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Classification of knotted tori in 2-metastable dimension

M. Cencelj, D. Repovš and M. B. Skopenkov

Abstract. This paper is devoted to the classical Knotting Problem: for a given manifold N and number m describe the set of isotopy classes of embeddings $N \to S^m$. We study the specific case of knotted tori, that is, the embeddings $S^p \times S^q \to S^m$. The classification of knotted tori up to isotopy in the metastable dimension range $m \geqslant p + \frac{3}{2}q + 2$, $p \leqslant q$, was given by Haefliger, Zeeman and A. Skopenkov. We consider the dimensions below the metastable range and give an explicit criterion for the finiteness of this set of isotopy classes in the 2-metastable dimension:

Theorem. *Assume that* $p + \frac{4}{3}q + 2 < m < p + \frac{3}{2}q + 2$ *and* $m > 2p + q + 2$. *Then the set of isotopy classes of smooth embeddings* $S^p \times S^q \to S^m$ *is infinite if and only if either* $q + 1$ *or* $p + q + 1$ *is divisible by* 4.

Bibliography: 35 titles.

Keywords: knotted torus, link, link map, embedding, surgery.

§ 1. Introduction

This paper is devoted to the classical Knotting Problem: for a given manifold N and a number m describe the set of isotopy classes of embeddings $N \to S^m$. For recent surveys see [1], [2]. This subject was actively studied in the 1960s [3]–[6], and there has been a renewed interest in it in the last years [7]–[10]. In this paper the results announced in [11], [12] are proved.

This problem generalizes the subject of classical knot theory. In contrast with knots in $\mathbb{R}^3$, a complete answer can sometimes be obtained in higher dimensions. We work in the smooth category except when explicitly indicated otherwise. Let us list some known results.

1.1. Knots. The classification of knots $S^q \to S^m$ in codimension at least 3, that is, for $m > q + 2$, has been reduced to a homotopy problem [4], [13]. In particular, the following complete rational classification is known.

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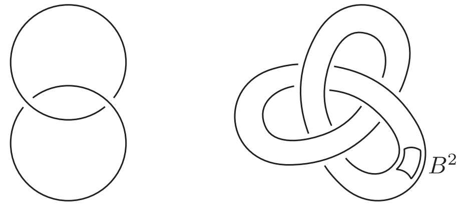
Theorem 1 ([4], Corollary 6.7). *Assume that $q+2<m<\frac{3}{2}q+2$. Then the set of isotopy classes of smooth embeddings $S^q \to S^m$ is infinite if and only if $q+1$ is divisible by* 4.

1.2. Links. The classification of links $S^p \sqcup S^q \to S^m$ is the next natural problem after knots. In codimension at least 3 there is an exact sequence involving the set of links up to isotopy and certain homotopy groups [5]. In a certain dimension range, called 2-metastable, there is an explicit description of the isotopy classes of links $S^p \sqcup S^q \to S^m$ modulo knots $S^p \to S^m$ and $S^q \to S^m$ in terms of homotopy groups of spheres and Stiefel manifolds [4]. A short proof of this result was given in [10].

1.3. Knotted tori. In this paper we study the classification of knotted tori, that is, smooth embeddings $S^p \times S^q \to S^m$. This theory generalizes the theory of 2-component links of the same dimension (see Fig. 1). The investigation of knotted tori is a natural next step (after link theory) towards the classification of embeddings of arbitrary manifolds [14], [15], by the handle decomposition theorem. This subject is also interesting because of many interesting examples [6], [16], [17]. Its systematic investigations began in [17].



$S^0 \times S^1 \to S^3$ $S^1 \times S^1 \to S^3$

Figure 1

There exists an explicit description of the set of isotopy classes of knotted tori in the metastable dimension $m \geqslant p+\frac{3}{2}q+2$, $p \leqslant q$ [3], [17] (up to small indeterminacy for $m<\frac{3}{2}p+\frac{3}{2}q+2$); see Fig. 2. This dimension restriction is a natural limit for classical classification methods for a $(p-1)$-connected $(p+q)$-manifold. In spite of many interesting approaches [18]–[20], little is known below the metastable dimension: all known explicit classification results concern knots and links (listed above), knotted tori in dimension $m=p+\frac{3}{2}q+\frac{3}{2}$ [21], 3-manifolds in $\mathbb{R}^6$ [22] and 4-manifolds in $\mathbb{R}^7$ [9].

Now let us state the main 'practical' result of the paper announced in [11]. It is an explicit criterion for finiteness of the knotted tori set in the 2-*metastable* range (see the shaded domain in Fig. 2, in which the number p is fixed, whereas the numbers q and m vary):

Theorem 2. *Assume that $p+\frac{4}{3}q+2<m<p+\frac{3}{2}q+2$ and $m>2p+q+2$. Then the set of isotopy classes of smooth embeddings $S^p \times S^q \to S^m$ is infinite if and only if either $q+1$ or $p+q+1$ is divisible by* 4.

Example 1. The set of isotopy classes of knotted tori $S^1 \times S^5 \to S^{10}$ is finite. All dimensions in the range $p=1$, $1 \leqslant q \leqslant 13$, $m>p+\frac{4}{3}q+2$, such that the set

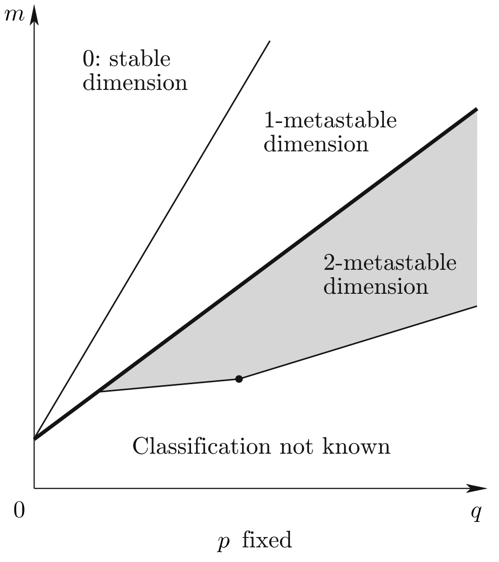


Figure 2

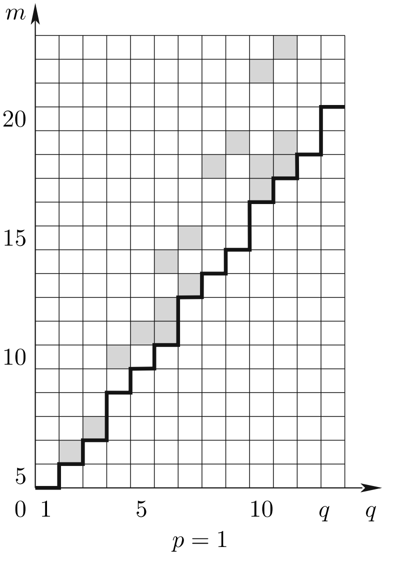


Figure 3

of isotopy classes of knotted tori is infinite, are shown in Fig. 3. Information in the figure is obtained from Theorem 2, the results in [2] (Theorems 3.10 and 2.9), Proposition 8 and Remark 2 below.

Our approach to the classification of embeddings is based on an analogue of the Koschorke exact sequence (Theorem 3 below), involving a new β-invariant of almost embeddings $S^p \times S^q \to S^m$. The exactness is proved using the Habegger-Kaiser techniques of studying the complement of an almost embedding.

1.4. Organization of the paper. In § 2 we state our main 'theoretical' result — Theorem 3. In § 3 we define the new β-invariant. In § 4, which is the main part of the paper, we prove the completeness of the β-invariant. In § 5 we prove Theorem 3, using the results in § 4. In § 6 we deduce Theorem 2 from Theorem 3. In § 7 we state some open problems. In § 8 we consider the embedded surgery of self-intersection sets which is used in § 4.

In a subsequent publication [23] we are going to generalize Theorem 2 for arbitrary $m > 2p+q+2$ by reducing the classification of knotted tori to the classification of links. A finiteness criterion for links is required for this purpose; we plan to obtain the criterion in [24].

§ 2. The main idea

In this section we state our main 'theoretical' result. We present an exact sequence, which reduces the classification of embeddings $S^p \times S^q \to S^m$ to an easier classification of all almost embeddings $S^p \times S^q \to S^m$.

We recall several well-known definitions required to state the result.

2.1. Isotopies and concordances. An embedding $f\colon X \times I \to S^m \times I$ is a *concordance* if $X \times 0 = f^{-1}(S^m \times 0)$ and $X \times 1 = f^{-1}(S^m \times 1)$. A concordance is an *isotopy* if $f(X \times t) \subset S^m \times t$ for each $t \in I$. A concordance or isotopy is *ambient* if $X = S^m$. We tacitly use the well-known facts that in codimension at least 3 *existence of concordance implies existence of an isotopy* and *any concordance or isotopy extends to an ambient one* [25].

2.2. Almost embeddings. Informally, an *almost embedding* is a map admitting only 'local' self-intersections; see Fig. 4. To give a formal definition, fix a base point $* \in S^p$ and a codimension 0 ball $B^{p+q} \subset S^p \times S^q$ such that $B^{p+q} \cap (* \times S^q) = \varnothing$; see Fig. 1, right-hand side.

A map $f\colon S^p \times S^q \to S^m$ is an *almost embedding* if the following two conditions hold:

(i) f is an embedding outside B^{p+q};

(ii) $fB^{p+q} \cap f(S^p \times S^q - B^{p+q}) = \varnothing$.

An *almost concordance* is defined analogously, except that the ball B^{p+q} is replaced by $B^{p+q} \times I$.

2.3. Commutative group structure. The parametric connected sum operation gives a natural commutative group structure on the set of embeddings $S^p \times S^q \to S^m$ up to concordance; see Fig. 5. This structure is well-defined for $m > 2p+q+2$ [21]. We shall give the formal definition in § 5.

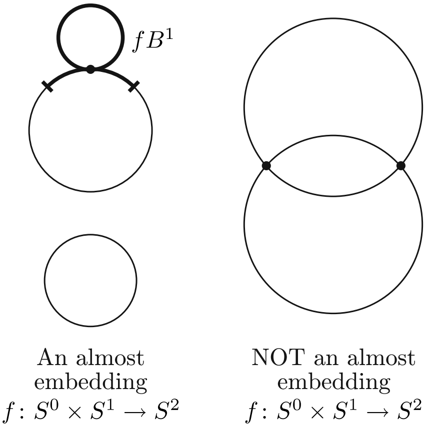


Figure 4

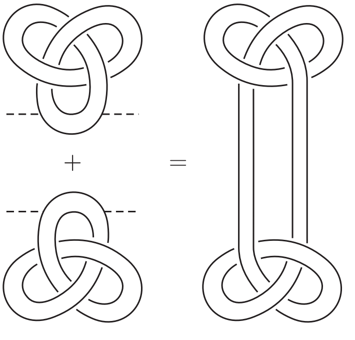


Figure 5

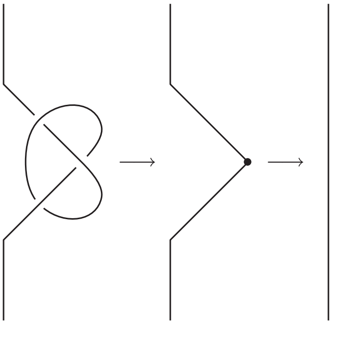


Figure 6

2.4. Action of knots on knotted tori. For $m > p+q+2$ the set of embeddings $S^{p+q} \to S^m$ up to concordance is a group with respect to the connected sum operation [4]. The same operation gives an action of this group on the set of embeddings $S^p \times S^q \to S^m$ up to concordance. In § 5 we shall prove that this action is injective for $m > 2p+q+2$. Notice that the set of orbits of the action is in a one-to-one correspondence with the set of embeddings $S^p \times S^q \to S^m$ up to concordance smooth outside a finite set; see Fig. 6.

2.5. Notation. $E^m(S^p \times S^q)/E^m(S^{p+q})$ denotes the group of all embeddings $S^p \times S^q \to S^m$ up to concordance and connected sums with embeddings $S^{p+q} \to S^m$.

$\overline{E}^m(S^p \times S^q)$ denotes the group of all almost embeddings $S^p \times S^q \to S^m$ up to almost concordance (with the parametric connected sum group structure).

$\Omega^m_{p,q} := \pi_{p+2q-m+1}(V_{N+m-p-q-1,N})$, where $V_{i,j}$ is the Stiefel manifold of j-frames at the origin of $\mathbb{R}^i$ and N is a large number. Equivalently, for $m \geqslant p+\frac{4}{3}q+2$ the group $\Omega^m_{p,q}$ is the normal bordism group $\Omega_{2p+3q-2m+2}(P^\infty, (m-p-q-1)\lambda)$ [26]. Many of the groups $\Omega^m_{p,q}$ are known [27], [28].

2.6. Let us state the main 'theoretical' result of the paper.

Theorem 3. *For every $m \geqslant p + \frac{4}{3}q + 2$ and $m > 2p + q + 2$ there exists an exact sequence*

$$E^m(S^p \times S^q)/E^m(S^{p+q}) \to \overline{E}^m(S^p \times S^q) \xrightarrow{\beta} \Omega^m_{p,q} \to E^{m-1}(S^p \times S^{q-1})/E^{m-1}(S^{p+q-1}) \to \overline{E}^{m-1}(S^p \times S^{q-1}) \to \Omega^{m-1}_{p,q-1} \to \cdots .$$

This theorem has a list of immediate corollaries. First, it provides estimates of the order or the rank of the group $E^m(S^p \times S^q)/E^m(S^{p+q})$ given such estimates for the group $\overline{E}^m(S^p \times S^q)$. Theorem 3 also easily implies the Haefliger formula for the group of links $S^q \sqcup S^q \to S^m$ in the 2-metastable range [5]. A short proof of this classical result (together with Theorem 3 for $p = 0$) was given in [10]. The exact sequence in Theorem 3 for $p = 0$ is analogous to the Koschorke exact sequence: cf. Theorems 3.1 and 3.5 in [10] and Theorem 3.1 in [29].

2.7. The beta-invariant. The map $\beta\colon \overline{E}^m(S^p \times S^q) \to \Omega^m_{p,q}$ in Theorem 3 is a new invariant and it is the main tool of the present paper. It generalizes both

- the normal bordism β-invariant of link maps [30]–[32];
- the β-invariant of knotted tori [14], [21]; cf. also [5], [6].

The idea of this invariant is the following. For any almost embedding $f\colon S^p \times S^q \to S^m$ we have by definition $fB^{p+q} \cap f(* \times S^q) = \varnothing$. The β-invariant measures the 'linking' of $f(* \times S^q)$ and (a certain retract of) fB^{p+q}.

The main idea of our paper is that from this point of view the study of almost embeddings $S^p \times S^q \to S^m$ is similar to the study of link maps $S^q \sqcup S^{p+q} \to S^m$. So we can apply the full strength of the Habegger-Kaiser technique from link map theory to our problem. Now let us concentrate on what is done in addition to the technique of the Habegger-Kaiser paper [30].

2.8. Sketch of the proof of Theorem 3. Let us outline the proof of exactness at the term $\overline{E}^m(S^p \times S^q)$. We need to prove that any almost embedding $f\colon S^p \times S^q \to S^m$ such that $\beta(f) = 0$ is almost concordant to an embedding.

It suffices to construct a ball $B^m \subset S^m$ such that $f^{-1}B^m = B^{p+q}$. Indeed, then the knot $f\colon \partial B^{p+q} \to \partial B^m$ is trivial by smoothing theory. Thus one can deform the restriction $f\colon B^{p+q} \to B^m$ to an embedding and obtain the desired embedding $S^p \times S^q \to S^m$.

To construct the ball B^m it suffices to span the meridians $f(*\times S^q)$ and $f(S^p \times *)$ by two discs D^{q+1} and D^{p+1} (called *webs*), whose interiors are disjoint and do not intersect the image $f(S^p \times S^q)$. Then the required ball B^m is the complement of a small tubular neighborhood of $D^{q+1} \cup D^{p+1}$ in S^m.

The existence of the web D^{p+1} is guaranteed by general position and the inequality $m > 2p + q + 2$. Let us show how to construct the web D^{q+1} under some additional assumptions. This is the most difficult step, which requires the conditions $\beta(f) = 0$ and $m \geqslant p + \frac{4}{3}q + 2$. By the results of Habegger-Kaiser [30] we may assume that $f|_{*\times S^q}$ is null-homotopic outside fB^{p+q}. So we can span $f(*\times S^q)$ by a (not necessarily embedded) disc D^{q+1} in $S^m - fB^{p+q}$. Then we can remove the self-intersection of the disc using Hudson's embedding theorem.

This completes the proof under the assumption that the interior of D^{q+1} does not intersect also $f(S^p \times S^q - B^{p+q})$ and the disc is smooth. The proof without these assumptions uses an appropriate relative version of this argument; see § 4. Let us emphasize that a relative version of the above construction is used in the formal proof of Theorem 3.

§ 3. The beta-invariant

In this section we shall give a detailed construction of the β-invariant of almost embeddings $S^p \times S^q \to S^m$.

3.1. Idea of the β-invariant. The visualization of this idea is based on an analogy of low-dimensional almost embeddings $f \sqcup g\colon S^1 \sqcup S^0 \to S^2$.

Fix an arc $B^1 \subset S^1$. A map $f \sqcup g\colon S^1 \sqcup S^0 \to S^2$ is an *almost embedding* if it is an embedding outside the arc B^1 and $fB^1 \cap (f(S^1 - B^1) \cup gS^0) = \varnothing$. An *almost isotopy* $f_t \sqcup g_t\colon S^1 \sqcup S^0 \to S^2$ is a homotopy in the class of almost embeddings.

A simple almost isotopy invariant of an almost embedding $f \sqcup g\colon S^1 \sqcup S^0 \to S^2$ is the linking number $\operatorname{lk}(f, g)$ which assumes its values in $\mathbb{Z}_2$. This invariant is incomplete: for example, the almost embedding in Fig. 7 is not almost isotopic to the 'unlink', although $\operatorname{lk}(f, g) = 0$.

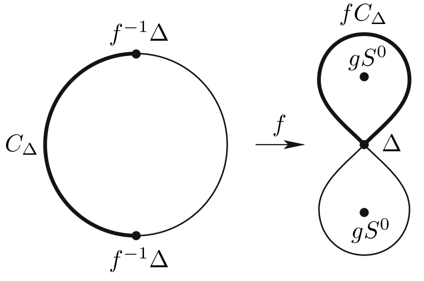


Figure 7

In the situation of Fig. 7 the following β-invariant of almost isotopy is useful. Take a double point Δ of the map $f\colon S^1 \to S^2$. Generically $f^{-1}\Delta$ consists of two points. Join these two points by an arc $C_\Delta \subset S^1$. The image fC_Δ is a cycle, and the number

$$\beta(f,g) = \sum_{\Delta} \operatorname{lk}(fC_\Delta, g) \pmod 2,$$

where the summation is over all double points of f, is an almost isotopy invariant. It is well-defined only if $\operatorname{lk}(f,g) = 0$. For example, for the almost embedding in Fig. 7 we have $\beta(f,g) = 1$ which proves that the shown almost embedding is indeed not almost isotopic to an 'unlink'.

3.2. Construction. To realize this idea in higher dimensions we construct:

(i) an analogue of the cycle fC_Δ, see Definition 2 below;

(ii) a generalization of the linking number $\operatorname{lk}(fC_\Delta, f(* \times S^q))$, see Definition 5 below.

Definition 1 [27]. Let $f\colon S^p \times S^q \to S^m$ be an almost embedding. Make f a general position immersion by an almost isotopy. Consider the diagram

$$\begin{array}{ccc} \widetilde{\Delta} & \xrightarrow{\ \widetilde{i}\ } & B^{p+q} \\ {\scriptstyle 2:1}\downarrow & & \downarrow{\scriptstyle f} \\ \Delta & \xrightarrow{\ i\ } & S^m \end{array}$$

Here $\widetilde{\Delta} = \operatorname{Cl}\{(x,y) \in B^{p+q} \times B^{p+q} \mid x \neq y, fx = fy\}$ and $\Delta = \widetilde{\Delta}/\mathbb{Z}_2$ are the double point manifolds, where $\mathbb{Z}_2$ acts on $\widetilde{\Delta}$ by interchanging the factors. The immersions $\widetilde{i}\colon \widetilde{\Delta} \to B^{p+q}$ and $i\colon \Delta \to S^m$ are given by the formulae $\widetilde{i}(x,y) = x$ and $i\{x,y\} = fx$. Denote by $\Sigma(f) = \operatorname{Im} \widetilde{i}$ the singular set of the map f.

Definition 2 [30]. Denote by $T(\lambda_\Delta)$ the mapping cone of the double covering $\widetilde{\Delta} \to \Delta$. Take a map $\bar{i}\colon C\widetilde{\Delta} \to B$ extending the map $\widetilde{i}\colon \widetilde{\Delta} \to B$. Define $\widetilde{f}\colon T(\lambda_\Delta) \to S^m$ to be the quotient map of the composition $f\bar{i}\colon C\widetilde{\Delta} \to S^m$.

Now let us define the generalized linking number. This is the announced step (ii) of the construction of the invariant. Extend $f|_{*\times S^q}$ to a general position immersion $\overline{f}\colon D^{q+1} \to S^m$ (web). Informally, the desired 'linking number' is the bordism class of the 'intersection' $\operatorname{Im} \widetilde{f} \cap \operatorname{Im} \overline{f}$ with a natural 'skew framing'. It assumes values in the normal bordism group $\Omega^m_{p,q}$ generalizing the group of framed links[1].

Definition 3 [27]. Let l and s be integer numbers. An $l\lambda$-*manifold* is a triple consisting of

1) a manifold M ('link');
2) a linear bundle λ_M on M;

[1] The construction of all framings below is obvious, so the reader may ignore the awkward steps 2) and 3) in all the definitions below. In fact these steps are not used in the paper except in the proof of Proposition 3 essentially borrowed from [30].

3) a stable isomorphism

$$\overline{g}_M\colon \nu(M) \cong l\lambda_M := \underbrace{\lambda_M \oplus \dots \oplus \lambda_M}_{l}$$

(‘skew framing’), where $\nu(M)$ is the stable normal bundle of M.

The *normal bordism group* $\Omega_s(P^\infty, l\lambda)$ is the set of s-dimensional $l\lambda$-manifolds up to bordism (with analogous ‘skew framing’ structure). The disjoint union commutative group structure is evidently defined on this set.

Hereafter set $s = 2p + 3q - 2m + 2$, $l = m - p - q - 1$ and $n = p + q$. By [27] we have $\Omega_s(P^\infty, l\lambda) \cong \pi_{l+s}(V_{N+l,N}) = \Omega^m_{p,q}$ for $s < l$ and a large integer N. Denote by $N(X, Y)$ the normal bundle of a manifold X immersed into a manifold Y and by ε the trivial line bundle.

Definition 4 [27]. The *double-point* $(m-n)\lambda$*-manifold* is a triple $(\Delta, \lambda_\Delta, \overline{g}_\Delta)$, where

1) Δ is the double point manifold;
2) λ_Δ is the line bundle associated with the double cover $\widetilde{\Delta} \to \Delta$;
3) $\overline{g}_\Delta\colon N(\Delta, S^m) \cong (m-n)\lambda_\Delta \oplus \varepsilon^{m-n}$ is constructed as follows. For each point $\{x, y\} \in \Delta$ we have canonical isomorphisms

$$N(\Delta, S^m)_{\{x,y\}} \cong N(\widetilde{\Delta}, B)_{(x,y)} \oplus N(\widetilde{\Delta}, B)_{(y,x)} \cong N(B, S^m)_y \oplus N(B, S^m)_x.$$

Let vectors $\{e^k_x\}_{k=1}^{m-n}$ form a trivialization of $N(B, S^m)$ at the point $x \in B$. Then $\{e^k_x, e^k_y\}$ form a ‘skew framing’ of the manifold Δ. Interchanging the points x and y in a pair $(x, y) \in \widetilde{\Delta}$ implies interchanging the vectors e^k_x and e^k_y. Thus the bundle $N(\Delta, S^m)$ can be decomposed into the sum of all line bundles $\langle e^k_x + e^k_y\rangle \cong \varepsilon$ and $\langle e^k_x - e^k_y\rangle \cong \lambda_\Delta$ for $k = 1, \dots, m-n$. This decomposition defines the required isomorphism $\overline{g}_\Delta\colon N(\Delta, S^m) \cong (m-n)\lambda_\Delta \oplus \varepsilon^{m-n}$.

Notice that the bundle λ_Δ can be identified with a subset of the mapping cone $T(\lambda_\Delta)$ of the covering $\widetilde{\Delta} \to \Delta$. Denote the restriction of the cycle $\widetilde{f}\colon T(\lambda_\Delta) \to S^m$ to the subset still by $\widetilde{f}\colon \lambda_\Delta \to S^m$. Denote by $\check{f}\colon \lambda_\Delta \to S^m$ a general position smooth map (not necessarily an immersion) sufficiently close to $\widetilde{f}\colon \lambda_\Delta \to S^m$.

Definition 5. The *beta-invariant* $\beta(f)$ of an almost embedding $f\colon S^p \times S^q \to S^m$ is the bordism class of the $l\lambda$-manifold $(\beta, \lambda_\beta, \overline{g}_\beta)$ defined as follows.

1) The manifold β is defined by

$$\beta = \{(x, y) \in D^{q+1} \times \lambda_\Delta : \overline{f}x = \check{f}y\}.$$

2) To construct the line bundle λ_β on the manifold β denote by $\mathrm{pr}\colon \beta \to \Delta$ the obvious composition $\beta \to D^{q+1} \times \lambda_\Delta \to 0 \times \Delta = \Delta$. Put $\lambda_\beta = \mathrm{pr}^*(\lambda_\Delta)$.

3) To construct the stable isomorphism $\overline{g}_\beta\colon \nu(\beta) \cong l\lambda_\beta$ restrict the stable isomorphism $\overline{g}_\Delta\colon \nu(\Delta) \cong (l+1)\lambda_\Delta$, in Definition 4, step 3) to the bundle $\langle e^1_x - e^1_y\rangle^\perp$. We get an isomorphism $\nu(\lambda_\Delta) \cong l\lambda_\Delta \oplus \varepsilon^{l+1}$. Identify $\nu(\lambda_\Delta)$ and $\nu(D^{q+1} \times \lambda_\Delta)$. Restricting the previous isomorphism to the manifold β we get an isomorphism

$$g_1 : \nu(D^{q+1} \times \lambda_\Delta)|_\beta \cong l\lambda_\beta \oplus \varepsilon^{l+1}.$$

Take a trivialization of the normal bundle $N(D^{q+1}, S^m)$. The trivialization gives an isomorphism

$$g_2 \colon \nu(\beta) \cong \nu(D^{q+1} \times \lambda_\Delta)|_\beta \oplus \varepsilon^{m-q-1}.$$

Put $\overline{g}_\beta = (g_1 \oplus \mathrm{id}) \circ g_2$.

Proposition 1. *There is a well-defined map $\overline{E}^m(S^p \times S^q) \to \Omega^m_{p,q}$ given by the formula $f \mapsto \beta(f)$.*

Proof. We need to check the following:

1) *The class of $\beta(f)$ does not depend on the choices in the construction.* Indeed, we made the following four choices. In the definition of the cycle $\widetilde{f}$ we took an extension $\bar{i} \colon C\widetilde{\Delta} \to D^{p+q}$. In Definition 4, step 3), we took a trivialization of the bundle $N(D^{p+q}, S^m)$. In Definition 5, step 1), we took an extension $\overline{f} \colon D^{q+1} \to S^m$. And in step 3) of the same definition we took a trivialization of $N(D^{q+1}, S^m)$. Clearly, these extensions and trivializations are unique up to homotopy, hence the class of $\beta(f)$ is well-defined.

2) *If f_1 and f_2 are almost concordant, then $\beta(f_1) = \beta(f_2)$.* Indeed, let $f \colon S^p \times S^q \times I \to S^m \times I$ be a general position almost concordance between f_1 and f_2. One can construct analogously as above the β-invariant $\beta(f)$ of this almost concordance. It will be a bordism between $\beta(f_1)$ and $\beta(f_2)$. Thus $\beta(f_1) = \beta(f_2)$.

Now we give a relative version of the above construction; see Fig. 8. A map $f \colon X \to Y$ is *proper* if $f^{-1}\partial Y = \partial X$.

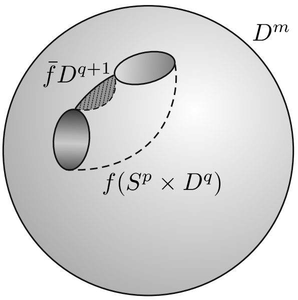


Figure 8

Definition 6. Fix a ball $B^{p+q} \subset S^p \times \operatorname{Int} D^q$ such that $B^{p+q} \cap (* \times D^q) = \varnothing$. A proper map $f \colon S^p \times D^q \to D^m$ is said to be a *proper almost embedding* if the following conditions hold:

(i) f is an embedding outside B^{p+q};

(ii) $fB^{p+q} \cap f(S^p \times D^q - B^{p+q}) = \varnothing$.

A *proper almost concordance* is defined analogously, except that the ball B^{p+q} is replaced by $B^{p+q} \times I$. The *standard embedding* $S^p \times D^q \to D^m$ is the composition $S^p \times D^q \subset D^{p+1} \times D^q \cong D^{p+q+1} \subset D^m$.

Hereafter $f \colon S^p \times D^q \to D^m$ will be assumed to be a proper almost embedding, unless stated otherwise.

Definition 7. A *proper almost embedding* $D^{p+q} \to D^m$ is a proper map whose restriction to the boundary is an embedding.

Definition 8. Fix the standard equatorial decomposition

$$\partial D^{q+1} = D^q_+ \bigcup_{\partial D^q_+ = \partial D^q_- = \partial D^q} D^q_-.$$

A *web* for a proper almost embedding $f\colon S^p \times D^q \to D^m$ is a map $\overline{f}\colon D^{q+1} \to D^m$ satisfying the following two conditions, see Fig. 8:

(i) $\overline{f}|_{D^q_+} = f|_{*\times D^q}$;

(ii) $\overline{f} D^q_- \subset \partial D^m$.

Definition 9. The definition of the (relative) *β-invariant* $\beta(f)$ of a proper almost embedding $f\colon S^p \times D^q \to D^m$ is completely analogous to the definition of the invariant $\beta(f)$ above, except that the map $f|_{*\times S^q}$ is replaced by $f|_{*\times D^q}$.

Given a proper map $g\colon D^q \to D^m$ missing the image $f(S^p \times D^q)$, define the β-invariant $\beta(f, g)$ analogously to $\beta(f)$, only replace the map by the map g.

An obvious consequence of the definitions is:

Proposition 2. *The map $g \mapsto \beta(f, g)$ induces a homeomorphism of groups $\pi_q(D^m - \operatorname{Im} f, \partial D^m - \operatorname{Im} f) \to \Omega^m_{p,q}$. Moreover, if a proper map $g\colon D^q \to D^m$ is sufficiently close to $f|_{*\times D^q}$, then $\beta(f, g) = \beta(f)$.*

§ 4. Completeness of the beta-invariant

In this section we prove the completeness of the relative β-invariant:

Theorem 4. *Assume that $m \geqslant p + \frac{4}{3}q + 2$. Then any proper almost embedding $f\colon S^p \times D^q \to D^m$ such that $\beta(f) = 0$ is properly almost concordant to a connected sum (relatively the boundary) of the standard embedding $S^p \times D^q \to D^m$ and a proper almost embedding $D^{p+q} \to D^m$.*

4.1. First let us state our central lemma which describes the homotopy groups of the complement of a proper almost embedding; cf. [30], Corollary 4.4.

Lemma 1 (Complement Lemma). *Assume that $m \geqslant p + \frac{4}{3}q + 2$. Then any proper almost embedding $f\colon S^p \times D^q \to D^m$ is properly almost concordant to a general position proper almost embedding $f'\colon S^p \times D^q \to D^m$ such that*

$$\pi_q(D^m - \operatorname{Im} f', \partial D^m - \operatorname{Im} f') \cong \Omega^m_{p,q}.$$

This isomorphism is given by the formula $g \mapsto \beta(f', g)$.

In particular, if $\beta(f) = 0$ then any proper map $g\colon D^q \to D^m - \operatorname{Im} f'$ sufficiently close to $f'|_{*\times D^q}$ is properly null-homotopic (outside $\operatorname{Im} f'$). This remark follows from Proposition 2 above and forms the basis of the following argument.

Proof of Theorem 4 (modulo Lemma 1). The proof consists of 3 steps.

1) *Construction of a web, whose interior misses* $\mathrm{Im}\, f$. Take a proper almost embedding $f\colon S^p \times D^q \to D^m$. Let $f'\colon S^p \times D^q \to D^m$ be a proper almost embedding (properly almost concordant to f) such that the isomorphism of Lemma 1 holds.

Without loss of generality assume that $f'(S^p \times D^q)$ is orthogonal to ∂D^m. Since the restriction of the normal bundle $N(f'(S^p \times D^q), D^m)$ to the disc $f'(* \times D^q)$ is trivial, there exists a unit vector field on $f'(* \times D^q)$ orthogonal to $f'(S^p \times D^q)$. Attach a collar neighbourhood to $f'(* \times D^q)$ along this vector field. One boundary component of this collar forms a proper map $g\colon D^q \to D^m - \mathrm{Im}\, f'$.

By Lemma 1 (and the paragraph after its statement) the map $g\colon D^q \to D^m - \mathrm{Im}\, f'$ is properly null-homotopic. Hence one can attach a disc (possibly with self-intersections, missing $\mathrm{Im}\, f'$ and with boundary in $\mathrm{Im}\, g \cup \partial D^m$) to the boundary component of the collar. The union of the disc and the collar is the image of the desired web $\overline{f}\colon D^{q+1} \to D^m$.

2) *Removing self-intersections of the web.* By [33], Theorem 2.1 there exists a piecewise smooth homeomorphism $h\colon D^m \to D^m$ such that $f'_{\mathrm{pl}} := hf'$ and $\overline{f}_{\mathrm{pl}} := h\overline{f}$ are piecewise linear. Denote by M^m the complement of a regular neighbourhood of $\mathrm{Im}\, f'_{\mathrm{pl}}$ in D^m. Then the pair $(M^m, \partial M^m \cap \partial D^m)$ is sufficiently highly connected (see Proposition 5 below). The following theorem allows one to remove the self-intersection of the web:

Theorem 5 (Embedding Theorem moving a part of the boundary). *Let*

$$(M^m, M^{m-1})$$

be a pair of piecewise linear manifolds such that $M^{m-1} \subset \partial M^m$. *Suppose this pair is* $(2q - m + 3)$*-connected and* $m \geqslant q + 4$. *Let*

$$\overline{f}_{\mathrm{pl}}\colon (D^{q+1}, D^q_-) \to (M^m, M^{m-1})$$

be a piecewise linear map which embeds D^q_+ *into* $\partial M^m - \mathrm{Int}\, M^{m-1}$. *Then* $\overline{f}_{\mathrm{pl}}$ *is homotopic* $\mathrm{rel}\, D^q_+$ *to a piecewise linear embedding*

$$\overline{f}_{\mathrm{emb}}\colon (D^{q+1}, D^q_-) \to (M^m, M^{m-1}).$$

This theorem is proved completely analogously to Theorem 9.2.1 in [25]. Since all the obstructions to smoothing the embedding $h^{-1}\overline{f}_{\mathrm{emb}}$ belong to trivial groups $H^k(D^{q+1}, D^q_-; C^{m-q}_{q-k})$ it follows by smoothing theory [33] that the piecewise linear embedding $h^{-1}\overline{f}_{\mathrm{emb}}$ is properly homotopic $\mathrm{rel}\, D^q_+$ to a web $\overline{f}'\colon D^{q+1} \to D^m$ which is a smooth embedding.

3) *Decomposition of* f' *into a connected sum.* Let B^m be the complement in D^m of the union of tubular neighbourhoods of $f'(S^p \times D^q - \mathrm{Int}\, B^{p+q})$ and $\overline{f}'D^{q+1}$. Then B^m is a smooth ball such that $(f')^{-1}B^m = B^{p+q}$. Denote the restriction $f'\colon B^{p+q} \to B^m$ by $g'\colon B^{p+q} \to B^m$. It is easy to see that f' is properly almost concordant to a connected sum (relative to the boundary) of $g'\colon B^{p+q} \to B^m$ and an embedding $S^p \times D^q \to D^m$. The latter is properly ambient concordant to the standard one (by easy Proposition 9, (a) below). Thus Theorem 4 modulo Lemma 1 follows.

The rest of § 4 is devoted to the proof of Complement Lemma 1. Our argument is parallel to [30], §§ 3, 4.

4.2. Hereafter we shall identify the disc D^m with the upper half-sphere of S^m and we shall fix the decomposition $S^m = D^m \cup CS^{m-1}$. Denote by $\{X, Y\}$ the set of stable homotopy classes of maps $X \to Y$.

Example 2. $\{(D^q; S^{q-1}), (D^m - \operatorname{Im} f, \partial)\} \cong \{S^q, S^m - f(S^p \times D^q)\}$ (because a pair (X, Y) is stably homotopy equivalent to the space $X \cup CY$).

The following proposition shows how to express the β-invariant via the homotopy class of a map $g\colon D^q \to D^m - \operatorname{Im} f$ in the group $\pi_q(D^m - \operatorname{Im} f, \partial)$. This can be considered as an alternative definition of the β-invariant $\beta(f, g)$.

Proposition 3 (cf. [30], Proposition 3.2). *Under the composition*

$$\pi_q(D^m - \operatorname{Im} f, \partial) \xrightarrow{\Sigma^\infty} \{S^q, S^m - \operatorname{Im} f\} \xrightarrow{\mathrm{SW}} \{\operatorname{Im} f, S^{m-q-1}\}$$
$$\xrightarrow{\widetilde{f}_*} \{T(\lambda_\Delta), S^{m-q-1}\} \xrightarrow{\mathrm{PT}} \Omega_s(P^\infty, l\lambda) \qquad (*)$$

the homotopy class of a proper map $g\colon D^q \to D^m - \operatorname{Im} f$ *is sent to* $\beta(f, g)$.

Here the first arrow is the iterated suspension map. The second map is the Spanier-Whitehead duality. The third arrow is induced by the map $\widetilde{f}$ defined in § 3. The fourth map is given by the Pontryagin-Thom construction (see [27] for details). This proposition is proved by a direct verification. In fact, it suffices to prove it for a map $f\colon B^{p+q} \to S^m$, which can be done analogously to [30], Proposition 3.2.

This assertion suggests that it is helpful to find the homotopy type of $\operatorname{Im} f$:

Proposition 4 (cf. [30], § 4). *Denote by* C *the mapping cone of* $f\colon \Sigma(f) \to f\Sigma(f)$. *Then* $\operatorname{Im} f \simeq C \vee S^p$.

The proof of this proposition immediately follows from the observation that both of these spaces can be obtained from $\mathrm{Cyl}(\Sigma(f) \to f\Sigma(f)) \bigcup_{\Sigma(f) \subset S^p \times D^q} (S^p \times D^q)$ by appropriate contractions (see Fig. 9).

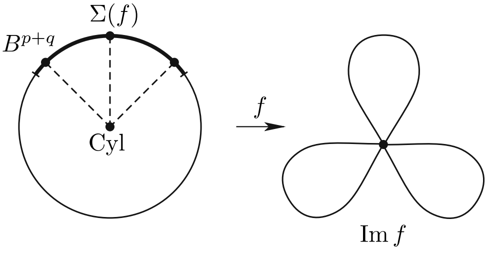


Figure 9

Let us begin the study of the homotopy type of the pair $(D^m - \operatorname{Im} f, \partial)$.

Proposition 5 (cf. [30], Lemma 4.2). *For a general position proper almost embedding* f *the pair* $(D^m - \operatorname{Im} f, \partial)$ *is* c*-connected, where* $c = \min\{m - p - 2, 2m - 2p - 2q - 3\}$.

Proof. In codimension at least 3 the pair $(D^m - \operatorname{Im} f, \partial)$ is generically 1-connected. By the homology excision theorem

$$H_i(D^m - \operatorname{Im} f, \partial) \cong H_i(S^m - \operatorname{Im} f).$$

By the Alexander duality $H_i(S^m - \operatorname{Im} f) \cong H^{m-i-1}(\operatorname{Im} f)$. By Proposition 4 we have $H^{m-i-1}(\operatorname{Im} f) = 0$ for $i \leqslant c$ because generically $\dim C = 2p + 2q - m + 1$. So by the Hurewicz theorem the pair $(D^m - \operatorname{Im} f, \partial)$ is c-connected.

Proof of Lemma 1. It suffices to show that for an appropriate choice of proper almost embedding f' all maps in Proposition 3 are bijective. We need to modify the initial proper almost embedding f only in step 4) below.

1) *The first map is bijective* by Proposition 5 and the Suspension Theorem, because the assumption $m \geqslant p + \frac{4}{3}q + 2$ implies $q/2 \leqslant c$.

2) *The second map is bijective* by the Spanier-Whitehead duality.

3) *The third map is bijective.* By Proposition 4 $\{\operatorname{Im} f, S^{m-q-1}\} \cong \{C, S^{m-q-1}\}$, hence it remains to check that $\{C, S^{m-q-1}\} \cong \{T(\lambda_\Delta), S^{m-q-1}\}$. Denote by λ_T the restriction of λ_Δ to the triple points, by C_T the image of $T(\lambda_T)$ under the map $T(\lambda_\Delta) \to C$, and by R the mapping cone of the map $T(\lambda_T) \to C_T$. Then R is a deformation retract of the mapping cone of the map $T(\lambda_\Delta) \to C$. Consider the Puppe exact sequence of the pair $(\operatorname{Cyl}(T(\lambda_\Delta) \to C), T(\lambda_\Delta))$:

$$\{R, S^{m-q-1}\} \to \{C, S^{m-q-1}\} \to \{T(\lambda_\Delta), S^{m-q-1}\} \to \{R, S^{m-q}\}.$$

Since R has dimension at most $3q + 3p - 2m + 2$, it follows by the assumption $m \geqslant p + \frac{4}{3}q + 2$ that $\{C, S^{m-q-1}\} \cong \{T(\lambda_\Delta), S^{m-q-1}\}$. Clearly, the obtained isomorphism $\{\operatorname{Im} f, S^{m-q-1}\} \cong \{T(\lambda_\Delta), S^{m-q-1}\}$ is induced by the map

$$\widetilde{f}\colon T(\lambda_\Delta) \to \operatorname{Im} f.$$

4) *Making the fourth map bijective.* One can see that the fourth map is bijective if the map $\Delta \to P^\infty$ classifying the bundle λ_Δ is $(s+1)$-connected. Indeed, by the Thom-Pontryagin construction $\{T(\lambda_\Delta), S^{m-q-1}\} \cong \Omega_s(\Delta, l\lambda_\Delta)$ and by homotopy lifting the induced map $\Omega_s(\Delta, l\lambda_\Delta) \to \Omega_s(P^\infty, l\lambda)$ is an isomorphism. (The definition of the group $\Omega_s(\Delta, l\lambda_\Delta)$ and the details can be found in [30], § 3 and [27].)

So it remains to make the classifying map $\Delta \to P^\infty$ $(s+1)$-connected by a proper almost concordance of f. This is made by the following theorem (applied here for $s = 2p + 3q - 2m + 2$ and $n = p + q$).

Theorem 6 (Surgery Theorem; cf. [30], Theorem 4.5). *Let M^m be an $(s+1)$-connected manifold and let $f\colon B^n \to M^m$ be a proper immersion such that the restriction $f|_{\partial B^n}$ is an embedding. Assume that $2s \leqslant 2n - m - 2$ and $0 \leqslant s \leqslant m - n - 3$. Then by a regular homotopy* rel ∂B^n *of the immersion $f\colon B^n \to M^m$ the classifying map $\Delta \to P^\infty$ of the covering $\widetilde{\Delta} \to \Delta$ can be made $(s+1)$-connected.*

The proof of Theorem 6 is completely analogous to the proof of Theorem 4.5 in [30]. We present it in § 8.

Thus we have proved, modulo Theorem 6, the completeness of the β-invariant (Theorem 4).

§ 5. The exact sequence

In this section we deduce Theorem 3 from the completeness of the β-invariant. Formally, Theorem 3 follows from Theorems 4 and 7, Lemma 2 and Proposition 7.

Theorem 7. *Assume that $m \geqslant p + \frac{4}{3}q + 2$. Then for each $x \in \Omega^m_{p,q}$ there exists a proper almost embedding $\omega_x \colon S^p \times D^q \to D^m$ such that $\beta(\omega_x) = x$.*

Proof. The construction of ω_x proceeds in 3 steps:

1) *Construction of a map $f' \sqcup g \colon S^p \times D^q \sqcup D^q \to D^m$ such that $\beta(f', g) = x$.* Start with the standard embedding $f \colon S^p \times D^q \to D^m$. By Complement Lemma 1 the map f is properly almost concordant to a proper almost embedding $f' \colon S^p \times D^q \to D^m$, such that $\beta(f', \cdot) \colon \pi_q(D^m - \operatorname{Im} f', \partial) \to \Omega^m_{p,q}$ is an isomorphism. Take a proper map $g \colon D^q \to D^m - \operatorname{Im} f'$ such that $\beta(f', g) = x$.

2) *Homotopical extension of $g \colon D^q \to D^m - \operatorname{Im} f'$ to a proper embedding $g' \colon S^p \times D^q \to D^m - \operatorname{Im} f'$.* Since $(D^m - \operatorname{Im} f', \partial)$ is sufficiently highly connected (see Proposition 5), it follows by the embedding theorem ([25], Theorem 8.2.1) that one can remove the self-intersection of $g \colon D^q \to D^m - \operatorname{Im} f'$. By Hirsch theory one can make $g \colon D^q \to D^m - \operatorname{Im} f'$ a smooth embedding with a trivial normal bundle. So one can extend $g \colon D^q \to D^m - \operatorname{Im} f'$ to a proper embedding $g' \colon S^p \times D^q \to D^m - \operatorname{Im} f'$ such that the image $\operatorname{Im} g'$ is contained in a tubular neighbourhood of $\operatorname{Im} g$.

3) *Making the S^p-parametric connected sum of f' and g'.* Fix a point $* \in \partial D^q$. By construction it follows that $f'|_\partial$ is concordant to the standard embedding.

Thus the sphere $f'(S^p \times *)$ can be spanned by a framed disc $D^{p+1} \subset \partial D^m$ (*web*) such that $\operatorname{Int} D^{p+1} \cap \operatorname{Im} f' = \varnothing$ and the first q vector fields of the framing of ∂D^{p+1} are tangent to $f'(S^p \times S^q)$. Generically the web does not intersect $\operatorname{Im} g$, and hence it does not intersect $\operatorname{Im} g'$ either. Clearly, the sphere $g'(S^p \times *)$ can also be spanned by a web $\overline{D}^{p+1} \subset \partial D^m$ whose interior does not intersect $\operatorname{Im}(f' \sqcup g')$ and D^{p+1}.

Join the centres of the webs D^{p+1} and $\overline{D}^{p+1}$ by an arc I in ∂D^m. Generically I intersects $\operatorname{Im}(f' \sqcup g') \cup D^{p+1} \cup \overline{D}^{p+1}$ only at the boundary ∂I. Let $\overline{D}^m$ be the union of small tubular neighbourhoods of D^{p+1}, I and $\overline{D}^{p+1}$ in the ball D^m. Clearly, the intersection $\operatorname{Im}(f' \sqcup g') \cap \overline{D}^m$ is standard. By performing the S^p-parametric connected sum of $f' \colon S^p \times D^q \to D^m$ and $g' \colon S^p \times D^q \to D^m$ in $\overline{D}^m$, we obtain the required proper almost embedding $\omega_x \colon S^p \times D^q \to D^m$. The proof is complete.

Let $\overline{E}^m(S^p \times D^q, S^p \times S^{q-1})$ denote the group of proper almost embeddings $S^p \times D^q \to D^m$ up to proper almost concordance and connected sums with proper almost embeddings $D^{p+q} \to D^m$ (with the parametric connected sum group structure).

Remark 1. The relative β-invariant is a map

$$\overline{E}^m(S^p \times D^q, S^p \times S^{q-1}) \to \Omega^m_{p,q}.$$

Theorems 4 and 7 together assert that this map is bijective for $m \geqslant p + \frac{4}{3}q + 2$.

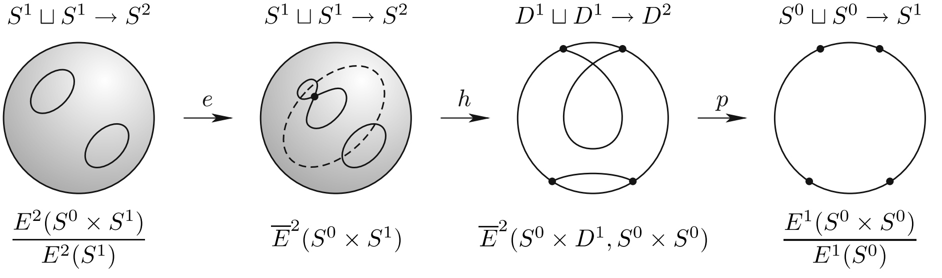


Figure 10

Lemma 2 (cf. [34] and [10], Theorem 1). *For every $m > 2p + q + 2$ there is an exact sequence (see Fig. 10)*

$$\frac{E^m(S^p \times S^q)}{E^m(S^{p+q})} \xrightarrow{e} \overline{E}^m(S^p \times S^q) \xrightarrow{h} \overline{E}^m(S^p \times D^q, S^p \times S^{q-1})$$
$$\xrightarrow{p} \frac{E^{m-1}(S^p \times S^{q-1})}{E^{m-1}(S^{p+q-1})} \to \cdots .$$

In the rest of § 5 we shall prove Lemma 2. We begin with the construction of the commutative group structure on the set of knotted tori (and also almost embeddings $S^p \times S^q \to S^m$). This construction is equivalent to the one from [21], § 2.

Definition 10. Fix a point $* \in S^q$. Without loss of generality assume that $* \times S^q \cap B^{p+q} = \varnothing$. A *web* of an almost embedding $f\colon S^p \times S^q \to S^m$ is a framed disc $D^{p+1} \subset S^m$ satisfying the following 3 conditions:

(i) $\partial D^{p+1} = f(S^p \times *)$;

(ii) $\operatorname{Int} D^{p+1} \cap \operatorname{Im} f = \varnothing$;

(iii) the first q vector fields of the framing of ∂D^{p+1} coincide with the obvious framing of the 'meridian' $f(S^p \times *)$ in the 'torus' $f(S^p \times S^q)$.

Webs of an almost concordance $f\colon S^p \times S^q \times I \to S^m \times I$ and of a proper almost embedding $f\colon S^p \times D^q \to D^m$ are defined analogously (the marked point $* \in D^q$ is the centre of the disc D^q).

The following assertion is equivalent to the Standardization Lemma in [21].

Proposition 6 ([21], Lemma 2.1). *If $m > 2p+q+1$, then for any almost embedding $f\colon S^p \times S^q \to S^m$ there exists a web. If $m > 2p + q + 2$, then for any almost concordance between the almost embeddings $f_1, f_2\colon S^p \times S^q \to S^m$ there exists a web extending given webs of f_1 and f_2.*

Proof. The bundle $N(f|_{S^p \times *}; S^m)$ is stably trivial and $m - p - q \geqslant p$, hence this bundle is trivial. Take a $(m - p - q)$-framing ξ of this bundle.

Take the section formed by the first vectors of ξ. Since $m \geqslant 2p + q + 2 \geqslant 2p + 2$, it follows that the embedding $f|_{S^p \times *}$ is unknotted in S^m. So there is an embedding $\overline{f}\colon D^{p+1} \subset S^m$ satisfying property (i) from the definition of the web above. Since $m \geqslant 2p + q + 2$, we may also assume property (ii) by general position.

By deleting the first vector from ξ we obtain an $(m - p - q - 1)$-framing ξ_1 of the boundary $\overline{f}(\partial D^{p+1})$ orthogonal to the disc $\overline{f}(D^{p+1})$. Denote by η the standard normal q-framing of $f(S^p \times *)$ in $f(S^p \times S^q)$. Then (ξ_1, η) is a normal $(m-p-1)$-framing on $\overline{f}(\partial D^{p+1})$ orthogonal to $\overline{f}(D^{p+1})$. Since $p < m-p-q-1$, the map $\pi_p(\mathrm{SO}_{m-p-q-1}) \to \pi_p(\mathrm{SO}_{m-p-1})$ is epimorphic. Hence we can change ξ_1 (and thus ξ) so that (ξ_1, η) extends to a normal framing on $\overline{f}(D^{p+1})$. Clearly, the obtained framing satisfies property (iii).

The second assertion of the proposition is proved analogously (see [21], the proof of Lemma 2.1).

The following definition is equivalent to one in [21].

Definition 11. Let $f_1, f_2 \colon S^p \times S^q \to S^m$ be a pair of almost embeddings. Without loss of generality we may assume that $\mathrm{Im}\, f_1 \subset D^m_+$, $\mathrm{Im}\, f_2 \subset D^m_-$. Take webs $D^{p+1} \subset D^m_+$ and $\overline{D}^{p+1} \subset D^m_-$ of these almost embeddings. Join the centres of the webs D^{p+1} and $\overline{D}^{p+1}$ by an arc I in S^m. Generically I intersects $\mathrm{Im}\, f_1 \cup \mathrm{Im}\, f_2 \cup D^{p+1} \cup \overline{D}^{p+1}$ only at the boundary ∂I. Let $\overline{D}^m$ be the union of small tubular neighbourhoods of D^{p+1}, I and $\overline{D}^{p+1}$ in S^m. Clearly, the intersection $(\mathrm{Im}\, f_1 \cup \mathrm{Im}\, f_2) \cap \overline{D}^m$ is standard.

By the *parametric connected sum* of f_1 and f_2 we mean the almost embedding $f_1 + f_2 \colon S^p \times S^q \to S^m$ obtained by the S^p-parametric connected sum of f_1 and f_2 inside $\overline{D}^m$.

Definition 12. Let $f \colon S^p \times S^q \to S^m$ be an almost embedding. By the *inverse* of f we mean the almost embedding given by the formula $(-f)(x, y) = \sigma_m f(x, \sigma_q y)$, where σ_k is the reflection of S^k across the hyperplane $x_1 = 0$.

Definition 13. By the *neutral element* we mean the standard embedding

$$S^q \times S^p \to D^{q+1} \times D^{p+1} \cong D^{p+q+2} \subset D^m \subset S^m.$$

The following important result is checked directly (see the details in [21]).

Theorem 8 (Group Structure Theorem). *A commutative group structure on the set of almost embeddings $S^p \times S^q \to S^m$ up to almost concordance is well-defined for $m > 2p + q + 2$ by the above construction.*

The analogues of this theorem for proper almost embeddings and smooth embeddings are also true and can be proved analogously.

Proposition 7. *For $m > 2p + q + 2$ the relative β-invariant*

$$\beta \colon \overline{E}^m(S^p \times D^q, S^p \times S^{q-1}) \to \Omega^m_{p,q}$$

is a homomorphism.

Proof. Consider a triple of proper almost embeddings f_1, f_2, and $f_3 = f_1 + f_2$, where '+' denotes the S^p-parametric connected sum relative to the boundary. One can see that up to homotopy $\widetilde{f}_3 = \widetilde{f}_1 \vee \widetilde{f}_2$ and $\overline{f}_3 = \overline{f}_1 \natural \overline{f}_2$, where '$\natural$' denotes the connected sum relative to the boundary. So $\beta(f_3) = \beta(f_1) \sqcup \beta(f_2)$, hence $\beta(f_1 + f_2) = \beta(f_1) + \beta(f_2)$.

Now we are going to prove that the action of embeddings $S^{p+q} \to S^m$ on the set of embeddings $S^p \times S^q \to S^m$ is injective.

Notations:

$E^m(S^{p+q})$ is the group of all embeddings $S^{p+q} \to S^m$ up to concordance.

$E^m(S^p \times S^q)$ is the group of all embeddings $S^p \times S^q \to S^m$ up to concordance (with the parametric connected sum group structure).

$\kappa^*\colon E^m(S^{p+q}) \to E^m(S^p \times S^q)$ is a map taking an embedding $g\colon S^{p+q} \to S^m$ to the connected sum of g and the standard embedding $S^p \times S^q \to S^m$. (The connected sum is made along a path I joining the images of the embeddings; these images are assumed to be separated by a hyperplane).

Proposition 8. *The map $\kappa^*\colon E^m(S^{p+q}) \to E^m(S^p \times S^q)$ is injective for $m > 2p + q + 2$.*

In fact, this proposition immediately implies the case of '$p+q+1$ divisible by 4' in Theorem 2 by Theorem 1.

Proof of Proposition 8. It suffices to construct a left inverse $\overline{\kappa}^*\colon E^m(S^p \times S^q) \to E^m(S^{p+q})$ of the map κ^*.

The map $\overline{\kappa}^*\colon E^m(S^p \times S^q) \to E^m(S^{p+q})$ is defined as follows. Take an embedding $f\colon S^p \times S^q \to S^m$. By Proposition 6 there exists a web D^{p+1} of this embedding. Perform embedded surgery on $S^p \times S^q$ along the framed disc D^{p+1}. Let $\overline{\kappa}^*(f)$ be the isotopy class of the embedding $S^{p+q} \to S^m$ obtained by the surgery.

The element $\overline{\kappa}^*(f)$ is well defined by the second assertion of Proposition 6. We have $\overline{\kappa}^*\kappa^* = \mathrm{id}$ because $\overline{\kappa}^*\kappa^*(f) = \overline{\kappa}^*(f \natural s) = f \natural \overline{\kappa}^*(s) = f \natural 0 = f$ for any $f \in E^m(S^{p+q})$.

Proposition 9. (a) *For each $m > 2p+q+2$ all proper embeddings $S^p \times D^q \to D^m$ are properly concordant.*

(b) ([21], the triviality criterion.) *For $m > 2p+q+2$ an embedding $S^p \times S^{q-1} \to S^{m-1}$ is concordant to the standard embedding if and only if it extends to a proper embedding $S^p \times D^q \to D^m$.*

Proof. (a) Take a proper embedding $f\colon S^p \times D^q \to D^m$. By an analogue of Proposition 6 there exists a web D^{p+1} of f. Let $\overline{D}^m$ be the tubular neighbourhood of D^{p+1}. Clearly, the restriction $f\colon f^{-1}\overline{D}^m \to \overline{D}^m$ is concordant to the standard embedding $S^p \times D^q \to D^m$. It remains to prove that f is concordant to this restriction. Let $c\colon S^p \times D^q \times I \to D^m \times I$ be the identical concordance (equal to the embedding $f\colon S^p \times D^q \to D^m$ for each $t \in I$). Let $h_m\colon D^m \times I \to D^m \times I$ be a diffeomorphism fixed on $D^m \times 0$ and taking $\overline{D}^m \times 1$ to $D^m \times 1$. Let $h_{p+q}\colon S^p \times D^q \times I \to S^p \times D^q \times I$ be a diffeomorphism fixed on $S^p \times S^q \times 0$ and taking $S^p \times D^q \times 1$ to $f^{-1}\overline{D}^m$. Then the composition $h_m c h_{p+q}$ is a proper concordance between $f\colon S^p \times D^q \to D^m$ and the restriction $f\colon f^{-1}\overline{D}^m \to \overline{D}^m$.

(b) follows directly from (a).

The analogue of this proposition for proper almost embeddings is also true, except that the ball B^{p+q} in the definition of a proper almost embedding should be replaced by a ball $\overline{B}^{p+q} \subset S^p \times D^q$ meeting the boundary at a common face. A *face* of the ball $\overline{B}^{p+q}$ is a ball contained in $\partial \overline{B}^{p+q}$.

Proof of Lemma 2 (cf. [10], the proof of Theorem 3.1).

1) *Construction of the homomorphisms.* Let e be the obvious map. Let p be the 'restriction-to-the-boundary' map. The homomorphism h is the 'cutting map' defined as follows. Take an almost embedding $f\colon S^p \times S^q \to S^m$. By Proposition 6

there exists a web $D^{p+1} \subset S^m$. Let $\overline{D}^m$ be a tubular neighbourhood of D^{p+1}. Set $h(f)$ to be the restriction $f\colon (S^p \times S^q - f^{-1} \operatorname{Int} \overline{D}^m) \to S^m - \operatorname{Int} \overline{D}^m$.

2) *Exactness at* $E^m(S^p \times S^q)/E^m(S^{p+q})$. The seqence is exact at $E^m(S^p \times S^q)/E^m(S^{p+q})$ because an embedding $f\colon S^p \times S^q \to S^m$ extends to a proper almost embedding $S^p \times D^{q+1} \to D^{m+1}$ if and only if f is almost concordant to the standard embedding (by the analogue of Proposition 9, (b)).

3) *Exactness at* $\overline{E}^m(S^p \times D^q, S^p \times S^{q-1})$. The inclusion $\operatorname{Im} h \subset \ker p$ follows by Proposition 9, (b). To prove that $\ker p \subset \operatorname{Im} h$ take a proper almost embedding $f\colon S^p \times D^q \to D^m$ such that $p(f) = 0$. Let $f|_\partial\colon S^p \times \partial D^q \to \partial D^m$ be the restriction of f to the boundary. By definition, there exists a smooth embedding $g\colon S^{p+q-1} \to S^{m-1}$ such that the connected sum $f|_\partial + g$ is concordant to the standard embedding. Extend $g\colon S^{p+q-1} \to S^{m-1}$ to a proper almost embedding $g'\colon D^{p+q} \to D^m$. Let $f + g'$ be the connected sum of f and g' relative to the boundary. By Proposition 9, (b) the map $f + g'\colon S^p \times D^q \to D^m$ extends to an almost embedding $f'\colon S^p \times S^q \to S^m$. Thus $f = h(f')$.

4) *Exactness at* $\overline{E}^m(S^p \times S^q)$. The inclusion $\operatorname{Im} e \subset \ker h$ follows by Proposition 9, (a). To prove that $\ker h \subset \operatorname{Im} e$ take an almost embedding $f\colon S^p \times S^q \to S^m$ such that $h(f) = 0$. Then, by definition, there exist a proper almost concordance c between $h(f)$ and a connected sum of the standard embedding $S^p \times D^q \to D^m$ and a proper almost embedding $g\colon D^{p+q} \to D^m$. Since the restriction of c to the boundary is a smooth concordance, by Proposition 8 it follows that the restriction of g to the boundary is unknotted. Thus we may assume that g is a smooth embedding. By the concordance extension theorem the restriction of c to the boundary extends to an ambient concordance of the disc $S^m - D^m$. So c can be extended to an almost concordance of f without adding new self-intersections. The latter is an almost concordance between f and an embedding $f'\colon S^p \times S^q \to S^m$. Hence $f = e(f')$, as required.

So the proof of Theorem 3 is complete.

§ 6. The finiteness criterion

In this section we deduce Theorem 2 from Theorem 3. Thus we need a classification of almost embeddings $S^p \times S^q \to S^m$. We summarize this classification and the above results in the following theorem.

Let $E^m(D^p \times S^q)$ denote the group of smooth embeddings $D^p \times S^q \to S^m$ up to concordance (with the D^p-parametric connected sum group structure).

Theorem 9. *For $m > p + \frac{4}{3}q + 2$ and $m > 2p + q + 2$ there exist the following exact sequences*:

1) $$0 \longrightarrow E^m(S^{p+q}) \longrightarrow E^m(S^p \times S^q) \longrightarrow \frac{E^m(S^p \times S^q)}{E^m(S^{p+q})};$$

2) $$\overline{E}^{m+1}(S^p \times S^{q+1}) \longrightarrow \Omega^{m+1}_{p,q+1} \longrightarrow \frac{E^m(S^p \times S^q)}{E^m(S^{p+q})} \longrightarrow \overline{E}^m(S^p \times S^q);$$

3) $$E^{m+1}(D^p \times S^{q+1}) \longrightarrow \pi_{p+q}(S^{m-q-1}) \longrightarrow \overline{E}^m(S^p \times S^q) \longrightarrow E^m(D^p \times S^q);$$

4) $$E^{m+1}(S^{q+1}) \longrightarrow \pi_q(V_{m-q,p}) \longrightarrow E^m(D^p \times S^q) \longrightarrow E^m(S^q).$$

The second terms in sequences 1)–4) in Theorem 9 are well known rationally.

Theorem 10. *Assume that* $p+\frac{4}{3}q+2\leqslant m<p+\frac{3}{2}q+2$, $m>2p+q+2$ *and* $m>n+2$. *Then*

1) $E^m(S^n)$ *is infinite if and only if* $m\leqslant\frac{3}{2}n+\frac{3}{2}$ *and* $n+1$ *is divisible by* 4;
2) $\Omega^{m+1}_{p,q+1}$ *is infinite if and only if* $m=p+\frac{3}{2}q+\frac{3}{2}$ *and* $q+1$ *is divisible by* 4;
3) $\pi_{p+q}(S^{m-q-1})$ *is infinite if and only if* $m=\frac{1}{2}p+\frac{3}{2}q+\frac{3}{2}$ *and* $p+q+1$ *is divisible by* 4;
4) $\pi_q(V_{m-q,p})$ *is infinite if and only if* $p\geqslant 1$, $\frac{3}{2}q+\frac{3}{2}\leqslant m\leqslant p+\frac{3}{2}q+\frac{1}{2}$ *and* $q+1$ *is divisible by* 4.

We argue by the following plan. First we prove Theorem 2 modulo Theorems 9 and 10. Then we prove Theorems 9 and 10 themselves, using some known results.

Proof of Theorem 2 (modulo Theorems 9 and 10). 1) *Case when* $q+1$ *and* $p+q+1$ *are not divisible by* 4. Recall that if $X\to Y\to Z$ is an exact sequence with finite X and Z, then Y is also finite. Applying this 4 times to the last 3 columns in Theorem 9 starting from the bottom, we are done, because by Theorem 10 the groups in the second column of Theorem 9 are finite when $q+1$ and $p+q+1$ are not divisible by 4.

2) *Case when* $p+q+1$ *is divisible by* 4. By Theorems 9, 1) and 10, 1) it follows directly that the group $E^m(S^p\times S^q)$ is in this case infinite.

3) *Case when* $q+1$ *is divisible by* 4, $m\leqslant\frac{3}{2}q+\frac{3}{2}$. By Theorem 10, 1), in this case the group $E^m(S^q)$ is infinite. Take an infinite order element x. The obstruction to existence of a $(p+1)$-framing on the embedding $x\colon S^q\to S^m$ belongs to the group $\pi_{q-1}(V_{m-q,p+1})$. By Theorem 10, 4) this group in our case is finite. So for some positive integer N the embedding Nx extends to a smooth embedding $HS^p\times S^q\to S^m$. Since the restriction of the embedding to the sphere $*\times S^q$ has infinite order it follows that H itself has infinite order.

4) *Case when* $q+1$ *is divisible by* 4, $\frac{3}{2}q+\frac{3}{2}<m\leqslant p+\frac{3}{2}q+\frac{1}{2}$, $p\geqslant 1$. It suffices to construct an embedding $T\colon S^p\times S^q\to S^m$ having infinite order in $E^m(S^p\times S^q)$. *Construction of the embedding* T. By Theorem 10, 4) the group $\pi_q(V_{m-q,p})$ is in this case infinite. Take an infinite order element x of this group. Consider the map $\tau\colon\pi_q(V_{m-q,p})\to E^m(D^p\times S^q)$ from Theorem 9, 4). This map takes the element x to the canonical p-frame $D^p\times S^q\to S^m$ of the standard sphere $S^q\subset S^m$. The complete obstruction to extension of this p-frame to a $(p+1)$-frame belongs to $\pi_{q-1}(S^{m-p-q-1})$. The latter group is in our case finite. So for some positive integer N the element $N\tau(x)$ can be extended to a smooth embedding $S^p\times S^q\to S^m$, which is the desired torus T.
Proof that T *has infinite order.* It suffices to prove that the element $\tau(x)\in E^m(D^p\times S^q)$, which is the restriction of T to $D^p\times S^q$, has infinite order. Suppose the contrary. Then $N\tau(x)=0$ for some positive integer N. So by Theorem 9, 4) Nx belongs to the image of the map $E^{m+1}(S^{q+1})\to\pi_q(V_{m-q,p})$. But the group $E^{m+1}(S^{q+1})$ is in our case finite. Thus x has finite order in contrast to our choice above. This contradiction proves that T has infinite order.

5) *Case when* $q+1$ *is divisible by* 4, $m=p+\frac{3}{2}q+\frac{3}{2}$.
Construction of the embedding W. By Theorem 10, 2) the group $\Omega^{m+1}_{p,q+1}$ is in this case infinite. Take an infinite order element x of this group. Let $W\colon S^p\times S^q\to S^m$ be an embedding realizing the image of x under the map $\Omega^{m+1}_{p,q+1}\to E^m(S^p\times S^q)/E^m(S^{p+q})$ from Theorem 9, 2).

Proof that W has infinite order. Consider the exact sequence 2 in Theorem 9. It suffices to prove that $\overline{E}^{m+1}(S^p \times S^{q+1})$ is in our case finite. Since $q+1$ is divisible 4, it follows by Theorem 10 that $\pi_{q+1}(V_{m-q,p})$ and $E^{m+1}(S^{q+1})$ are finite. An easy computation shows that $\pi_{p+q+1}(S^{m-q-1})$ is also finite in our case. So by Theorem 9, 3), 4) it follows that $\overline{E}^{m+1}(S^p \times S^{q+1})$ is finite.

In the rest of the section we shall prove Theorems 9 and 10.

Assertions 1) and 2) of Theorem 9 are reformulations of Proposition 8 and Theorem 3 which were proved in § 5. Theorem 9, 4) is proved by a direct verification analogously to [4], Corollary 5.9. We sketch the proof below for the convenience of the reader. Theorem 9, 3) is proved in [21], Restriction Lemma 5.2 for $p \geqslant 1$, $m > 2p+q+2$ and $m \geqslant \frac{1}{2}p + \frac{3}{2}q + 2$. The additional restriction $m \geqslant \frac{1}{2}p + \frac{3}{2}q + 2$ in the proof of this assertion in [21] can be dropped. We sketch an alternative proof to keep the paper self-contained.

Sketch of the proof of assertion 3) *in Theorem* 9. (a) *Definition of the groups*

$$\widetilde{E}^m(S^p \times S^q) \quad \text{and} \quad \widetilde{E}^m(S^p \times D^q, S^p \times S^{q-1}).$$

A map $f\colon S^p \times S^q \to S^m$ is said to be a *weak almost embedding* if it is an embedding outside the fixed ball $B^{p+q} \subset S^p \times S^q$ (the intersection $fB^{p+q} \cap f(S^p \times S^q - B^{p+q})$ is allowed to be nonempty). A *weak almost concordance* is defined analogously. Denote by $\widetilde{E}^m(S^p \times S^q)$ the group of weak almost embeddings up to weak almost concordance. Identify the groups $\widetilde{E}^m(S^p \times S^q)$ and $E^m(D^p \times S^q)$. Clearly, these groups are isomorphic for $m > 2p+q+2$.

Fix a ball $\overline{B}^{p+q} \subset (S^p - *) \times D^q$ meeting the boundary in its face.

A proper map $f\colon S^p \times D^q \to D^m$ is said to be a *proper weak almost embedding* if the following two conditions hold:

(i) f is an embedding outside $\overline{B}^{p+q}$;

(ii) $f(S^p \times \partial D^q \cap \overline{B}^{p+q}) \cap f(S^p \times \partial D^q - \overline{B}^{p+q}) = \varnothing$.

A proper weak almost concordance is defined analogously. Denote by

$$\widetilde{E}^m(S^p \times D^q, S^p \times S^{q-1})$$

the group of proper weak almost embeddings up to proper weak almost concordance.

(b) *Existence of an exact sequence.* For every $m > 2p+q+2$ there exists an exact sequence

$$\overline{E}^m(S^p \times S^q) \xrightarrow{e} \widetilde{E}^m(S^p \times S^q) \xrightarrow{h} \widetilde{E}^m(S^p \times D^q, S^p \times S^{q-1}) \xrightarrow{p} \overline{E}^{m-1}(S^p \times S^{q-1}) \to \cdots .$$

Here e, h and p are the obvious forgetful, cutting and restriction homomorphisms, respectively. This assertion is proved completely analogously to Lemma 2.

(c) *Definition of the homomorphism*

$$\lambda\colon \widetilde{E}^m(S^p \times D^q, S^p \times S^{q-1}) \to \pi_{p+q-1}(S^{m-q-1}).$$

Take a proper weak almost embedding $f\colon S^p \times D^q \to D^m$. By definition $f\partial\overline{B}^{p+q} \cap f(* \times D^q) = \varnothing$. Notice that $D^m - f(* \times D^q) \simeq S^{m-q-1}$. Let $\lambda(f)$ be the homotopy class of the restriction $f\colon \partial\overline{B}^{p+q} \to D^m - f(* \times D^q)$.

(d) λ *is injective.* Take a proper weak almost embedding $f\colon S^p\times D^q\to D^m$ such that $\lambda(f)=0$. Then $f|_{\partial\overline{B}^{p+q}}$ extends to a map $g\colon \overline{B}^{p+q}\to D^m$ missing $f(*\times D^q)$. Since $f|_{S^p\times D^q-\overline{B}^{p+q}}$ is an embedding, it follows that the intersection of $g\overline{B}^{p+q}$ with $f(S^p\times D^q-\overline{B}^{p+q})$ can be removed by a homotopy relative to the boundary. Thus we may assume that g misses $f(S^p\times D^q-\overline{B}^{p+q})$. Perform a proper weak almost concordance which replaces $f|_{\overline{B}^{p+q}}$ by g. By an analogue of Proposition 9, (a) the obtained map is properly weakly almost concordant to the standard embedding $S^p\times D^q\to D^m$.

(e) λ *is surjective.* Take an element $x\in\pi_{p+q-1}(S^{m-q-1})$. Take the standard embedding $f\colon S^p\times D^q\to D^m$. Realize the element x by a map $g\colon S^{p+q-1}\to\partial D^m-f(*\times D^q)$. Since $f|_{S^p\times D^q-\overline{B}^{p+q}}$ is an embedding it follows that the intersection of $g\overline{B}^{p+q}$ with $f(S^p\times D^q-\overline{B}^{p+q})$ can be removed by a homotopy relative to the boundary. Thus we may assume that g misses $f(S^p\times D^q-\overline{B}^{p+q})$. Extend the map $g\colon S^{p+q-1}\to\partial D^m$ to a proper map $g'\colon D^{p+q}\to D^m$. Let μ_x be the connected sum (relative to the boundary) of g' and f. Clearly, $\lambda(\mu_x)=x$. This completes the proof of assertion 3).

Sketch of the proof of assertion 4) *in Theorem* 9. (a) *Definition of homomorphisms.* The map $i^*\colon E^m(D^p\times S^q)\to E^m(S^q)$ is restriction-induced. Here $0\times S^q$ is identified with S^q in the obvious way.

The map $\mathrm{Ob}\colon E^m(S^q)\to\pi_{q-1}(V_{m-q,p})$ is the complete obstruction to the existence of a p-framing on an embedding $S^q\to S^m$. This obstruction is defined as follows. Take an embedding $f\colon S^q\to S^m$. Take a (unique up to homotopy) $(m-q)$-framing of the disc fD^q_+. Take a (unique up to homotopy) p-framing of the disc fD^q_-. Thus the sphere fS^{q-1} is equipped both with the p-framing and the $(m-q)$-framing. Using the $(m-q)$-framing identify each fibre of the normal bundle to fD^q_+ with the space $\mathbb{R}^{m-q}$. To each point $x\in S^{q-1}$ assign the p-framing at the point fx. This leads to a map $S^{q-1}\to V_{m-q,p}$. By definition $\mathrm{Ob}(f)\in\pi_{q-1}(V_{m-q,p})$ is the homotopy class of this map.

The map $\tau\colon\pi_q(V_{m-q,p})\to E^m(D^p\times S^q)$ is defined as follows. Represent $f\in\pi_q(V_{m-q,p})$ as a smooth map $f\colon D^p\times S^q\to D^{m-q}$ linear in each fibre $D^p\times *$. Define $\tau(f)$ to be the composition $D^p\times S^q\to D^{m-q}\times S^q\to S^m$ of the embedding $f\times\mathrm{pr}_2$ and the standard embedding s.

(b) *Exactness at* $E^m(D^p\times S^q)$ *and* $E^m(S^q)$ is checked directly.

(c) *Proof of the exactness at* $\pi_q(V_{m-q,p})$. Let $f\colon S^{q+1}\to S^{m+1}$ be an embedding. Then f is isotopic to a standardized embedding $f'\colon S^{q+1}\to S^m$, that is, an embedding satisfying the conditions

- $f'\colon D^{q+1}_-\to D^{m+1}_-$ is the restriction of the standard inclusion $S^{q+1}\to S^{m+1}$;
- $f'(D^{q+1})_+\subset D^{m+1}_+$.

Take a p-framing of $f'(D^{q+1})$. Clearly, the embedding $\tau\mathrm{Ob}f'\colon D^p\times S^q\to S^m$ extends to the embedding $D^p\times D^{q+1}\to D^{m+1}$ determined by the p-framing. So $\tau\mathrm{Ob}f'$ is isotopic to the standard embedding $D^p\times S^q\to S^m$. Thus $\mathrm{Im}\,\tau\subset\ker\mathrm{Ob}$. Analogously $\mathrm{Im}\,\tau\supset\ker\mathrm{Ob}$.

Theorem 10 can easily be reduced to known results. Assertion 1) is proved in [4], Corollary 6.7. Assertion 3) follows from the Serre theorem.

Sketch of the proof of assertion 4) *in Theorem* 10. The assumptions $m > 2p+q+2$ and $m < p+\frac{3}{2}q+2$ together imply that $m \leqslant 2q$. We are going to prove assertion 4) with these assumptions replaced by the only assumption $m \leqslant 2q$. We use induction on p.

(a) *Case of* $q+1$ *not divisible by* 4. Since $m \leqslant 2q$, it follows that $\pi_q(V_{m-q,1}) \cong \pi_q(S^{m-q-1})$ is finite. Using the homotopy exact sequence of the 'restriction' bundle $S^{m-p-q} \to V_{m-q,p} \to V_{m-q,p-1}$ tensored by $\mathbb{Q}$, we get inductively that $\pi_q(V_{m-q,p})$ is finite.

(b) *Case of* $q+1$ *divisible by* 4 *and either* $m < \frac{3}{2}q + \frac{3}{2}$ *or* $m > p + \frac{3}{2}q + \frac{1}{2}$. In this case the groups $\pi_q(S^{m-q-i})$ are still finite for each $i = 1, 2, \dots, p$. Similarly as above we get that $\pi_q(V_{m-q,p})$ is finite.

(c) *Case of* $q+1$ *divisible by* 4 *and* $\frac{3}{2}q + \frac{3}{2} \leqslant m \leqslant p + \frac{3}{2}q + \frac{1}{2}$. Take i such that $m = i + \frac{3}{2}q + \frac{1}{2}$. Consider the exact homotopy sequence above for $p = i$. Analogously as above it can be shown that for $q+1$ divisible by 4 and $m \leqslant 2q$ the group $\pi_{q+1}(V_{m-q,i-1})$ is finite. Thus the group $\pi_q(V_{m-q,i})$ is infinite. By induction $\pi_q(V_{m-q,p})$ is also infinite.

Sketch of the proof of assertion 2) *in Theorem* 10. Use the notation $s = 2p+3q-2m+2$ and $l = m-p-q-1$. Then the group in question is $\pi_{s+l+1}(V_{M+l,M})$. Our restriction $p+\frac{4}{3}q+2 \leqslant m < p+\frac{3}{2}q+2$ is equivalent to the restriction $-1 \leqslant s \leqslant l-3$.

(a) *Case* $s = -1$. By tables in [28] the group $\pi_l(V_{M+l,M})$ is infinite if and only if l is divisible by 2. Together with condition $s = -1$ this is equivalent to the conditions $m = p + \frac{3}{2}q + \frac{3}{2}$, $q+1$ is divisible by 4.

(b) *Case* $0 \leqslant s \leqslant l-3$. Let us prove by induction on s that the group $\pi_{s+l+1}(V_{M+l,M})$ is finite. The base $s = 0$ follows from tables in [28]. For $s > 0$ consider the homotopy exact sequence of the 'restriction' bundle $S^l \to V_{M+l,M} \to V_{M+l,M-1}$ tensored by $\mathbb{Q}$. In this sequence $\pi_{s+l+1}(S^l)$ is finite because $0 \leqslant s \leqslant l-3$. By the inductive hypothesis the group

$$\pi_{s+l+1}(V_{M+l,M-1}) \cong \pi_{(s-1)+(l+1)+1}(V_{M+l+1,M})$$

is finite. Hence the group $\pi_{s+l+1}(V_{M+l,M})$ is also finite.

Remark 2. Analogously using the bundle

$$\mathrm{SO}_{m-q-p-1} \to \mathrm{SO}_{m-q} \to \pi_q(V_{m-q,p+1}),$$

one can prove the following assertion: *suppose that* $m \geqslant p + \frac{3}{2}q + 2$; *then the group* $\pi_q(V_{m-q,p+1})$ *is infinite if and only if either* $m = 2q+1$, q *odd, or* $m = p+2q+1$, q *even* [24].

Thus the proof of Theorem 2 is complete.

§ 7. Concluding remarks

Let us give a counterexample which shows that Theorem 3 and Lemma 1 cannot be proved using the ordinary Whitney trick or the approach of [35].

Example 3. For some p, q, m satisfying the inequalities of Theorem 3 there exists a proper almost embedding $f\colon S^p \times D^q \to D^m$ such that $\beta(f) = 0$ but f admits no webs. In particular, the isomorphism of Lemma 1 does not hold for $f' = f$.

For a proof take $m = p + \frac{3}{2}q + \frac{3}{2}$ and choose p and q so that $l = m - p - q - 1$ is odd. Take a generic proper almost embedding $f\colon S^p \times D^q \to D^m$ such that Δ is connected whereas $\widetilde{\Delta}$ is not connected (for example, start with the standard embedding and perform the finger Whitney moves). Our proof of Lemma 1 in § 4 shows that, in fact $\pi_q(D^m - \operatorname{Im} F, \partial) \cong \Omega_s(\Delta; l\lambda_\Delta)$. The latter group is isomorphic to $\mathbb{Z}$ for $s = 0$ and l odd by [30], the end of § 4. On the other hand, $\Omega_0(P^\infty; l\lambda) = \mathbb{Z}_2$. So the map $\beta(f, \cdot)\colon \mathbb{Z} \to \mathbb{Z}_2$ is not injective.

Then there exists a proper map $g\colon D^q \to D^m - \operatorname{Im} f$ such that $\beta(f, g) = 0$, but g is not null-homotopic. Perform the construction from the proof of Theorem 7 steps 2) and 3). We get a new proper almost embedding f such that $\beta(f) = 0$. On the other hand, the map $g\colon D^q \to D^m - \operatorname{Im} f$ is close to $f|_{*\times D^q}$, but it is not null-homotopic. Thus there exists no web for f.

There are several directions to study knotted tori further:

(i) Explicit classification results. How many embeddings $S^1 \times S^5 \to S^{10}$ are there up to isotopy?

(ii) Weakening the dimension restrictions. Is it possible to drop the restrictions $m > 2p + q + 2$ or $m > p + \frac{4}{3}q + 2$ in Theorem 2? (Cf. [4].)

(iii) Arbitrary manifolds. It would be interesting to generalize the β-invariant and Theorem 3 to embeddings of arbitrary manifolds [1], [21].

(iv) Rational classification of embeddings. For a given manifold N and a number m determine whether the set of embeddings $N \to S^m$ up to isotopy is finite.

§ 8. Appendix. Surgery on the double point manifold

Here we perform a surgery on the double point manifold Δ to make the classifying map $\Delta \to P^\infty$ of the covering $\widetilde{\Delta} \to \Delta$ sufficiently highly connected. This is required for the proof of Complement Lemma 1 stated in § 4. Our exposition is completely analogous to [30], Appendix A, although more general, explicit and detailed.

Let $f\colon D^n \to M^m$ be a general position proper immersion such that $f|_{\partial D^n}\colon \partial D^n \to \partial M^m$ is an embedding. The embedding theorem in [25], cf. Theorem 5 above, allows us to remove the self-intersection of f by a homotopy $\operatorname{rel} \partial D^n$ under certain conditions. In the dimension range where the embedding theorem is not true we give an approach to 'simplify' the double points of f.

In this section we prove that the classifying map $\Delta \to P^\infty$ can be made $(s+1)$-connected by a homotopy $\operatorname{rel} \partial D^n$ of the map $f\colon D^n \to M^m$, provided that

(i) M^m is $(s+1)$-connected;

(ii) $2s \leqslant 2n - m - 2$;

(iii) $0 \leqslant s \leqslant m - n - 3$.

This is a restatement of Surgery Theorem 6 above.

Proof of Theorem 6. The procedure of making $\Delta \to P^\infty$ $(s+1)$-connected is done in two steps.

Step 1. Making Δ connected and $\pi_1(\Delta) \to \pi_1(P^\infty)$ surjective (in other words, making $\widetilde{\Delta}$ connected).

Step 2. Killing the elements of $\ker(\pi_i(\Delta) \to \pi_i(P^\infty))$ for each $i = 1, \dots, s$.

These two steps are sufficient for the proof of the theorem because the map $\pi_{i+1}(\Delta) \to \pi_{i+1}(P^\infty) = 0$ is surjective for $1 \leqslant i \leqslant s$.

In both steps 1 and 2 we carry out the following Whitney-Haefliger trick, performing a surgery on Δ. First let us construct the Habegger-Kaiser standard model for doing surgery on a framed i-sphere of double points of an n-disc immersed into S^m.

Standard model for doing surgery [30]. We will make use of the 'model' manifold

$$\mathbb{R}^m = \mathbb{R} \times \mathbb{R}^{i+1} \times \mathbb{R}^{2n-m-i} \times \mathbb{R}^{m-n-1} \times \mathbb{R}^{m-n-1}$$

and of two embeddings g_+ and g_- of $\mathbb{R}^n = \mathbb{R}^{i+1} \times \mathbb{R}^{2n-m-i} \times \mathbb{R}^{n-m-1}$ into $\mathbb{R}^m$ intersecting transversally along $0 \times S^i \times \mathbb{R}^{2n-m-i} \times 0 \times 0$. For example, one may take

$$g_-(x,y,z) = (|x|^2 - 1, x, y, 0, z), \qquad g_+(x,y,z) = (1 - |x|^2, x, y, z, 0).$$

The sphere S^i bounds a ball $D^{i+1} \subset \mathbb{R}^{i+1} \subset \mathbb{R}^n$. Furthermore, the sphere $S^{i+1} = D_+^{i+1} \cup D_-^{i+1}$, where $D_\pm^{i+1} = g_\pm(D^{i+1} \times 0 \times 0)$, bounds a ball $D^{i+2} \subset \mathbb{R} \times \mathbb{R}^{i+1} \subset \mathbb{R}^m$ with corners along S^i. Pushing one of the two caps $D_\pm^{i+2}$ across D^{i+2} does the required surgery. More precisely, the double points of the resulting regular homotopy form the trace of this surgery.

Now we are going to make some preparations for doing surgery, which are a bit different in Steps 1 and 2.

Step 1: *making* Δ *and* $\widetilde{\Delta}$ *connected.* If $\Delta = \varnothing$, then we first create a nonempty self-intersection (for example, by the Whitney finger moves). Assume $\Delta \neq \varnothing$. Take a pair of points (a,b), (c,d) belonging to distinct components of $\widetilde{\Delta}$. One can assume that they are outside the triple point set. Consider the spheres $S^0 = \{\{a,b\},\{c,d\}\}$, $S^0_+ = \{(a,b),(c,d)\}$, and $S^0_- = \{(b,a),(d,c)\}$, and let η be the trivialization of the normal bundle $N(S^0, \Delta)$. The surgery on S^0 (see the completion of the proof below) will connect distinct components of $\widetilde{\Delta}$ because $\dim \Delta = 2n - m \geqslant 2s + 2 \geqslant 2$.

Step 2: *killing the elements of* $\ker(\pi_i(\Delta) \to \pi_i(P^\infty))$. Assume that $g\colon S^i \to \Delta$ represents an element of the kernel $\ker(\pi_i(\Delta) \to \pi_i(P^\infty)$, $1 \leqslant i \leqslant s$. Since $2i \leqslant \dim \Delta - 1$ (because $2s \leqslant 2n - m - 2$), it follows that g can be assumed to be an embedding. By general position the triple point set has dimension $\leqslant 3n - 2m$. Since $s \leqslant m - n - 3$, it follows that $i + 3n - 2m \leqslant \dim \Delta - 1$, so generically $\operatorname{Im} g$ does not contain triple points.

Since the composition $S^i \to \Delta \to P^\infty$ is trivial, it follows that $(m-n)\lambda_\Delta|_{S^i}$ is trivial. Then the normal bundle $\eta = N(S^i, \Delta)$ is stably trivial and hence trivial. Since $S^i \to P^\infty$ is trivial, it follows that S^i is also trivially covered in $\widetilde{\Delta}$. Denote by S^i_+ and S^i_- the two copies of S^i in $\widetilde{\Delta}$.

Completion of the proof: *surgery on* S^i. First let us span the spheres S^i_+ and S^i_- by two disjoint discs D_+^{i+1} and D_-^{i+1} in the ball D^n. To do this take a trivialization of the normal bundle $N(\widetilde{\Delta}, D^n)$ (for instance, constructed in the definition of the double point $(m-n)\lambda$-manifold in § 3). Push the spheres S^i_+ and S^i_- along the first vector field of the trivialization. Since $s \leqslant m - n - 3$ and $2s \leqslant 2n - m - 1$, it follows that $2s \leqslant 2n - m - 1$. Hence $i + 1 + \dim \Delta \leqslant n - 1$ and $2(i+1) \leqslant n - 1$. Then by general position the pushed spheres can be spanned by discs missing the image of $\widetilde{\Delta}$.

Consider the obvious decomposition $N(D_+^{i+1}, D^n)|_{S^i_+} = \eta \oplus \varepsilon^{m-n-1}$, where $\eta = N(S^i, \Delta)$. This decomposition gives an $(m-n-1)$-framing of the sphere S^i_+. We wish to extend this $(m-n-1)$-framing over D^{i+1}_+. The complete obstruction lies in $\pi_i(V_{n-i-1,m-n-1})$. The latter group vanishes for $2i \leqslant 2n-m-1$, which follows from $2s \leqslant 2n-m-2$. Thus we obtain a decomposition $N(D^{i+1}_+, D^n) = \eta_+ \oplus \varepsilon^{m-n-1}$, where η_+ is the extension of the bundle η over D^{i+1}_+ complementary to the $(m-n-1)$-framing. Define the bundle η_- analogously.

Next we extend the embedding of $S^{i+1} = D^{i+1}_+ \cup D^{i+1}_-$ to an embedding of D^{i+2} into S^m. To do this push D^{i+1}_+ and D^{i+1}_- along the first vector field of a trivialization of the bundle $N(D^n, S^m)$. In this way we obtain an embedding of a collar neighbourhood of the sphere S^{i+1} into S^m. Since $n+s+3 \leqslant m$ and $s \leqslant n-1$, it follows that $i+2+n \leqslant m-1$ and $2(i+2) \leqslant m-1$. Thus by general position the collar can be extended to an embedded disc D^{i+2} in S^m whose interior does not intersect fD^n.

Finally, consider the following $(m-n-1)$-framing of the sphere S^{i+1}. On the disc D^{i+1}_+ take the $(m-n-1)$-framing complementary to η_+. On the disc D^{i+1}_- take the $(m-n-1)$-framing obtained from the trivialization of $N(D^n, S^m)$ by forgetting the first vector field. By the construction above it follows that these two framings coincide on S^i. Thus we obtain an $(m-n-1)$-framing of S^{i+1}. Let us extend it to D^{i+2}. The complete obstruction lies in $\pi_{i+1}(V_{m-i-2,m-n-1})$. The latter group vanishes for $2i+2 \leqslant n-1$ because $3s \leqslant n-4$. Let η' be the complementary bundle to the obtained $(m-n-1)$-framing over D^{i+2}. We have a splitting $\eta' = \eta_- \oplus \varepsilon^{m-n-1}$ on D^{i+1}_-. Extending it to D^{i+2} we get a splitting $N(D^{i+2}, S^m) = \eta'' \oplus \varepsilon^{m-n-1} \oplus \varepsilon^{m-n-1}$ for some bundle η''.

Thus the relevant framing information along D^{i+2} agrees with that of the standard model. So there is a diffeomorphism between a neighbourhood of D^{i+2} and the Euclidean m-space taking a restriction of our immersion to the standard model. So one can perform the surgery and kill the spheroid $g\colon S^i \to \Delta$. The theorem is thus proved.

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M. Cencelj
Institute of Mathematics, Physics and Mechanics,
Ljubljana, Slovenia;
University of Ljubljana, Slovenia
E-mail: matija.cencelj@guest.arnes.si

D. Repovš
Institute of Mathematics, Physics and Mechanics,
Ljubljana, Slovenia;
University of Ljubljana, Slovenia
E-mail: dusan.repovs@guest.arnes.si

M. B. Skopenkov
Institute for Information Transmission Problems,
the Russian Academy of Sciences, Moscow;
King Abdullah University of Science and
Technology, Thuwal, Saudi Arabia
E-mail: skopenkov@rambler.ru