathfrak{Z}_p(0, y) = \sup_{y\in\mathbb{R}}\left\{-\widehat{e}_p(y) + O_p\sigma_p^{-1} \sum_{q=1}^{n}\left(\left|\bar{\vartheta}_{pq}(y)\right|\mathscr{T}_q^{\bar{\Theta}} + \left|\vartheta_{pq}(y)\right|\mathscr{T}_q^{\Theta} + \mathscr{T}_q^{\widehat{\Theta}}\left|h_{pq}(y)\right|\int_0^{\infty}\left|\psi_{pq}(r)\right|dr\right)\sigma_q\right\} < -\kappa_p < 0. \tag{54}$$

Next, we shall show that there exists $0 < \nu_0 < m$ such that $\mathfrak{Z}_p(\nu) < 0$, for all $0 < \nu < \nu_0$. Also, we have

$$\mathfrak{Z}_p(w, y) - \mathfrak{Z}_p(0, y) = w + O_p\sigma_p^{-1}\sum_{q=1}^{n}\left(\left|\vartheta_{pq}(y)\right|\mathscr{T}_q^{\Theta}\left(\sum_{\varsigma=0}^{\infty}\frac{(w\zeta_{pq})^{\varsigma}}{\Gamma(\varsigma\lambda + 1)} - 1\right) + \mathscr{T}_q^{\Theta}\left|h_{pq}(y)\right|\left(\int_0^{\infty}\left|\psi_{pq}(r)\right|\sum_{\varsigma=0}^{\infty}\frac{(wr)^{\varsigma}}{\Gamma(\varsigma\lambda + 1)}dr - \int_0^{\infty}\left|\psi_{pq}(r)\right|dr\right)\right)\sigma_q. \tag{55}$$

If we take ϑ_p and h_p nonnegative numbers

$$\vartheta_p = \sup_{q=1}^{n}\left|\vartheta_{pq}(y)\right|\mathscr{T}_q^{\Theta}, h_p = \sup_{q=1}^{n}\left|h_{pq}(y)\right|\mathscr{T}_q^{\widehat{\Theta}}, \tag{56}$$

then

$$\left|\mathfrak{Z}_p(w, y) - \mathfrak{Z}_p(0, y)\right| \le w + O_p\sigma_p^{-1}\vartheta_p\sum_{q=1}^{n}\left(\sum_{\varsigma=0}^{\infty}\frac{(w\zeta_{pq})^{\varsigma}}{\Gamma(\varsigma\lambda + 1)} - 1\right)\sigma_q + O_q\sigma_p^{-1}h_p\sum_{q=1}^{n}\left(\int_0^{\infty}\left|\psi_{pq}(r)\right|\sum_{\varsigma=0}^{\infty}\frac{(wr)^{\varsigma}}{\Gamma(\varsigma\lambda + 1)}dr - \int_0^{\infty}\left|\psi_{pq}(r)\right|dr\right)\sigma_q, \tag{57}$$

for all $w \in (0, m]$ and $y \in \mathbb{R}$. Now, for every $\rho > 0$, by continuity, there exists $0 < \delta_p^1 < m$ such that the following holds:

$$w < \delta_p^1 \Rightarrow w + O_p\sigma_p^{-1}\vartheta_p\sum_{q=1}^{n}\left(\sum_{\varsigma=0}^{\infty}\frac{(w\zeta_{pq})^{\varsigma}}{\Gamma(\varsigma\lambda + 1)} - 1\right)\sigma_q < \frac{\rho}{2}. \tag{58}$$

In the sequel, invoking condition $(\mathbf{I}_3)$, the Lebesgue dominated convergence theorem (LDCT), and the integrability of the function $|\psi_{pq}(r)\sum_{\varsigma=0}^{\infty}(\varsigma r)^{\varsigma}/\Gamma(\varsigma\lambda + 1)|$ on the interval $(0, \infty)$, we get $0 < \delta_p^2 < m$ such that $w < \delta_p^2$ implies

$$O_p\sigma_p^{-1}h_p\sum_{q=1}^{n}\left(\int_0^{\infty}\left|\psi_{pq}(r)\right|\sum_{\varsigma=0}^{\infty}\frac{(wr)^{\varsigma}}{\Gamma(\varsigma\lambda + 1)}dr - \int_0^{\infty}\left|\psi_{pq}(r)\right|dr\right)\sigma_q < \frac{\rho}{2}. \tag{59}$$

If we now take $0 < \nu < \min(\delta_p^1, \delta_p^2)$ and $\rho < \kappa_p/2$, we can conclude that for every $y \in \mathbb{R}$,

$$\mathfrak{Z}_p(\nu, y) \le \mathfrak{Z}_p(0, y) + \left(\mathfrak{Z}_p(\nu, y) - \mathfrak{Z}_p(0, y)\right) < -\kappa_p + \rho < -\frac{\kappa}{2}. \tag{60}$$

□

6. On the Mittag-Leffler Stability and Attractiveness of Unique Solutions

Theorem 23. *If we assume conditions $(\boldsymbol{I}_1)$, $(\boldsymbol{I}_3)$, $(\boldsymbol{I}_5)$ and $(\boldsymbol{I}_6)$, then the Mittag-Leffler stability for any solution z^* of equation (2) is established by the initial condition $\{z^*(y): y \le 0\}$. If we add condition $(\boldsymbol{I}_2)$, then the Mittag-Leffler stability for the solution $z^* \in \mathscr{BC}(\mathbb{R}, \mathbb{R}^n)$ of equation (2) is also established. Also, if we add condition $(\boldsymbol{I}_7)$ to conditions $(\boldsymbol{I}_1)$, $(\boldsymbol{I}_3)$, $(\boldsymbol{I}_5)$, $(\mathbf{I}_6)$, and $(\mathbf{I}_2)$, then the Mittag-Leffler stability for the solution $z^* \in \mathscr{PAP}(\mathbb{R}, \mathbb{R}^n, \mu)$ of equation (2) is also established.*

Proof. Let $z(t)=\{z_p(y)\}_{p=1}^{n}$ be a solution of equation (2) with initial value $\xi(y)=\{\xi_p(y)\}_{p=1}^{n}$. Set

$$f(y)=\left\{f_p(y)\right\}_{p=1}^{n}=\left\{\sigma_i^{-1}\left(z_p(y)-z_p^*(y)\right)\right\}_{p=1}^{n}. \tag{61}$$

Then,

$$\begin{aligned}f_p'(b)+e_p(b)f_p(b)&=\sigma_p^{-1}\sum_{q=1}^{n}\bar{\vartheta}_{pq}(b)\left(\bar{\Theta}_q(z_q(b))-\bar{\Theta}_q\left(z_q^*(b)\right)\right)\\&+\sigma_p^{-1}\sum_{q=1}^{n}\bar{\vartheta}_{pq}(b)\times\left(\Theta_q\left(z_q\left(b-\zeta_{pq}\right)\right)\right.\\&\left.-\Theta_q\left(z_q'\left(b-\zeta_{pq}\right)\right)\right)+\sigma_p^{-1}\sum_{q=1}^{n}h_{pq}(b)\\&\times\int_0^{\infty}\psi_{pq}(v)\left(\widehat{\Theta}_q\left(z_q(b-v)\right)\right.\\&\left.-\widehat{\Theta}_q\left(z_q^*(b-v)\right)\right)dv.\end{aligned} \tag{62}$$

Let $\|\xi-z^*\|_\sigma=\sup_{y\in(-\infty,0]}\max_{p=1,2,3,\cdots,n}\sigma_p^{-1}|\xi_p(y)-z_p^*(y)|$, and $\aleph>0$ be such that

$$\aleph>\sum_{p=1}^{n}\mathrm{O}_p+1. \tag{63}$$

Then, for all $\mathrm{y}\in(-\infty,0]$, we have

$$\|\mathrm{f}(\mathrm{y})\|\le\|\xi-\mathrm{z}^*\|_\sigma. \tag{64}$$

Given ν_0 in Lemma 22, we assume that $0<\nu<\min\{\min_{p=1,2,\cdots,n}\widehat{e}_p^+,\nu_0\}$. Then for all $y,\rho>0$, we have

$$\|f(y)\|<\aleph\left(\|\xi-z^*\|_\sigma+\rho\right)\sum_{\varsigma=0}^{\infty}\frac{(-\nu y)^\varsigma}{\Gamma(\varsigma\lambda+1)}. \tag{65}$$

Otherwise, for $p=1,2,3,\cdots,n$ and $\mathrm{u}>0$, given the continuity, we have

$$\begin{cases}\left|f_p(\varkappa)\right|=\aleph\left(\|\xi-z^*\|_\sigma+\rho\right)\sum_{\varsigma=0}^{\infty}\frac{(-\nu\varkappa)^\varsigma}{\Gamma(\varsigma\lambda+1)},\\ \|f(y)\|<\aleph\left(\|\xi-z^*\|_\sigma+\rho\right)\sum_{\varsigma=0}^{\infty}\frac{(-\nu y)^\varsigma}{\Gamma(\varsigma\lambda+1)},\quad\text{for all }y\in(-\infty,\varkappa).\end{cases} \tag{66}$$

Now, first, we multiply both sides of (62) by

$$\sum_{\varsigma=0}^{\infty}\frac{\left(\int_0^b e_p(r)dr\right)^\varsigma}{\Gamma(\varsigma\lambda+1)}. \tag{67}$$

Then, we integrate the obtained equation with respect to b on $[0,\mathrm{u}]$. Finally, we multiply by

$$\sum_{\varsigma=0}^{\infty}\frac{\left(-\int_0^{\mathrm{y}}\mathrm{e}_p(\mathrm{r})dr\right)^\varsigma}{\Gamma(\varsigma\lambda+1)}. \tag{68}$$

Therefore, we have

$$\begin{aligned}f_p(\varkappa)&=f_p(0)\sum_{\varsigma=0}^{\infty}\frac{\left(-\int_0^{\varkappa}e_p(r)dr\right)^\varsigma}{\Gamma(\varsigma\lambda+1)}\\&+\sigma_p^{-1}\int_0^{\varkappa}\sum_{\varsigma=0}^{\infty}\frac{\left(-\int_b^{\varkappa}e_p(r)dr\right)^\varsigma}{\Gamma(\varsigma\lambda+1)}\sum_{q=1}^{n}\left[\bar{\vartheta}_{pq}(b)\left(\bar{\Theta}_q(z_q(b))-\bar{\Theta}_q\left(z_q^*(b)\right)\right)\right.\\&+\vartheta_{pq}(b)\left(\Theta_q\left(z_q\left(b-\zeta_{pq}\right)\right)-\Theta_q\left(z_q^*\left(b-\zeta_{pq}\right)\right)\right)\\&\left.+h_{pq}(b)\int_0^{\infty}\psi_{pq}(v)\left(\widehat{\Theta}_q(z_q(b-v))-\widehat{\Theta}_q\left(z_q^*(b-v)\right)\right)dv\right]db.\end{aligned} \tag{69}$$

Next, by $(\mathbf{I}_1)$ and $(\mathbf{I}_5)$we obtain that

$$\begin{aligned}\left|f_p(\varkappa)\right|&\le O_p\left(\left|f_p(0)\right|\sum_{\varsigma=0}^{\infty}\frac{\left(-\int_0^{\varkappa}\widehat{e}_p(r)dr\right)^\varsigma}{\Gamma(\varsigma\lambda+1)}+\sigma_p^{-1}\int_0^{\varkappa}\sum_{\varsigma=0}^{\infty}\frac{\left(-\int_b^{\varkappa}\widehat{e}_p(r)dr\right)^\varsigma}{\Gamma(\varsigma\lambda+1)}\right.\\&\cdot\sum_{q=1}^{n}\left[\left|\bar{\vartheta}_{pq}(b)\right|\mathcal{T}_q^{\bar{\Theta}}\sigma_q\left|f_q(b)\right|+\left|\vartheta_{pq}(b)\right|\mathcal{T}_q^{\Theta}\sigma_q\left|f_q\left(b-\zeta_{pq}\right)\right|\right.\\&\left.\left.+\mathcal{T}_q^{\widehat{\Theta}}\left|h_{pq}(b)\right|\int_0^{\infty}\left|\psi_{pq}(v)\right|\sigma_q\left|f_q(b-v)\right|dv\right]db\right).\end{aligned} \tag{70}$$

Now, by (64) and (66), we get

$$\begin{aligned}\left|f_p(\varkappa)\right|&\le O_p\left(\|\xi-z^*\|_\sigma+\rho\right)\left\{\sum_{\varsigma=0}^{\infty}\frac{\left(-\int_0^{\varkappa}\widehat{e}_p(r)dr\right)^\varsigma}{\Gamma(\varsigma\lambda+1)}\right.\\&+\aleph\sigma_p^{-1}\int_0^{\varkappa}\sum_{\varsigma=0}^{\infty}\frac{\left(-\int_b^{\varkappa}\widehat{e}_p(r)dr\right)^\varsigma}{\Gamma(\varsigma\lambda+1)}\\&\cdot\sum_{q=1}^{n}\sigma_q\left[\left|\bar{\vartheta}_{pq}(b)\right|\mathcal{T}_q^{\bar{\Theta}}\sum_{\varsigma=0}^{\infty}\frac{(-\nu b)^\varsigma}{\Gamma(\varsigma\lambda+1)}\right.\\&+\left|\vartheta_{pq}(b)\right|\mathcal{T}_q^{\Theta}\sum_{\varsigma=0}^{\infty}\frac{\left(-\nu\left(b-\zeta_q\right)\right)^\varsigma}{\Gamma(\varsigma\lambda+1)}\\&\left.\left.+\mathcal{T}_q^{\Theta}\left|h_{pq}(b)\right|\int_0^{\infty}\left|\psi_{pq}(r)\right|\sum_{\varsigma=0}^{\infty}\frac{(-\nu(b-r))^\varsigma}{\Gamma(\varsigma\lambda+1)}dr\right]db\right\}\end{aligned} \tag{71}$$

Since $b>0$ and $0<\nu<\nu_0$, we have by virtue of Lemma22,

$$
\begin{aligned}
\left|f_p(\varkappa)\right| &\le O_p\left(\|\xi - z^*\|_\sigma + \rho\right)\left\{\sum_{\varsigma=0}^{\infty}\frac{\left(-\int_0^{\varkappa}\widehat{e}_p(r)dr\right)^{\varsigma}}{\Gamma(\varsigma\lambda+1)}\right.\\
&\quad\left. + \aleph O_p^{-1}\int_0^u\sum_{\varsigma=0}^{\infty}\frac{\left(-\int_b^{\varkappa}\widehat{e}_p(r)dr\right)^{\varsigma}}{\Gamma(\varsigma\lambda+1)}\left(\widehat{e}_p(b)-\nu\right)\sum_{\varsigma=0}^{\infty}\frac{(-\nu b)^{\varsigma}}{\Gamma(\varsigma\lambda+1)}\right\}\\
&= O_p\left(\|\xi - z^*\|_\sigma + \rho\right)\left\{\sum_{\varsigma=0}^{\infty}\frac{\left(-\nu\varkappa\left(-\int_0^{\varkappa}\left(\widehat{e}_p(r)-\nu\right)dr\right)\right)^{\varsigma}}{\Gamma(\varsigma\lambda+1)}\right.\\
&\quad\left. + \aleph O_p^{-1}(-\nu\varkappa)^{\varsigma}\int_0^{\varkappa}\sum_{\varsigma=0}^{\infty}\frac{\left(-\int_b^{\varkappa}\left(\widehat{e}_p(r)-\nu\right)dr\right)^{\varsigma}}{\Gamma(\varsigma\lambda+1)}\left(\widehat{e}_p(b)-\nu\right)db\right\}\\
&= O_p\left(\|\xi - z^*\|_\sigma + \rho\right)\sum_{\varsigma=0}^{\infty}\frac{(-\nu\varkappa)^{\varsigma}}{\Gamma(\varsigma\lambda+1)}\left\{\left(-\int_0^{\varkappa}\left(\widehat{e}_p(r)-\nu\right)dr\right)^{\varsigma}\right.\\
&\quad\left. + \aleph O_p^{-1}\int_0^{\varkappa}\left(-\int_b^{\varkappa}\left(\widehat{e}_p(r)-\nu\right)dr\right)^{\varsigma}\left(\widehat{e}_p(b)-\nu\right)db\right\}\\
&= O_p\left(\|\xi - z^*\|_\sigma + \rho\right)\sum_{\varsigma=0}^{\infty}\frac{(-\nu\varkappa)^{\varsigma}}{\Gamma(\varsigma\lambda+1)}\left\{\left(-\int_0^{\varkappa}\left(\widehat{e}_p(r)-\nu\right)dr\right)^{\varsigma}\right.\\
&\quad\left. + \aleph O_p^{-1}\left[\left(-\int_b^{\varkappa}\left(\widehat{e}_p(r)-\nu\right)dr\right)^{\varsigma}\right]_0^{\varkappa}\right\}\\
&= O_p\left(\|\xi - z^*\|_\sigma + \rho\right)\sum_{\varsigma=0}^{\infty}\frac{(-\nu\varkappa)^{\varsigma}}{\Gamma(\varsigma\lambda+1)}\left\{\left(-\int_0^{\varkappa}\left(\widehat{e}_p(r)-\nu\right)dr\right)^{\varsigma}\right.\\
&\quad\left. + \aleph O_p^{-1}\left(0-\left(-\int_0^{\varkappa}\left(\widehat{e}_p(r)-\nu\right)dr\right)^{\varsigma}\right)\right\}\\
&= \aleph\left(\|\xi - z^*\|_\sigma + \rho\right)\sum_{\varsigma=0}^{\infty}\frac{(-\nu\varkappa)^{\varsigma}}{\Gamma(\varsigma\lambda+1)}\\
&\quad\cdot\left\{-\left(1-\frac{O_p}{\aleph}\right)\sum_{\varsigma=0}^{\infty}\frac{\left(-\int_0^{\varkappa}\left(\widehat{e}_p(r)-\nu\right)dr\right)^{\varsigma}}{\Gamma(\varsigma\lambda+1)}\right\}.
\end{aligned}
\tag{72}
$$

We recall that $\nu < \widehat{e_p}$. This implies that $\widehat{e}_p(\mathrm{r}) - \nu > 0$ for all $\mathrm{r} \in (0,\infty)$. Hence, by (61), we see that

$$
\left|\mathrm{f}_p(\varkappa)\right| < \aleph\left(\|\xi - \mathrm{z}^*\|_\sigma + \rho\right)\sum_{\varsigma=0}^{\infty}\frac{(-\nu\varkappa)^{\varsigma}}{\Gamma(\varsigma\lambda+1)}, \tag{73}
$$

which contradicts (66) and we can conclude that what was claimed in (65) is true.

We now assume that y is constant and ρ tends to zero. Then we get

$$
\|f(y)\| \le \aleph\|\xi - z^*\|_\sigma\sum_{\varsigma=0}^{\infty}\frac{(-\nu\varkappa)^{\varsigma}}{\Gamma(\varsigma\lambda+1)}. \tag{74}
$$

so the proof is complete. □

Corollary 24. *If we assume that conditions* $(\boldsymbol{I}_1), (\boldsymbol{I}_2), (\boldsymbol{I}_3), (\boldsymbol{I}_5)$, *and* $(\boldsymbol{I}_6)$ *hold, then the Mittag-Leffler attractiveness for unique solution* $z^* \in \mathscr{BC}(\mathbb{R}, \mathbb{R}^n)$ *of equation (2) holds.*

Corollary 25. *If we assume that conditions* $(\boldsymbol{I}_4)$, *and* $(\boldsymbol{I}_7)$ *hold, then the Mittag-Leffler attractiveness for unique solution* $z^* \in \mathscr{PAP}(\mathbb{R}, \mathbb{R}^n)$ *of equation (2) holds.*

7. Applications

We shall provide two examples (see Figures 1–3).

Example 26. Let $n = 2, p, q = 1, 2,$

- $(\mathbf{P}_1)$ We consider Lipschitz functions $\bar{\Theta}_p(r) = 0, \Theta_p(r) = \widehat{\Theta}_p(r) = 2/5 \arctan r$, with the Lipschitz constants $\mathscr{T}_p^{\bar{\Theta}} = 0, \mathscr{T}^{\Theta} = \mathscr{T}_p^{\widehat{\Theta}} = 2/5$.
- $(\mathbf{P}_2)$ Then, ψ_{pq} is bounded and continuous,

$$
\psi_{pq}(y) = \frac{2}{5}\sum_{\varsigma=0}^{\infty}\frac{(-8y)^{\varsigma}}{\Gamma(\varsigma\lambda+1)} \tag{75}
$$

- $(\mathbf{P}_3)$ Furthermore, the next sum is integrable on $[0, +\infty)$ for $n = 1$,

$$
\left|\psi_{pq}\right|\sum_{\varsigma=0}^{\infty}\frac{(ny)^{\varsigma}}{\Gamma(\varsigma\lambda+1)} \tag{76}
$$

- $(\mathbf{P}_4)$ (see [1]) If $\phi \in \mathscr{N}$, where $d\phi(y) = \eta_\psi(y)dy$, and

$$
\eta_\psi(y) = \sum_{\varsigma=0}^{\infty}\frac{(\psi y)^{\varsigma}}{\Gamma(\varsigma\lambda+1)}, \psi_{pq}(y) = \sum_{\varsigma=0}^{\infty}\frac{(-8y)^{\varsigma}}{\Gamma(\varsigma\lambda+1)}, \tag{77}
$$

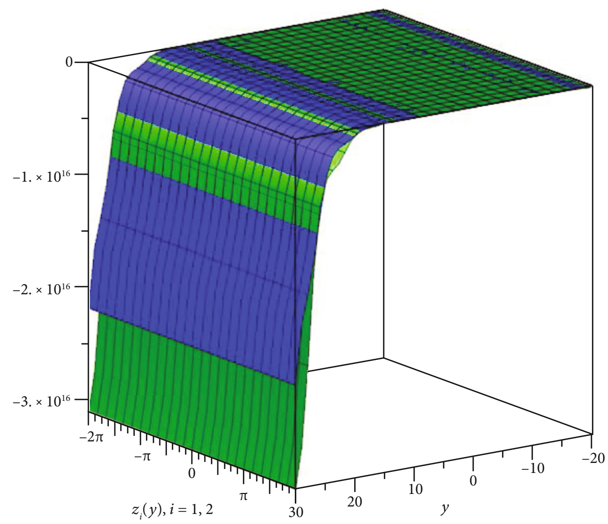
with $\vartheta_0 \coloneqq 2 - \psi > 0, -u \le b < y \le u$, then for $\omega = 1$ we have that $L = \varnothing$.

- $(\mathbf{P}_5)$ For $e_1(y) = (2/5)(1 + (3/2)\sin y), e_2(y) = (2/5)(1 + (7/6)\cos y), \widehat{e}_p(y) = (2/5)$, we have

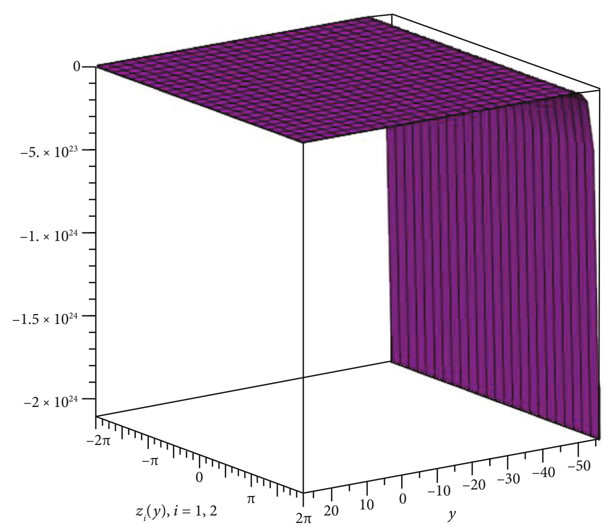
$$
\mathrm{O}_p = \sum_{\varsigma=0}^{\infty}\frac{(4/15)^{\varsigma}}{\Gamma(\varsigma\lambda+1)} \tag{78}
$$

- $(\mathbf{P}_6)$ For $\sigma_p = 1, R = \pi, \zeta_{pq} = 3/(p+q), \bar{\vartheta}_{pq}(y) = 0, \vartheta_{pq}(y) = (8/10)\sin y, h_{pq}(t) = (9/10)\cos 2\mathrm{y}, \zeta_\ell = 9/100$, we have

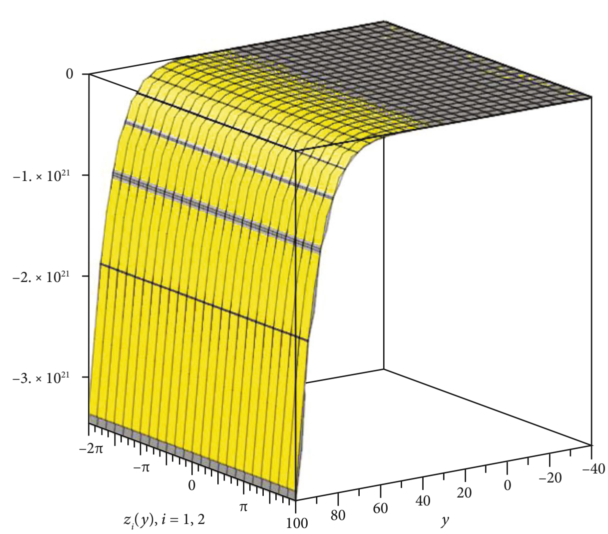
$$
\sup_{y\in\mathbb{R}}\left\{-\widehat{e}_p(y) + O_p\left[\sigma^{-1}\sum_{q}^{n}\left(\left|\bar{\vartheta}_{pq}(y)\right|\mathscr{T}_q^{\Theta} + \left|\vartheta_\sigma(y)\right|\mathscr{T}_q^{\Theta} + \widehat{\mathscr{T}}_q^{\widehat{\Theta}}\left|h_p(y)\right|\left|\int_0^{\infty}\left|\psi_{pq}(r)\right|dr\right)\sigma_q\right]\right\}, \le -\kappa_p \le 0 \tag{79}
$$



(a) Numerical solutions of CNNs for $y \in (-20, 30)$

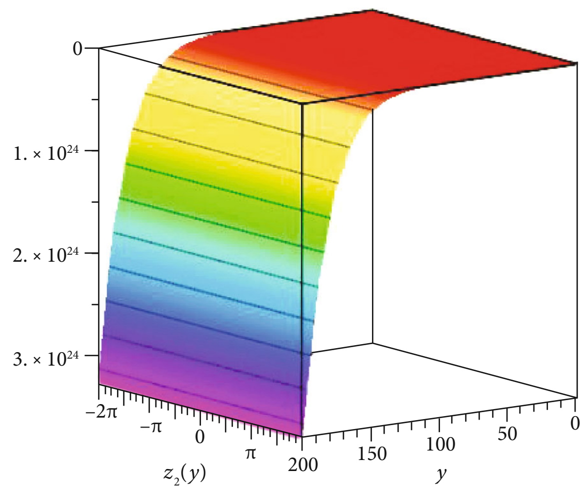


(b) Numerical solutions of CNNs for $y \in (28, -56)$

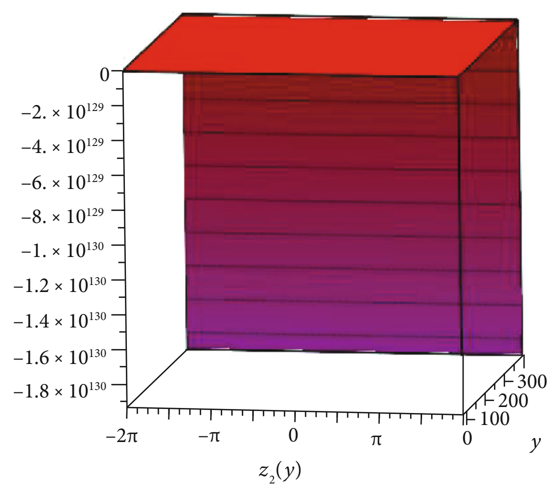


(c) Numerical solutions of CNNs for $y \in (100, -40)$

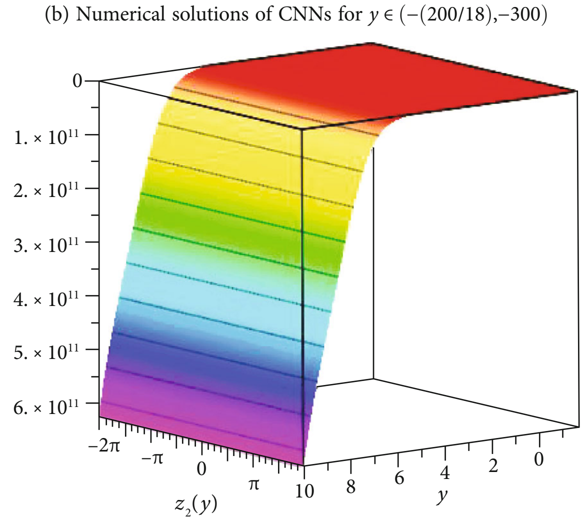
Figure 1: Graphs related to numerical solutions of CNNs (1) for different values.



(a) Numerical solutions of CNNs for $y \in (-(10/7), 200)$

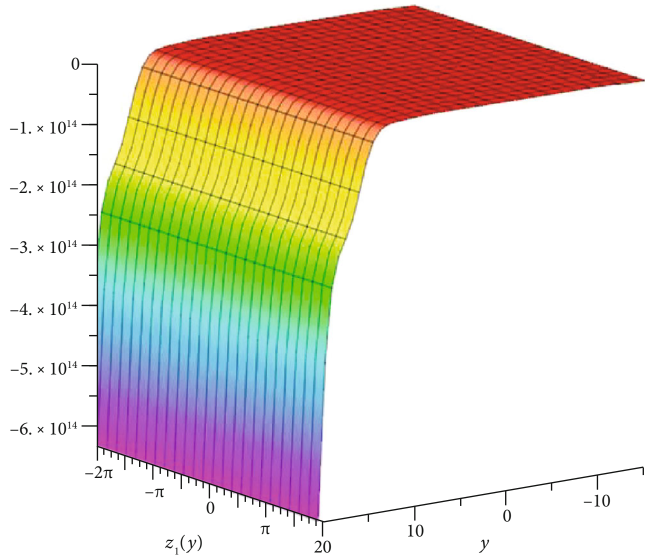


(b) Numerical solutions of CNNs for $y \in (-(200/18), -300)$

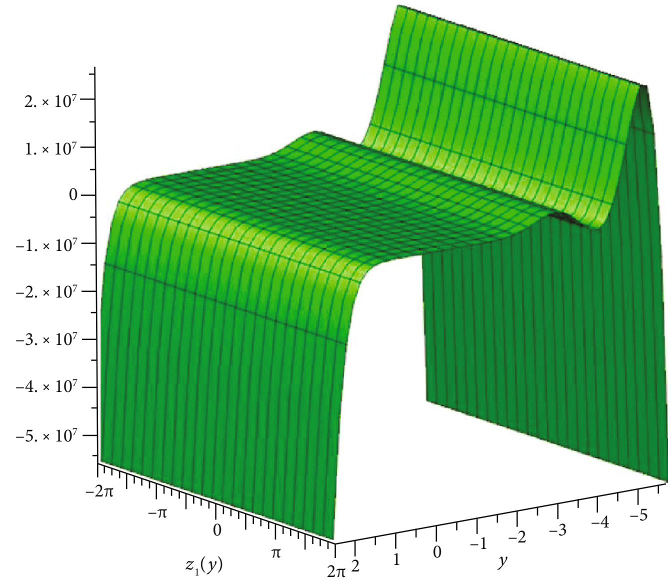


(c) Numerical solutions of CNNs for $y \in (-(10/6), 10)$

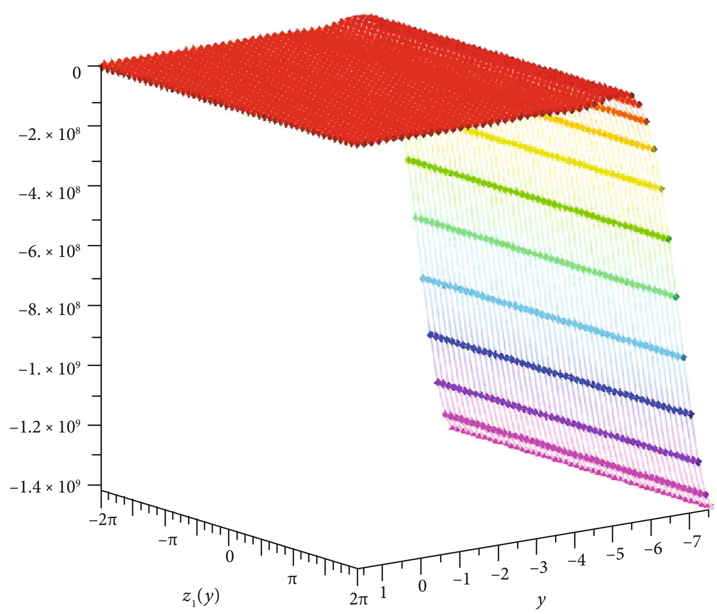
Figure 2: Graphs related to numerical solutions of CNNs (1) for different values.



(a) Numerical solutions of CNNs for $y \in (20,-15)$



(b) Numerical solutions of CNNs for $y \in (15/6,-40/7)$



(c) Numerical solutions of CNNs for $y \in (10/6,-60/8)$

Figure 3: Graphs related to numerical solutions of CNNs (1) for different values.

($\mathbf{P}_7$) Also,

$$0 < \max_{p=1,2,3,n,n} \left\{ \frac{\widehat{e}_p^+}{\widehat{e}_p^-} - \frac{\kappa_p}{\widehat{e}_p^+} \right\} < 1. \tag{80}$$

Let $L_p(y) = (20 + p)|\cos y| + U(y)$, where

$$U(b) = \begin{cases} e^{-b}, & b \leq 0 \\ 1, & b < 0, \end{cases} \tag{81}$$

and $L_p \in \mathscr{PAP}(\mathbb{R}, \mathbb{R}, \rho)$. Then, all solutions of (1) are in the Mittag-Leffler form and they converge to a unique solution of equation (1) such that $z^*(y) \in \mathscr{PAP}(\mathbb{R}, \mathbb{R}^2, \phi)$, when $y \longrightarrow +\infty$ with convergence rate $\nu \approx 0.05 < \widehat{e}_p$.

Example 27. Assume conditions ($\boldsymbol{P}_1$) to ($\boldsymbol{P}_7$) hold and consider functions from Example 26. For $\psi > 1, \nu > 0$ and Dirac measure $\delta_{1/n}$, define the following measure:

$$d\phi_{\psi,\nu}(y) = \sum_{\varsigma=0}^{\infty} \frac{(\psi y)^{\varsigma}}{\Gamma(\varsigma\lambda + 1)} dy + \nu \sum_{n=1}^{\infty} \frac{1}{n^2} \delta_{1/n}. \tag{82}$$

Consider the interval $I = (-1 - |\zeta|, 1 + |\zeta|)$, for $\zeta \in \mathbb{R}$ and $\vartheta = 1$. Then for all $R \in \mathscr{A}$ and $\zeta \in \mathbb{R}$, there exist $\vartheta > 0$ and a bounded interval L such that $\phi(R + \zeta) \leq \vartheta\phi(R)$ and $R \cap L = \varnothing$.

Let $L_p(y) = (20 + p)|\cos y| + U(y)$, where

$$U(b) = \begin{cases} e^{-b}, & b \leq 0 \\ 0, & b > 0, \end{cases} \quad (83)$$

and $L_p \in \mathscr{PAP}(\mathbb{R}, \mathbb{R}, \rho)$. Then, all solutions of (1) are in the Mittag-Leffler form and they converge to a unique solution of equation (1) such that $z^*(y) \in \mathscr{PAP}(\mathbb{R}, \mathbb{R}^2, \phi)$, when $y \longrightarrow +\infty$ with convergence rate $\nu \approx 0.05$.

8. Conclusion

In this work, we considered differential systems of cellular neural networks (CNNs) with mixed delays. We also considered general measurement theory whose general form is $d\phi = \eta(y)dy + d\phi_1$. We first investigated the existence of a unique solution of this system and proved that the solutions of equation (1) are ϕ-pseudo almost periodic. Then we studied the Mittag-Leffler stability and the Mittag-Leffler attractiveness of these solutions. We obtained our results by considering new conditions and using the fixed point contraction mapping theorem. Also, two examples were given to illustrate our results.

Data Availability

The numerical data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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