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On extending actions of groups

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Abstract. Problems of dense and closed extension of actions of compact transformation groups are solved. The method developed in the paper is applied to problems of extension of equivariant maps and of construction of equivariant compactifications.

Bibliography: 27 titles.

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§ 1. Introduction

Let a compact group G act on a space $\mathbb{X}$. The diagram $\mathscr{D}$ of the form $\mathbb{X} \xrightarrow{p} X \overset{i}{\hookrightarrow} Y$ is called *admissible* if $p\colon \mathbb{X} \to X$ is the orbit projection and i is a topological embedding of the orbit space X into a hereditarily paracompact space Y. We say that

1) the problem of extending the action is solvable for the admissible diagram $\mathscr{D}$ if there exists an equivariant embedding $j\colon \mathbb{X} \hookrightarrow \mathbb{Y}$ into a G-space $\mathbb{Y}$ (which is called a *solution* of the problem of extending the action for the given diagram) covering i, that is, the embedding $\widetilde{j}\colon X \hookrightarrow p(\mathbb{Y})$ of orbit spaces induced by j coincides with i (in particular, $p(\mathbb{Y}) = Y$);

2) the problem of extending the action (denoted briefly by *PEA*) is solvable if there exists a solution of the PEA for each admissible diagram $\mathscr{D}$, that is, the diagram $\mathscr{D}$ can be involved in a commutative square diagram:

$$\begin{array}{ccc} \mathbb{X} & \overset{j}{\hookrightarrow} & \mathbb{Y} \\ p\downarrow & & \downarrow p \\ X & \overset{i}{\hookrightarrow} & Y \end{array}$$

3) the PEA is solvable for the class $\mathscr{C}$ if for each admissible diagram $\mathscr{D}$ such that $\mathbb{X}$, X and Y belong to $\mathscr{C}$, there exists a solution of the PEA with $\mathbb{Y} \in \mathscr{C}$.

Note that the problem of extending the action is a part of the more general problem of extension for complete preimages (see [1], [2]). The problem of extending actions of groups is naturally split into the closed PEA and the dense PEA depending on the type of the embedding i. On the other hand, it is clear that the

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simultaneous solvability of the closed and dense PEAs implies the solvability of the PEA in general.

Shchepin was the first to pose the closed problem of extending the action of groups in view of the following fact established by him:

Proposition 1. *Let G be a compact group. If the closed problem of extending actions of groups is solvable for the class of metric G-spaces, then*

(1) *the orbit space of each metric G-A[N]E-space $\mathbb{E}$ is an A[N]E-space.*

Proof. Let $Z \hookleftarrow A \xrightarrow{\varphi} E$ be a partial map with $Z \in \mathscr{M}$, where $\mathscr{M}$ is the class of all metric G-spaces. We denote by $\mathbb{A}$ the fibrewise product $A_\varphi \times_p \mathbb{E}$. Since $\mathbb{A} \times Z \in \mathscr{M}$, the hypothesis implies that the embedding $A \hookrightarrow Z$ is covered by a closed G-embedding $\mathbb{A} \hookrightarrow \mathbb{Z} \in \mathscr{M}$. The constructed partial G-map $\mathbb{Z} \hookleftarrow \mathbb{A} \xrightarrow{\varphi'} \mathbb{E} \in G$-A[N]E, where φ' is parallel to φ, can be G-extended to $\mathbb{Z}$ (to a G-neighbourhood). Passing to the orbit spaces we get the desired extension of φ.

As was shown in [3], the preservation of equivariant extensor properties by the orbit functor implies the solvability of the closed PEA and so this problem obtains the dual expression.

Proposition 2. *Let G be a compact group. Then the validity of* (1) *implies the solvability of the closed PEA for all metric G-spaces.*

Proof. There exists by Theorem 18 an isovariant G-map $f\colon \mathbb{X} \to \mathbb{T}_G \in G$-AE. In view of (1), $T_G = \mathbb{T}_G/G \in$ AE and therefore the partial map $Y \hookleftarrow X \xrightarrow{\widetilde{\alpha}} T_G$ has an extension $\widehat{\alpha}$. It is easy to verify that the fibrewise product $Y_{\widehat{\alpha}} \times_p \mathbb{T}_G$ is the desired metric G-space $\mathbb{Y}$.

Since property (1) was established in [3], [4] for arbitrary compact groups (in [5] it was established for compact metric groups[1]), by now the closed PEA has been solved in the class of metric spaces with the action of an arbitrary compact group.

The dense problem of extending actions of groups arises naturally in the theory of equivariant compactifications and was first posed by Zambakhidze and Smirnov. It turns out that the dense PEA is intimately connected with the existence of various invariant metrics on a G-space.

Proposition 3. *Let G be a compact group. The dense problem of extending the action of groups is solvable in the class of separable metric G-spaces if and only if*

(2) *for each separable metric G-space $\mathbb{X}$ and each compatible metric d on the orbit space X there exists a compatible invariant metric ρ on $\mathbb{X}$ such that $\mathscr{F}_d = \mathscr{F}_{\widetilde{\rho}}$ (here $\widetilde{\rho}([x],[x']) = \min\{\rho(g\cdot x, g\cdot x') \mid g \in G\}$ is the induced metric on X, and $\mathscr{F}_d, \mathscr{F}_{\widetilde{\rho}}, \dots$ are the sets of all Cauchy sequences on X with respect to the corresponding metrics).*

Proof. We consider a metric compactum K containing the completion $\mathrm{c}_d X$ of X with respect to d. It is clear that $\mathscr{F}_d \subset \mathscr{F}_\delta$, where δ is a metric on K.

Since the dense problem of extending actions of groups is solvable, the embedding $X \hookrightarrow K$ is covered by a G-embedding $\mathbb{X} \hookrightarrow \mathbb{K}$ into a G-compactum $\mathbb{K}$. Consider

[1] One should note the earlier approach of Jaworowski [6] for finite groups $G < S_n$, which is applicable also to compact Lie groups [7].

the invariant metric σ, which exists on $\mathbb{K}$. It is clear that $\mathscr{F}_{\widetilde{\sigma}} = \mathscr{F}_{\delta} \supset \mathscr{F}_d$, and the invariant metric $\rho(x, x') = \sigma(x, x') + d(p(x), p(x'))$ on the space $\mathbb{X}$ is compatible with its topology. We shall check that $\mathscr{F}_d = \mathscr{F}_{\widetilde{\rho}}$, where $\widetilde{\rho} = \widetilde{\sigma} + d$. Since $d \leqslant \widetilde{\rho}$, it follows that $\mathscr{F}_{\widetilde{\rho}} \subset \mathscr{F}_d$. The reverse embedding $\mathscr{F}_d \subset \mathscr{F}_{\widetilde{\rho}}$ easily follows from the embedding $\mathscr{F}_d \subset \mathscr{F}_{\widetilde{\sigma}}$ already obtained.

We prove the converse result. Let d' be a metric on Y and d the restriction of d' to a dense subset X. By hypothesis there exists an invariant metric ρ on $\mathbb{X}$ such that $\mathscr{F}_d = \mathscr{F}_{\widetilde{\rho}}$. Let $\mathrm{c}_\rho\mathbb{X}$ be the G-completion with respect to the metric ρ, which is a separable metric G-space. It is easy to verify that $\mathrm{c}_\rho\mathbb{X}$ is naturally homeomorphic to $(\mathrm{c}_\rho\mathbb{X})/G$. Since $\mathscr{F}_d \subset \mathscr{F}_{\widetilde{\rho}}$, Y naturally lies in $\mathrm{c}_{\widetilde{\varrho}}X = (\mathrm{c}_\varrho\mathbb{X})/G$. We take the inverse image of Y with respect to the orbit projection $p\colon c_\varrho\mathbb{X} \to c_\varrho\mathbb{X}/G$ as a solution $s\colon \mathbb{X} \hookrightarrow \mathbb{Y}$ of the dense PEA.

By now the dense PEA has been solved for the class of metric spaces with action of an arbitrary zero-dimensional compact group [8], [9]. The main aim of the article is the direct proof of the following results giving an answer in the affirmative both to the dense PEA and the closed PEA under maximally wide assumptions.

Theorem 1. *The problem of extending the action of compact groups is solvable for all admissible diagrams.*

We note that for some nonmetrizable group G and metric spaces $\mathbb{X}$ and Y, the equivariant embedding $j\colon \mathbb{X} \hookrightarrow \mathbb{Y}$ solving the PEA can be nonmetrizable. The following result improves this defect.

Theorem 2. *Let G be a compact group and $\mathbb{X} \xrightarrow{p} X \xhookrightarrow{i} Y$ a diagram such that all orbits of $\mathbb{X}$ are metrizable, and also*

(3) *Y is stratifiable and the embedding i is closed*;

or

(4) *$Y \in \mathscr{M}$.*

Then there exist a metric G-space $\mathbb{Z}$ and an equivariant embedding $j\colon \mathbb{X} \hookrightarrow \mathbb{Y}$ into a G-space $\mathbb{Y} \subset Y \times \mathbb{Z}$, which is a solution of the PEA for this diagram.

In particular, Theorem 2, (3) implies the direct proof of the solvability of the closed PEA both in the class of metrizable G-spaces and the class of stratifiable G-spaces (since a countable product preserves the class of stratifiable spaces). We can easily derive from Theorem 2, (4) the following facts on equivariant compactifications for compact acting groups:

(5) if $\mathbb{X}$ is a metric separable G-space with $\dim X = k$, then there exists an equivariant compactification $\mathbb{X} \hookrightarrow \mathbb{Y}$ such that $\mathbb{Y}$ is a metric compactum with $\dim Y = k$;

(6) if $\mathbb{X}$ is a metric G-space with $\dim X = k$, then there exists an equivariant completion $\mathbb{X} \hookrightarrow \mathbb{Y}$ with $\dim Y = k$.

We observe that the PEA ceases to be solvable in some simple situations. Let us take $\mathbb{X} = (\operatorname{Con} \mathbb{Z}_2)^{\omega_1}$ with the natural action of the simplest nontrivial group $G = \mathbb{Z}_2$ and an arbitrary embedding $i\colon X \hookrightarrow I^{\omega_1}$ into a Tychonoff cube of uncountable weight ω_1. Then the PEA is unsolvable for the diagram $\mathbb{X} \xrightarrow{p} X \xhookrightarrow{i} I^{\omega_1}$.

Theorem 1 and the method developed for its proof admit various modifications. We endow the set $\mathscr{S}$ of all solutions of the closed PEA for the admissible diagram

$\mathscr{D} = (\mathbb{X} \xrightarrow{p} X \overset{i}{\hookrightarrow} Y)$ with the following partial order: s_1 majorizes s_2 (for brevity $s_1 \geqslant s_2$), where $\{s_i : \mathbb{X} \hookrightarrow \mathbb{Y}_i\} \subset \mathscr{S}$, if there exists a G-map $h : \mathbb{Y}_1 \to \mathbb{Y}_2$ making commutative the associated diagram: $h \circ s_1 = s_2$ and such that $h\restriction_{\mathbb{X}} = \mathrm{Id}_{\mathbb{X}}$ and $\widetilde{h} = \mathrm{Id}_Y$.

Theorem 3. *Let G be a compact group. Then for each finite set of solutions $\{s_i\}_{i=1}^n \subset \mathscr{S}$ of the closed PEA there exists a solution $s \in \mathscr{S}$ such that $s \geqslant s_i$ for $i \leqslant n$.*

Theorem 4. *Let G be a compact group. Then for each admissible diagram $\mathscr{D}$ there exists a solution $s : \mathbb{X} \hookrightarrow \mathbb{Y}$ of the closed PEA such that* $\operatorname{type}\mathbb{Y} \geqslant \operatorname{type}\mathbb{X}$. *If G is a Lie group, then this solution $s : \mathbb{X} \hookrightarrow \mathbb{Y}$ can be chosen so that* $\operatorname{type}\mathbb{Y} = \operatorname{type}\mathbb{X} \cup \{(G)\}$.

The proofs of Theorem 3 and Theorem 4 are omitted because of restrictions on the volume of the article. In their turn, Theorems 1, 3 and 4 imply several results of equivariant extensor theory (which, as can be shown, are equivalent to the corresponding facts on extending the action).

Theorem 5. *Let G be a compact group. If the stratifiable G-space $\mathbb{X}$ is an equivariant absolute extensor for the class of stratifiable spaces, then its orbit space X is an absolute extensor for the class of stratifiable spaces.*

In [7] it was also pointed out that the Whitehead-Borsuk-Hanner theorem for stratifiable spaces [10] implies Theorem 5. Therefore there is a hope that the gap in the Whitehead-Borsuk-Hanner theorem [10] communicated to us by Cauty can be overcome.

Theorem 6. *Let G be a compact group. If a metric G-space $\mathbb{X}$ is an equivariant absolute extensor, then for any finite collection of G-extensions $h_i : \mathbb{Z} \to \mathbb{X}$ of the partial G-map $\mathscr{M} \ni \mathbb{Z} \hookleftarrow \mathbb{A} \xrightarrow{f} \mathbb{X}$ there exists a G-extension $h : \mathbb{Z} \to \mathbb{X}$ of f such that $(G_{h(z)}) \geqslant (G_{h_i(z)})$ for all $z \in \mathbb{Z}$ and i.*

Theorem 7. *Let G be a compact group, $\mathbb{X} \in G$-ANE and $\mathscr{C} \subset \operatorname{Orb} G$. Then*

(7) $\mathbb{X}^{\mathscr{C}} \subset \mathbb{X}$ *is* G-ANE,

(8) $\mathbb{X}^{\mathscr{C}}/G \subset X$ *is* ANE.

Theorem 7, (7) for a compact Lie group and a one-element collection $\mathscr{C}$ of orbit types was proved before by Murayama [11]; Theorem 7, (8) for $G = \mathrm{S}^1$ and $\mathbb{X} = \exp G$ was proved by Torunczyk and West [12]. The proofs of Theorems 3–7 will be published elsewhere.

In completion we give the result revealing the role of the set of extensor points in the theory of G-extensors. We say that a G-embedding $\mathbb{Y} \hookrightarrow \mathbb{X}$ is *equivariant homotopy dense* if there exists a G-homotopy $F : \mathbb{X} \times [0,1] \to \mathbb{X}$ such that $F_0 = \mathrm{Id}$ and $F_t(\mathbb{X}) \subset \mathbb{Y}$ for each $t > 0$.

Theorem 8 (on equivariant homotopy density). *Let $\mathbb{X}$ be a metric G-ANE-space. Then the subspace $\mathbb{X}_{\mathscr{E}}$ of all extensor points is equivariant homotopy dense in $\mathbb{X}$.*

As an immediate consequence, we get the following result:

Theorem 9. *Let $\mathbb{X}$ be a metric G-*ANE*-space. Then each G-subspace $\mathbb{Y}$ such that $\mathbb{X}_{\mathscr{E}} \subset \mathbb{Y} \subset \mathbb{X}$ is G-*ANE.

§ 2. Preliminary facts and results

All spaces (and maps) throughout that do not arise as a result of some constructions are assumed to be hereditarily paracompact (respectively, continuous), and all acting groups are assumed to be compact.

For $B \subset A$ we use the standard notations: $\mathrm{Cl}_A B$ for the closure and $\mathrm{Int}_A B$ for the interior. We use the notation $f\restriction_A$ for the restriction of the map f to $A \subset X$. If the set A in question is clear, we shall simply omit the symbol A. If there is no danger of ambiguity, we shall leave out the definitions of some concepts arising in the natural manner.

An *action of a compact group G on a space* $\mathbb{X}$ is a homomorphism $T\colon G \to \mathrm{Aut}\,\mathbb{X}$ of G into the group $\mathrm{Aut}\,\mathbb{X}$ of all autohomeomorphisms of $\mathbb{X}$ such that the map $G \times \mathbb{X} \to \mathbb{X}$ given by $(g, x) \mapsto T(g)(x) \risingdotseq g \cdot x$ is continuous (here and throughout the paper the sign $\risingdotseq$ is used for the introduction of the new objects placed to the left of it). A space $\mathbb{X}$ with a fixed action of G is called a *G-space*.

For any point $x \in \mathbb{X}$ the following subset $G_x = \{g \in G \mid g \cdot x = x\}$ is a closed subgroup of G and is called the *stabilizer of x*. The *orbit of $x \in \mathbb{X}$* is $G(x) = \{g \cdot x \mid g \in G\} \subset \mathbb{X}$. The set of orbits is denoted by $X \risingdotseq \mathbb{X}/G$, and the natural map $\pi\colon \mathbb{X} \to X$, $\pi(x) = G(x)$, is called the *orbit projection*. The *orbit space* of $\mathbb{X}$ is the set X of orbits equipped with the quotient topology induced by π. (See [13] for more details about G-spaces.)

A map $f\colon \mathbb{X} \to \mathbb{Y}$ of G-spaces is called *equivariant* or a *G-map* if $f(g{\cdot}x) = g{\cdot}f(x)$, $g \in G$, $x \in \mathbb{X}$. Each G-map $f\colon \mathbb{X} \to \mathbb{Y}$ induces a map $\widetilde{f}\colon X \to Y$ of the orbit spaces by the formula $\widetilde{f}(G(x)) = G(f(x))$. An equivariant homeomorphism is called an *equimorphism*. An equivariant map $f\colon \mathbb{X} \to \mathbb{Y}$ is said to be *isovariant* if $G_x = G_{f(x)}$ for all $x \in \mathbb{X}$.

Theorem 10 (an equimorphism criterion [14]). *An isovariant map $f\colon \mathbb{X} \to \mathbb{Y}$ inducing a homeomorphism $\widetilde{f}\colon X \to Y$ of the orbit spaces is an equimorphism.*

Note that all G-spaces and G-maps generate a category denoted by G-TOP, or EQUIV provided that the group G is clear. If '$*{*}*$' is a known notion from nonequivariant topology, then 'G-$*{*}*$' or 'Equiv-$*{*}*$' means the corresponding equivariant analogue.

A subset $A \subset \mathbb{X}$ is called *invariant* or a *G-subset* if $G \cdot \mathbb{A} = \mathbb{A}$. For each closed subgroup $H < G$ (see the footnote[2]) we define the following sets:

$$\mathbb{X}^H = \{x \in \mathbb{X} \mid H \cdot x = x\}$$

(which is called an *H-fixed-point set*) and

$$\mathbb{X}_H = \{x \in \mathbb{X} \mid G_x = H\}.$$

[2]Further we shall denote in this way a closed subgroup; the notation for a normal closed subgroup is $H \triangleleft G$.

It is clear that

$$\mathbb{X}^{(H)} \rightleftharpoons \bigcup \{ \mathbb{X}_K \mid \text{ there is } K' < G \text{ such that } K' \supseteq H \text{ and } K' \text{ is conjugate to } K \}$$

coincides with $G \cdot \mathbb{X}^H$, and

$$\mathbb{X}_{(H)} \rightleftharpoons \bigcup \{ \mathbb{X}_K \mid K < G \text{ and } K \text{ is conjugate to } H \}$$

coincides with $G \cdot \mathbb{X}_H$.

The set of all conjugacy classes of closed subgroups of G is denoted by $\mathrm{Orb}(G)$ and is called the *set of orbit types*. We endow $\mathrm{Orb}(G)$ with the following partial order: $(K) \leqslant (H) \iff K \subset g^{-1} \cdot H \cdot g$ for some $g \in G$. We denote the set $\{(G_x) \mid x \in \mathbb{X}\} \subset \mathrm{Orb}(G)$ of orbit types of $\mathbb{X}$ by $\mathrm{type}\, \mathbb{X}$. If $\mathscr{C} \subset \mathrm{Orb}(G)$, then $\mathbb{X}^{\mathscr{C}} \rightleftharpoons \{ x \mid (G_x) \geqslant (H) \text{ for some } (H) \in \mathscr{C} \} \subset \mathbb{X}$.

We introduce several concepts related to extension of G-maps partially defined on G-spaces in some class $\mathscr{K}$ from the increasing chain: the class of metric G-spaces $\subset$ the class of stratifiable G-spaces[3] $\subset$ the class of hereditarily paracompact G-spaces[4]. A G-space $\mathbb{X}$ is called an *absolute equivariant neighbourhood extensor for* $\mathscr{K}$, $\mathbb{X} \in G\text{-ANE}(\mathscr{K})$, if each G-map $\varphi\colon \mathbb{A} \to \mathbb{X}$ defined on a closed G-subset $\mathbb{A}$ of a G-space $\mathbb{Z} \in \mathscr{K}$ and called a *partial G-map* can be G-extended to a G-neighbourhood $\mathbb{U} \subset \mathbb{Z}$ of $\mathbb{A}$, $\widehat{\varphi}\colon \mathbb{U} \to \mathbb{X}$, $\widehat{\varphi}\restriction_{\mathbb{A}} = \varphi$. If it is always possible to G-extend φ to $\mathbb{U} = \mathbb{Z}$, then $\mathbb{X}$ is called an *equivariant absolute extensor for* $\mathscr{K}$, $\mathbb{X} \in G\text{-AE}(\mathscr{K})$. If the acting group G is trivial (that is, no action on the spaces is considered), then these notions are transformed into the notions of absolute [neighbourhood] extensors for $\mathscr{K}$, $\mathrm{A[N]E}(\mathscr{K})$. Since we are mainly interested in equivariant absolute (neighbourhood) extensors for the class of metric G-spaces, they will be briefly denoted by G-A[N]E. Important examples of G-A[N]E-spaces are Banach G-spaces [15] and linear normed G-spaces (for compact Lie groups G) [11]. Important for us will be the question whether a homogeneous space G/H belongs to the class of G-ANE-spaces. It is known that $G/H \in G$-ANE if and only if G/H is a finite-dimensional locally connected space. In particular, if G is a compact Lie group, then $G/H \in G$-ANE, which is in fact equivalent to the slice theorem.

The following construction of the equivariant absolute extensor for an arbitrary compact group G goes back to [16]. Recall that G acts on the space $\mathbb{X} = \mathrm{C}(G, Y)$ of all continuous maps endowed with the compact-open topology by the formula $(g \cdot f)(h) = f(g^{-1} \cdot h)$, where $f \in \mathrm{C}(G, Y)$ and $g, h \in G$. If Y is metrizable, then $\mathbb{X}$ is also metrizable.

Theorem 11. *If a metric space Y is an* AE*-space for the class $\mathscr{P}$ of paracompact spaces, then $\mathrm{C}(G, Y)$ is a G-*AE*-space for $\mathscr{P}$.*

By [17] each Banach space B is an AE-space for the class of paracompact spaces. Hence it follows by Theorem 11 that $\mathrm{C}(G, B)$ is G-AE for $\mathscr{P}$.

A metric G-space $\mathbb{X}$ is called an *equivariant absolute neighbourhood retract*, $\mathbb{X} \in$ G-ANR, if for each closed G-embedding of $\mathbb{X}$ into a metric G-space $\mathbb{Z}$ there

[3] A space X is stratifiable if there exists a family $\{f_U : X \to [0,1] \mid U \subset X \text{ is an open subset}\}$ of continuous functions such that $f_U^{-1}(0,1] = U$ and $f_U \leqslant f_V$ if and only if $U \subset V$. It is known that each CW-complex is stratifiable.

[4] A space is called hereditarily paracompact if each subspace of it is paracompact (this is equivalent to the paracompactness of each open subspace of it).

exists a neighbourhood G-retraction $r\colon \mathbb{U} \to \mathbb{X}$, $r \circ r = r$. If it is always possible to choose a G-retraction r defined on $\mathbb{U} = \mathbb{Z}$, then $\mathbb{X}$ is called an *equivariant absolute retract*, $\mathbb{X} \in$ G-AR.

Theorem 12 (on approximate G-extension of maps). *Suppose that a compact G-subspace $\mathbb{X}$ of the Euclidean G-space $\mathbb{R}^n$ is a G-ANE. Then for each cover $\omega \in \operatorname{cov}\mathbb{X}$ and each G-map $\varphi\colon \mathbb{A} \to \mathbb{X}$ of a (not necessarily closed) G-subset $\mathbb{A}$ of a hereditarily paracompact G-space $\mathbb{Z}$ there exists a G-map $\psi\colon \mathbb{U} \to \mathbb{X}$ defined on an open G-set $\mathbb{U}$ containing $\mathbb{A}$ such that* $\operatorname{dist}(\varphi, \psi\restriction_{\mathbb{A}}) \prec \omega$.

Proof. Since $\mathbb{X} \in$ G-ANE, there exists an equivariant retraction $r'\colon \mathbb{V}' \to \mathbb{X}$ defined on a G-neighbourhood $\mathbb{V}'$, $\mathbb{X} \subset \mathbb{V}' \subset \mathbb{R}^n$. It is clear that there exist a G-neighbourhood $\mathbb{V}$, $\mathbb{X} \subset \mathbb{V} \subset \mathbb{V}'$, and $\varepsilon > 0$ such that $N(\mathbb{X};\varepsilon) \subset \mathbb{V}$ and also $\{r(N(x;\varepsilon)) \mid x \in \mathbb{V}\} \prec \omega$, where $r \rightleftharpoons r'\restriction_{\mathbb{V}}$.

Lemma 1. *For each map $\psi\colon A \to \mathbb{R}^n$ defined on a subset A of a hereditarily paracompact space Z there exists a map $\chi\colon \mathscr{U} \to \mathbb{R}^n$ defined on an open subset $\mathscr{U}$ containing A such that* $\operatorname{dist}(\chi\restriction_A, \psi) < \varepsilon$.

Proof. From the hereditary paracompactness of Z it easily follows that there exists a locally finite cover $\omega = \{W_\alpha\} \in \operatorname{cov}\mathscr{U}$ of a neighbourhood $\mathscr{U}$ of A such that $\operatorname{diam}\psi(W_\alpha \cap A) < \varepsilon/3$. Fix points $r_\alpha \in \psi(W_\alpha \cap A)$ and a partition of unity $\{\lambda_\alpha\colon W_\alpha \to [0,1]\}_{\alpha\in\Delta}$ subordinated to the cover ω. Then the required map χ is given by the formula $\chi(z) = \sum_{\alpha\in\Delta} \lambda_\alpha(z)\cdot r_\alpha \in \mathbb{R}^n$.

There exists by Lemma 1 a map $\chi\colon \mathbb{W} \to \mathbb{R}^n$ defined on a G-neighbourhood $\mathbb{W}$, $\mathbb{A} \subset \mathbb{W} \subset \mathbb{Z}$, such that $\operatorname{dist}(\chi\restriction_{\mathbb{A}}, \varphi) < \varepsilon$ (see the footnote[5]). In general the map χ is not equivariant, but the continuous map $\theta\colon \mathbb{W} \to \mathbb{R}^n$ given by the formula

$$\theta(z) = \int_G g^{-1}\cdot\chi(g\cdot z)\,\partial\mu,$$

where μ is a Haar measure on G, is already equivariant. Since $\varphi(g\cdot z) = g\cdot\varphi(z)$ for all $z \in \mathbb{A}$ and $\operatorname{dist}(\chi\restriction_{\mathbb{A}}, \varphi) < \varepsilon$, it follows that $\operatorname{dist}(\varphi, \theta\restriction_{\mathbb{A}}) < \varepsilon$. Hence $\theta(\mathbb{A}) \subset \mathrm{N}(\mathbb{X};\varepsilon) \subset \mathbb{V}$.

It is clear that $\mathbb{U} \rightleftharpoons \theta^{-1}(\mathbb{V})$ contains $\mathbb{A}$, and $r\circ\theta$ is the required G-map $\psi\colon \mathbb{U} \to \mathbb{X}$.

The proof of the following metatheorem due to Palais [18] is based on the stabilization of nested sequences of compact Lie groups.

Proposition 4. *Let $\mathscr{P}(H)$ be a property that depends on a compact Lie group H. Suppose that $\mathscr{P}(H)$ holds for the trivial group $H = \{e\}$ and $\mathscr{P}(H)$ holds if $\mathscr{P}(K)$ is true for each proper subgroup $K < H$. Then $\mathscr{P}(H)$ is true for all compact Lie groups H.*

By the *fibrewise product of spaces C and B with respect to maps* $g\colon C \to A$ and $f\colon B \to A$ we mean the subset $\{(c,b) \mid g(c) = f(b)\} \subset C \times B$, which is denoted by $C_g \times_f B$. The projections of $D = C_g \times_f B$ onto the factors C and B are denoted by $\check{f}\colon D \to C$ and $\check{g}\colon D \to B$. The maps $\check{f}$ and $\check{g}$ are called the *maps parallel to* f and g, respectively, and we write for brevity $\check{f} \parallel f$ and $\check{g} \parallel g$. Note that the map $f\circ\check{g} = g\circ\check{f}\colon D \to A$ is the product of g and f in the category TOP_A of all

[5] In view of the closedness of the orbit projection $\pi\colon \mathbb{Z} \to Z$ one can insert between $\mathbb{A}$ and an arbitrary neighbourhood $\mathscr{U}$ an invariant neighbourhood $\mathbb{W}$, $\mathbb{A} \subset \mathbb{W} \subset \mathscr{U}$.

spaces over A. The most important example of the fibrewise product in the theory of compact transformation groups is supplied by isovariant maps.

Proposition 5. *Let $h\colon \mathbb{Y} \to \mathbb{X}$ be an isovariant map and $\widetilde{h}\colon Y \to X$ the map of the orbit spaces induced by h. Then $\mathbb{Y}$ is the fibrewise product $Y_{\widetilde{h}} \times_{\pi_{\mathbb{X}}} \mathbb{X}$; moreover, h and $\widetilde{h}$, as well as the orbit projections $\pi_{\mathbb{Y}}$ and $\pi_{\mathbb{X}}$ are parallel.*

We denote by $\mathscr{D}$ a commutative square diagram in the category EQUIV with closed $\mathbb{A} \subset \mathbb{Z}$:

$$\begin{array}{ccc} \mathbb{A} & \xrightarrow{\varphi} & \mathbb{X} \\ \cap & & \downarrow f \\ \mathbb{Z} & \xrightarrow{\psi} & \mathbb{Y} \end{array}$$

We shall say that

1) the G-map φ is a partial lifting of the G-map ψ with respect to f;

2) the problem of extension of a partial lifting for $\mathscr{D}$ is globally (locally) solvable if there exists a G-map $\widehat{\varphi}\colon \mathbb{Z} \to \mathbb{X}$ ($\widehat{\varphi}\colon \mathbb{U} \to \mathbb{X}$, where $\mathbb{U} \subset \mathbb{Z}$ is a neighbourhood of $\mathbb{A}$), extending φ such that $f \circ \widehat{\varphi} = \psi$ ($f \circ \widehat{\varphi} = \psi\restriction_{\mathbb{U}}$);

3) $\widehat{\varphi}\colon \mathbb{Z} \to \mathbb{X}$ $[\widehat{\varphi}\colon \mathbb{U} \to \mathbb{X}]$ is a global (local) lifting of the G-map ψ.

If $\mathbb{Z} = \mathbb{Y}$, then the problem of global extension of a partial lifting for $\mathscr{D}$ is transformed into the *problem of global extension of a partial section of f*.

Definition 1. A morphism $f\colon \mathbb{X} \to \mathbb{Y}$ in the category EQUIV is called *equivariantly soft* (*locally equivariantly soft*) if for each commutative square diagram $\mathscr{D}$ in the category EQUIV the problem of extending a partial lifting is globally (locally) solvable.

Finally, we give a sufficient condition under which the topology is generated by subspaces. Let the topological space (D, τ_D) be represented as a union $A \cup B$ of its subspaces. Consider the *weak topology τ_w on D generated by A and B*: $U \in \tau_w$ if and only if $A \cap U \subset A$ and $B \cap U \subset B$ are open. We say that the subspaces A and B generate the topology of D if the weak topology τ_w coincides with τ_D. It is easy to check that the subspaces A and B generate the topology of D if A and B are simultaneously closed or simultaneously open.

We say that A' and $B' \subset D$ are separated if $B \cap \operatorname{Cl}_D A' = A' \cap \operatorname{Cl}_D B' = \varnothing$. By [19], Theorem 2.1.7 the separation of A' and $B' \subset D$ is equivalent to the following: $A' \subset \operatorname{Int}_D(D \setminus B')$ and $B' \subset \operatorname{Int}_D(D \setminus A')$. Apparently, the following sufficient condition to generate the topology of D by its subspaces is known to experts.

Proposition 6. *If $A \setminus B$ and $B \setminus A$ are separated, then the subspaces A and B generate the topology of D.*

The converse is valid for metric space D: let $a \in A \setminus B$ and let the sequence $\{x_n\} \subset B \setminus A$ converge to a. Then $D \setminus \{x_n\}$ is open in both subspaces but not open in D.

In this paper Proposition 6 will be used in the following situation.

Theorem 13. *Let $D = Y \cup Q \cup U$, moreover, let Q and U be open subspaces of D. Then $Y \cup Q$ and $U \cup U^\star$ generate the topology of D, where $U^\star \rightleftharpoons \operatorname{Cl}_D(U) \cap Y$ is the set of points adherent to U lying in Y.*

Proof. Since $U \subset D$ is open, it follows that $(U \cup U^{\star}) \setminus (Y \cup Q) \subset U$. Therefore, it is sufficient to show that $E \rightleftharpoons (Y \cup Q) \setminus (U \cup U^{\star})$ is contained in $\operatorname{Int}_D(Y \cup Q)$. It is easy to check that $Y \setminus (U \cup U^{\star}) = Y \setminus \operatorname{Cl}_D U$. Therefore,

$$E = (Y \setminus \operatorname{Cl}_D U) \cup (Q \setminus (U \cup U^{\star})) \subset (D \setminus \operatorname{Cl}_D U) \cup Q.$$

Hence E is contained in the open set $(D \setminus \operatorname{Cl}_D U) \cup Q$, which obviously lies in $Y \cup Q$.

§ 3. *P*-orbit projection

Let $P \lhd G$, $G/P = H$, and let $\pi\colon G \to H$, $\pi(g) = g \cdot P$ be the natural epimorphism. On the G-space $\mathbb{X}$ we consider the equivalence relation $x \sim x' \iff x' \in P \cdot x$. Then the quotient space $\mathbb{X}/P$ defined by this relation coincides with $\{P \cdot x \mid x \in \mathbb{X}\}$. It is clear that $\mathbb{Y} = \mathbb{X}/P$ is an H-space: $(g \cdot P) \cdot (P \cdot x) = P \cdot (g \cdot x)$. If $y = P \cdot x$, then the stabilizer H_y coincides with $G_x \cdot P$. The quotient map $f\colon \mathbb{X} \to \mathbb{X}/P$ is called a *P-orbit projection*. If $P = G$, then f coincides with the orbit projection $p\colon \mathbb{X} \to \mathbb{X}/G$. Since the composite of the P-orbit projection f and the H-orbit projection of $\mathbb{Y}$ is perfect, f is a perfect surjection and has the following properties:

(1) $f(gx) = \pi(g) \cdot f(x)$ for all $x \in \mathbb{X}$ and $g \in G$;

(2) $\pi(G_x) = H_{f(x)}$ for all $x \in \mathbb{X}$;

(3) if $f(x) = f(x')$, then x and x' belong to the same G-orbit.

The following fact shows that these properties characterize a P-orbit projection completely.

Proposition 7. *Let $\pi\colon G \to H$ be an epimorphism of compact groups and $P = \operatorname{Ker} \pi$. A perfect surjection $f\colon \mathbb{X} \to \mathbb{Y}$ from a G-space $\mathbb{X}$ onto an H-space $\mathbb{Y}$ is a P-orbit projection if and only if f has properties* (1)–(3).

Proof. We consider the P-orbit projection $\varphi\colon \mathbb{X} \to \mathbb{X}/P = \mathbb{Z}$ and define the map $\theta\colon \mathbb{Z} \to \mathbb{Y}$ by the formula $\theta(P \cdot x) = f(x)$. It follows by (1) that θ is well defined and equivariant. Since f is perfect, θ is a perfect surjection and therefore the induced map $\widetilde{\theta}$ of orbit spaces is also perfect and surjective.

Let $z = P \cdot x$ and $z' = P \cdot x'$. If $\theta(z) = \theta(z')$, then $f(x) = f(x')$. In view of (3), x and x' lie on the same G-orbit. Therefore, z and z' lie on the same G/P-orbit. Hence $\widetilde{\theta}$ is a homeomorphism.

The map θ preserves the orbit type of points as on the one hand $H_{\theta(z)} = H_{f(x)}$ and on the other hand $H_z = G_x \cdot P = \pi(G_x)$. By (2) we have $H_{\theta(z)} = H_z$. Therefore, θ is an isovariant map inducing on the orbit spaces a homeomorphism. By Theorem 10 the map θ is an equimorphism.

For a compact group G we consider the Lie series $\{P_\alpha \lhd G\}$ of normal closed subgroups indexed by ordinals $\alpha < \omega$ (see [20]). This means that

(5) $P_1 = G$, $P_\beta < P_\alpha$ for all $\alpha < \beta$, $P_\alpha / P_{\alpha+1}$ is a compact Lie group for all $\alpha < \omega$, $P_\alpha = \bigcap\{P'_\alpha \mid \alpha' < \alpha\}$ for each limit ordinal α, and also $\bigcap\{P_\alpha \mid \alpha < \omega\} = \{e\}$.

In this case G is the limit $\varprojlim\{G/P_\alpha, \varphi_\alpha^\beta\}$ of the inverse system of quotient groups $\{G/P_\alpha\}$ and natural epimorphisms $\varphi_\alpha^\beta\colon G/P_\beta \to G/P_\alpha$, $\alpha < \beta$. The known result of [21], Ch. 2, § 6, Theorem 11 can be slightly generalized.

Lemma 2. *Let $f^\beta_\alpha\colon \mathbb{X}_\beta \to \mathbb{X}_\alpha$, $\alpha < \beta$, be the natural projection from $\mathbb{X}_\beta = \mathbb{X}/P_\beta$ in $\mathbb{X}_\alpha = \mathbb{X}/P_\alpha$. Then f^β_α is a P_α/P_β-orbit projection and the map $f\colon \mathbb{X} \to \varprojlim \{\mathbb{X}_\alpha, f^\beta_\alpha\}$ defined by the formula $f(x) = \{P_\alpha \cdot x\}$ is an equimorphism.*

The proof of Lemma 2 consists in a straightforward application of the equimorphism criterion (Theorem 10).

The converse result is also valid.

Lemma 3. *Let $\{P_\alpha \lhd G\}$ be the Lie series, P^β_α, $\alpha < \beta$, the kernel of the homomorphism φ^β_α, and $g^\beta_\alpha\colon \mathbb{Z}_\beta \to \mathbb{Z}_\alpha$ a P^β_α-orbit projection with $g^\beta_\alpha \circ g^\gamma_\beta = g^\gamma_\alpha$ for all $\alpha < \beta < \gamma$. Then $\mathbb{Z} = \varprojlim \{\mathbb{Z}_\alpha, g^\beta_\alpha\}$ is a G-space and $\mathbb{Z}_\alpha = \mathbb{Z}/P_\alpha$.*

§ 4. Extensor subgroups

Definition 2. A closed subgroup $H < G$ of a compact group G is called a $\mathscr{P}$-subgroup if the homogeneous space G/H is finite-dimensional and locally connected.

Pontryagin [20] proved that $H < G$ is a $\mathscr{P}$-subgroup if and only if one of the following properties holds:

(1) there exists a normal subgroup $P \lhd G$ such that $P < H$ and G/P is a compact Lie group;

(2) G/H is a topological manifold.

Therefore, G/H is metrizable for each $\mathscr{P}$-subgroup $H < G$. It is known [20] that each compact group contains arbitrarily small normal $\mathscr{P}$-subgroups. Hence the following fact is valid.

Proposition 8. *Let G be a compact group and $\mathbb{X}$ a G-space. Then*

(3) *for each neighbourhood $\mathscr{O}(H) \subset G$ of the subgroup $H < G$ there exists a $\mathscr{P}$-subgroup $H' < G$ such that $H \subset H' \subset \mathscr{O}(H)$;*

(4) *for each neighbourhood $\mathscr{O}(x)$ of the point $x \in \mathbb{X}$ there exists a normal $\mathscr{P}$-subgroup $P \lhd G$ such that the P-orbit projection $p\colon \mathbb{X} \to \mathbb{X}/P \rightleftharpoons \mathbb{Y}$ has a small preimage of $y \rightleftharpoons P \cdot x$, $p^{-1}(y) \subset \mathscr{O}(x)$. Moreover, there exists a neighbourhood $\mathscr{W} \subset \mathbb{X}/P$ of y such that $p^{-1}(\mathscr{W}) \subset \mathscr{O}(x)$.*

Property (3) easily implies that

(5) G/H is metrizable if and only if $H < G$ is an intersection of countably many $\mathscr{P}$-subgroups.

If $H < G$ is a $\mathscr{P}$-subgroup, then, in view of (1) and the slice theorem [13], it follows that G/H is G-ANR. The proof of the converse fact is based on the existence of a metric G-space, in which there is a nowhere dense orbit equimorphic to G/H such that the stabilizers of the remaining points are $\mathscr{P}$-subgroups. Hence it follows that

(6) $H < G$ is a $\mathscr{P}$-subgroup if and only if G/H is a metric G-ANE.

This equivalence (6) expressing the main property of $\mathscr{P}$-subgroups justifies their alternative name of *extensor subgroups*. We also draw the reader's attention to a result in [22], from which it follows that

(7) $H < G$ is a $\mathscr{P}$-subgroup if and only if G/H is locally contractible.

We list the basic properties of extensor subgroups. One can show that the subgroup $H < G$ is extensor if and only if G admits an orthogonal action on $\mathbb{R}^n$ such that G/H is equimorphic to the orbit of a point. It is clear that each subgroup $H < G$ in a compact Lie group G as well as each clopen subgroup $H < G$ are extensor subgroups.

It easily follows by (1) that the property of being an extensor subgroup is inherited after a passage to the larger subgroup. As is well-known, a compact group is a Lie group if and only if it contains no small subgroups [20]. Hence it follows that the quotient group $G/(P_1 \cap P_2)$, $P_i \triangleleft G$, is a Lie group if and only if each G/P_i is a Lie group.

Proposition 9. *The intersection of finitely many extensor subgroups is an extensor subgroup.*

This proposition cannot be improved: if an extensor subgroup $H < G$ is an intersection of a family $\{H_\alpha < G\}$ of extensor subgroups, then H is an intersection of finitely many subgroups H_{α_i}.

Since each closed subgroup of a compact Lie group is again a compact Lie group [20], it easily follows that

(8) if $H < G$ is an extensor subgroup and $K < G$, then $H \cap K < K$ is an extensor subgroup.

The proof of the next fact follows from the definition of $\mathscr{P}$-subgroups and Hurewicz's theorem on closed maps that lower dimension [23].

Proposition 10. *Let $K < L < G$ and $K < L$ be an extensor subgroup. Then $K < G$ is an extensor subgroup if and only if $L < G$ is an extensor subgroup.*

If G is not a compact Lie group, then by (7) we obtain $G \notin \mathrm{ANE}$. Let $\mathbb{X} \in \mathrm{ANE}$ be a free G-space (which exists). It is easy to see that each invariant open set of it is not homeomorphic to a product $G \times U$, and therefore in this case the slice theorem fails [24]. But if we weaken the requirement on a slice, then one can assert the following.

Theorem 14 (on an approximate slice of G-space [25], [26]). *Let a compact group G act on a G-space $\mathbb{X}$. Then for each neighbourhood $\mathscr{O}(x)$ of $x \in \mathbb{X}$ there exists a neighbourhood $\mathscr{V} = \mathscr{V}(e)$ of the unit $e \in G$, an extensor subgroup $K < G$, $G_x < K$, and an invariant neighbourhood $\mathbb{U} = \mathbb{U}(x)$ admitting a slice map $\alpha\colon \mathbb{U} \to G/K$ such that $x \in \alpha^{-1}(\mathscr{V} \cdot [K]) \subset \mathscr{O}(x)$.*

Proof. Let $P \triangleleft G$ be a normal $\mathscr{P}$-subgroup and $\mathscr{W}$ a neighbourhood of $y = P \cdot x$ taken from Proposition 8, (4). Since $\mathbb{Y} = \mathbb{X}/P$ is naturally endowed with an action of the compact Lie group $G' = G/P$ and $G'_y = G_x \cdot [P] < G'$, there exists by the slice theorem for compact Lie groups a neighbourhood $\mathscr{V}(e)$ and a slice map $\alpha\colon \mathbb{V} \to G'/G'_y \cong G/(G_x \cdot P)$ defined on a G'-neighbourhood $\mathbb{V}$ of the orbit $G'(y)$ such that $\alpha^{-1}(\mathscr{V}(e) \cdot [G'_y]) \subset \mathscr{W}$. We can easily check that $K \rightleftharpoons G_x \cdot P < G$ is an extensor subgroup and the composite $\alpha \circ p\colon p^{-1}(\mathbb{V}) \to G/K \in G\text{-ANE}$ is the desired slice map.

Definition 3. A conjugacy class (H) is said to be *extensor* if $H < G$ is an extensor subgroup. We denote by $\mathbb{X}_{\mathscr{E}}$ the set of all extensor points of $\mathbb{X}$, that is, of points $x \in \mathbb{X}$ for which $G_x < G$ is an extensor subgroup.

We say that a G-subspace $\mathbb{Y} \subset \mathbb{X}$ is G-dense if $\mathbb{Y}^H \subset \mathbb{X}^H$ is dense for each subgroup $H < G$. In [25] with the help of Theorem 14 on an approximate slice it was shown that

(9) $\mathbb{X}_{\mathscr{E}} \subset \mathbb{X}$ is G-dense if and only if $\mathbb{X}$ is a G-ANE for the class of metric G-spaces with zero-dimensional orbit space, $\mathbb{X} \in G$-ANE(0);

(10) a normed linear G-space $\mathbb{L}$ is a G-AE if and only if $\mathbb{L}_{\mathscr{E}} \subset \mathbb{L}$ is G-dense (the equivariant Dugundji theorem).

There exists an example of a normed linear G-space $\mathbb{L} \notin G$-AE for which $\mathbb{L}_{\mathscr{E}} \subset \mathbb{L}$ is dense (but not G-dense).

The following results known for compact Lie groups G (see [14], Example 7.6.4) can be extended to a more general setting.

Theorem 15. *Let H and K be subgroups of a compact group G such that $H < K$ is an extensor subgroup. Then the natural projection $p\colon G/K \to G/H$ is locally equivariantly soft.*

§ 5. The tube structure on orbit projections

We consider the epimorphism $\pi\colon G \to H$ of compact groups with kernel P being a Lie group. Let $K < G$ be an extensor subgroup and $\pi(K) \rightleftharpoons L < H$. Since $K < \pi^{-1}(L) < G$, it follows by Proposition 10 that

(1) $K < \pi^{-1}L$ and $\pi^{-1}(L) < G$ are extensor subgroups,

and hence

(2) $G/K \in G$-ANE and $G/\pi^{-1}(L) \cong H/L \in H$-ANE.

Let $\kappa\colon G/K \to H/L$ be the composite of the natural epimorphism $\alpha\colon G/K \to G/\pi^{-1}L$, which is generated by the embedding $K < \pi^{-1}L$, and the isomorphism $\beta\colon G/\pi^{-1}L \to H/L$. By (1) and Theorem 15 we have

(3) the maps $\alpha\colon G/K \to G/\pi^{-1}L$ and $\kappa = \beta \circ \alpha\colon G/K \to H/L$ are locally equivariantly soft.

Since $\kappa(g \cdot [K]) = \pi(g) \cdot [L]$ and $\pi(K) = L$, it follows that

(4) $\kappa(g \cdot [K]) = [L]$ if and only if $g \in P \cdot K$.

We say that the P-orbit projection $f\colon \mathbb{X} \to \mathbb{Y}$ has a κ-tube structure generated by the slice maps $\varphi\colon \mathbb{X} \to G/K \in G$-ANE and $\psi\colon \mathbb{Y} \to H/L \in H$-ANE if they close the following diagram $\mathscr{A}$ up to a commutative diagram: $\kappa \circ \varphi = \psi \circ f$,

$$\begin{array}{ccc} \mathbb{X} & \xrightarrow{\varphi} & G/K \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \kappa} \\ \mathbb{Y} & \xrightarrow{\psi} & H/L \end{array}$$

The κ-tube structure on f is said to be *nontrivial* if κ is not a bijection. This is equivalent to K either being a proper subgroup of $\pi^{-1}L$ or $P \setminus K \neq \varnothing$.

It is clear that $\mathbb{X} = G \times_K \mathbb{S}$ and $\mathbb{Y} = H \times_L \mathbb{T}$, where $\mathbb{S} = \varphi^{-1}([K])$ is a K-space and $\mathbb{T} = \psi^{-1}([L])$ is an L-space. In view of the commutativity $\kappa \circ \varphi = \psi \circ f$ it can be easily seen that $f(\mathbb{S}) \subset \mathbb{T}$.

Let $y \in \mathbb{T}$, $x \in f^{-1}(y)$ and $\varphi(x) = g \cdot K$. Then $[L] = \psi(y) = \psi(f(x)) = \kappa(\varphi(x)) = \kappa(g \cdot K)$. By (4), $g \in P \cdot K$. Now it is easy to check that $x' \rightleftharpoons g^{-1} \cdot x \in \mathbb{S}$ and $f(x') = y$. Hence the following result is proved.

Proposition 11. $f(\mathbb{S}) = \mathbb{T}$ *and the map* $f\restriction$ *is perfect.*

Proposition 12. *Let $f\colon \mathbb{X} \to \mathbb{Y}$ be a P-orbit projection. If $x \notin \mathbb{X}^P$, then the restriction of f to the orbit $G(x)$ has a nontrivial tube structure.*

Proof. Since $P \setminus G_x \neq \varnothing$, there exists by Proposition 8, (3) an extensor subgroup $K<G$ such that $P\setminus K\neq\varnothing$ and $G_x<K$. It is clear that $G_{f(x)}$ coincides with $G_x \cdot P$. The embeddings of subgroups $G_x < K < K\cdot P$ and $G_x < G_{f(x)} < K \cdot P$ naturally generate G-maps

$$\varphi\colon G(x) \to G/K, \quad \psi\colon G(f(x)) \to G/(K\cdot P) \quad \text{and} \quad \kappa\colon G/K \to G/(K\cdot P),$$

which generate on the P-orbit projection $f{\restriction}\colon G(x) \to G(f(x))$ a nontrivial tube structure.

The following two Lemmas 4 and 5 show that if the P-orbit projection f has a nontrivial tube structure, then f is generated by a Q-orbit projection with the simpler group G. They explain the origin of the term ‘a tube structure on a map’.

Lemma 4. *Let f have a nontrivial tube structure given by an epimorphism κ. Then $f(\mathbb{S}) = \mathbb{T}$ and the map $f{\restriction}\colon \mathbb{S} \to \mathbb{T}$ is a Q-orbit projection, where Q is the kernel of the epimorphism $\pi' = \pi{\restriction}\colon K \to L$ and a proper subgroup of P.*

Proof. Recall that $f{\restriction}\colon \mathbb{S} \to \mathbb{T}$ is a perfect surjection. Let us verify properties (1)–(3) in Proposition 7 for $f{\restriction}$: property (1) and a part of (3) obviously hold. Property (2) holds since for $s \in \mathbb{S}$ we have $K_s = G_s$ and $L_{f(s)} = H_{f(s)}$.

Since $K \subsetneqq \pi^{-1}L$, P is not contained in K. But $Q = \operatorname{Ker}\pi'$ coincides with $K \cap P$ and hence is a proper subgroup of P. Thus, $Q < P$, but $Q \neq P$.

We now consider the converse situation: there exist an epimorphism $\pi'\colon K \to L$ of compact groups and a Q-orbit projection $f'\colon \mathbb{S} \to \mathbb{T}$, where $\mathbb{S}$ is a K-space, $\mathbb{T}$ is an L-space and $Q = \operatorname{Ker}\pi'$. Let $K < G$ and $L < H$ be extensor subgroups and $\pi\colon G\to H$ an epimorphism extending π'. We denote by $\kappa\colon G/K \to H/L$ the composite of the natural epimorphisms $\alpha\colon G/K \to G/\pi^{-1}L$ and $\beta\colon G/\pi^{-1}L \to H/L$. Then the formula

$$f([g,s]_K) = [\pi(g), f'(s)]_L$$

correctly defines the map $f = \pi \times f'\colon G\times_K \mathbb{S} \to H \times_L \mathbb{T}$. It is straightforward to check that if f' is perfect, then so is f.

Lemma 5. *The map $f\colon G\times_K \mathbb{S} \to H\times_L \mathbb{T}$ is a P-orbit projection for $P = \operatorname{Ker}\pi$. Furthermore, the natural slice maps $\varphi\colon G\times_K \mathbb{S} \to G/K$, $\psi\colon H\times_L\mathbb{T} \to H/L$, and also the epimorphism $\kappa\colon G/K \to H/L$ define a tube structure on the P-orbit projection f, that is, κ closes the following diagram up to a commutative one: $\kappa\circ\varphi = \psi\circ f$.*

Proof. Let $x = [g,s]_K$ and $x' = [g',s']_K \in G\times_K \mathbb{S}$. If $f(x) = f(x')$, then $\pi(g') = \pi(g)\cdot l^{-1}$ and $f'(s') = l\cdot f'(s)$ for $l \in L$.

Since π' is an epimorphism, $l = \pi(k)$, $k\in K$, and therefore $f'(s') = \pi(k)\cdot f'(s) = f'(k\cdot s)$. Since f' is a Q-orbit projection, s' and $k\cdot s$ lie on the same K-orbit. Hence it is easily deduced that x' and x lie on the same G-orbit.

All the other characterization properties for P-orbit projections from Proposition 7 are checked straightforwardly and we leave this to the reader.

§ 6. Theorem on the slice of a G-map

Assertions on *extension of tube structure on maps* play an important role in inductive arguments. Their proofs essentially use the equivariant softness of G-maps of homogeneous spaces (see Theorem 15).

Theorem 16 (on extension of tube structure on maps). *Let $\mathscr{B}$ be a commutative diagram*

$$\begin{array}{ccc} \mathbb{X} = \operatorname{Cl}\mathbb{X} & \overset{j}{\hookrightarrow} & \mathbb{W} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \widehat{f}} \\ \mathbb{Y} = \operatorname{Cl}\mathbb{Y} & \overset{i}{\hookrightarrow} & \mathbb{Z} \end{array}$$

in which $f\colon \mathbb{X} \to \mathbb{Y}$ and $\widehat{f}\colon \mathbb{W} \to \mathbb{Z}$ are P-orbit projections. If f has a κ-tube structure generated by the slice maps $\varphi\colon \mathbb{X} \to G/K$ and $\psi\colon \mathbb{Y} \to H/L$, then there exist invariant neighbourhoods $\mathbb{B}$, $\mathbb{Y} \subset \mathbb{B} \subset \mathbb{Z}$, and $\mathbb{A} \rightleftharpoons \widehat{f}^{-1}(\mathbb{B})$, $\mathbb{X} \subset \mathbb{A} \subset \mathbb{W}$, such that the P-orbit projection $\widehat{f}\restriction\colon \mathbb{A} \to \mathbb{B}$ has a κ-tube structure generated by the slice maps $\widehat{\varphi}\colon \mathbb{A} \to G/K$ and $\widehat{\psi}\colon \mathbb{B} \to H/L$, which are G-extensions of φ and ψ, respectively.

Proof. Let $\mathscr{A}$ be a diagram that generates the κ-tube structure on f. Because the subgroup $K < G$ is extensor, as was mentioned before, G admits an orthogonal action on $\mathbb{R}^N$ such that G/K is equimorphic to an orbit in $\mathbb{R}^N$ (note also that $\mathbb{R}^N \in G$-AE; see [13]). Hence we can assume without loss of generality that

(1) G/K is an orbit in $H/L \times \mathbb{R}^N$; moreover, κ coincides with the restriction of the projection $\mathrm{pr}_1\colon H/L \times \mathbb{R}^N \to H/L$ to G/K.

Since by (3) from the previous section the map $\kappa\colon G/K \to H/L$ is locally equivariantly soft, there exists a fibrewise equivariant retraction $r\colon \mathbb{V} \to G/K$ of an invariant neighbourhood $\mathbb{V} \subset H/L \times \mathbb{R}^N$ of the orbit G/K, that is, $\mathrm{pr}_1 \circ r = \mathrm{pr}_1$ (see also Theorem 15). In view of $H/L \in H$-ANE, there exists an H-map $\widetilde{\psi}\colon \mathbb{B}' \to H/L$ given on an invariant neighbourhood $\mathbb{B}'$, $\mathbb{Y} \subset \mathbb{B}' \subset \mathbb{Z}$, such that $\widetilde{\psi} = \operatorname{ext}\psi$.

We represent the map $\varphi\colon \mathbb{X} \to G/K \hookrightarrow H/L \times \mathbb{R}^N$ as (φ_1, φ_2). Since $\mathbb{R}^N \in G$-AE, there exists a G-map $\chi\colon \mathbb{W} \to \mathbb{R}^N$ extending $\varphi_2\colon \mathbb{X} \to \mathbb{R}^N$.

Let $\mathbb{A}' \rightleftharpoons \widehat{f}^{-1}(\mathbb{B}')$. Consider the G-map $\sigma\colon \mathbb{A}' \to H/L \times \mathbb{R}^N$ given by the formula $\sigma = (\widetilde{\psi} \circ \widehat{f}) \times \chi$. It is clear that $\mathbb{A} \rightleftharpoons \sigma^{-1}(\mathbb{V})$ and $\mathbb{B} \rightleftharpoons \widehat{f}(\mathbb{A})$ are invariant neighbourhoods of $\mathbb{X}$ and $\mathbb{Y}$, respectively. One can check using the fibrewise retraction σ that $\widehat{\varphi} \rightleftharpoons r \circ \sigma\colon \mathbb{A} \to G/K$ and $\widehat{\psi} \rightleftharpoons \widetilde{\psi}\restriction_{\mathbb{B}}$ generate a κ-tube structure on $\widehat{f}\restriction_{\mathbb{A}}$, that is, $\kappa \circ \widehat{\varphi} = \widehat{f}\restriction \circ \widehat{\psi}$.

Consider now the situation when the embeddings j and i from Theorem 16 are not necessarily closed. In this case the conclusion of the theorem remains true, provided that one requires no matching of the present tube structures and the constructed ones.

Theorem 17 (on expansion of tube structure on a map). *Let $\mathscr{B}$ be a commutative diagram*

$$\begin{array}{ccc} \mathbb{X} & \overset{j}{\hookrightarrow} & \mathbb{W} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \widehat{f}} \\ \mathbb{Y} & \overset{i}{\hookrightarrow} & \mathbb{Z} \end{array}$$

in which $f\colon \mathbb{X} \to \mathbb{Y}$ *and* $\widehat{f}\colon \mathbb{W} \to \mathbb{Z}$ *are* P*-orbit projections for a compact Lie group* P*. If* f *has a nontrivial tube structure, then there exist invariant neighbourhoods* $\mathbb{B}$, $\mathbb{Y} \subset \mathbb{B} \subset \mathbb{Z}$, *and* $\mathbb{A} \rightleftharpoons \widehat{f}^{-1}(\mathbb{B})$, $\mathbb{X} \subset \mathbb{A} \subset \mathbb{W}$, *such that the* P*-orbit projection* $\widehat{f}{\restriction}\colon \mathbb{A} \to \mathbb{B}$ *also has a nontrivial tube structure.*

Proof. In the main, this proceeds along the lines of Theorem 16 on extension of tube structure on maps if we apply Theorem 12 on approximate G-extension of maps where appropriate. Let us consider a commutative diagram $\mathscr{A}$,

$$\begin{array}{ccc} \mathbb{X} & \xrightarrow{\varphi} & G/K \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \kappa} \\ \mathbb{Y} & \xrightarrow{\psi} & H/L \end{array}$$

generating a nontrivial tube structure on f. We shall assume that condition (1) holds. Fix the fibrewise (with respect to pr_1) equivariant retraction $r\colon \mathbb{V} \to G/K$ of some invariant neighbourhood $\mathbb{V} \subset H/L \times \mathbb{R}^N$ of the orbit G/K, $\mathrm{pr}_1 \circ r = \mathrm{pr}_1$. Since $H/L \in H$-ANE, there exists by Theorem 12 an H-map $\widetilde{\psi}\colon \mathbb{B}' \to H/L$ given on an invariant neighbourhood $\mathbb{B}'$, $\mathbb{Y} \subset \mathbb{B}' \subset \mathbb{Z}$, such that ψ and $\widetilde{\psi}{\restriction}_{\mathbb{Y}}$ are arbitrarily close.

We represent the map $\varphi\colon \mathbb{X} \to G/K \hookrightarrow H/L \times \mathbb{R}^N$ in the diagonal form (φ_1, φ_2). Since $\mathbb{R}^N \in G$-AE, there exists by Theorem 12 a G-map $\chi\colon \mathbb{A}' \to \mathbb{R}^N$ given on an invariant neighbourhood $\mathbb{A}'$, $\mathbb{X} \subset \mathbb{A}' \subset \mathbb{W}$, such that φ_2 and $\chi{\restriction}_{\mathbb{X}}$ are arbitrarily close. Without loss of generality we can assume that $\mathbb{A}' = \widehat{f}^{-1}(\mathbb{B}')$.

Now consider the G-map $\sigma\colon \mathbb{A}' \to H/L \times \mathbb{R}^N$ given by the formula $\sigma = (\widetilde{\psi} \circ \widehat{f}) \times \chi$. If the approximating maps $\widetilde{\psi}$ and χ constructed above were sufficiently close to ψ and φ_2, respectively, then the image $\sigma(\mathbb{A}')$ lies in $\mathbb{V}$. It is clear that $\mathbb{A} \rightleftharpoons \sigma^{-1}(\mathbb{V})$ and $\mathbb{B} \rightleftharpoons \widehat{f}(\mathbb{A})$ are invariant neighbourhoods of $\mathbb{X}$ and $\mathbb{Y}$, respectively. We assert that $\widetilde{\varphi} \rightleftharpoons r \circ \sigma\colon \mathbb{A} \to G/K$ and $\widetilde{\psi}{\restriction}_{\mathbb{B}}$ set a nontrivial tube structure on $\widehat{f}{\restriction}_{\mathbb{A}}$, that is, $\kappa \circ \widetilde{\varphi} = \widehat{f}{\restriction} \circ \widetilde{\psi}{\restriction}_{\mathbb{B}}$.

Corollary 1. *Suppose that a* P*-orbit projection* $f\colon \mathbb{X} \to \mathbb{Y}$ *for a compact Lie group* P *has a nontrivial tube structure and* $\mathbb{Y}$ *is contained in* $\mathbb{Z}$*. Then there exists a commutative diagram* $\mathscr{A}$ *generating a nontrivial tube structure on* f *in which* ψ *has an* H*-extension onto some invariant neighbourhood of* $\mathbb{Y}$*.*

For the proof of Corollary 1 one should repeat the reasoning from Theorem 17 ignoring references to the G-map $\chi\colon \mathbb{A}' \to \mathbb{R}^N$. In more detail, consider the G-map $\sigma_1\colon \mathbb{X} \to H/L \times \mathbb{R}^N$ given by the formula $\sigma_1\colon \mathbb{X} \to H/L \times \mathbb{R}^N$. If the constructed approximating map $\widetilde{\psi}$ is sufficiently close to ψ, then the image $\sigma_1(\mathbb{X})$ lies in $\mathbb{V}$. We assert that $\varphi_1 \rightleftharpoons r \circ \sigma_1\colon \mathbb{X} \to G/K$ and $\psi_1 \rightleftharpoons \widetilde{\psi}{\restriction}_{\mathbb{X}}$ set a nontrivial tube structure on f, that is, $\kappa \circ \varphi_1 = \widehat{f} \circ \psi_1$.

§ 7. Theorem on equivariant homotopic density

We defer the proof of Theorem 8 till the end of the section because we require some auxiliary facts. For a compact group G we denote by $\mathbb{T}$ the discrete union of all homogeneous spaces $G/H \in G$-ANE. It is clear that the metric cone $\operatorname{Con}\mathbb{T}$ over $\mathbb{T}$ has extensor type and its orbit space is the cone over a discrete space.

Theorem 18. *Let $\mathbb{X}$ be a metric G-subspace of a G-space $\mathbb{Y}$ whose orbit space Y is metrizable. Then for each nested family $\{\mathbb{V}_n \subset \mathbb{Y}\}_{n=1}^{\infty}$ of invariant neighbourhoods of $\mathbb{X}$ there exists a G-map $f\colon \mathbb{Y} \to \mathbb{T}_G$, where $\mathbb{T}_G = (\operatorname{Con}\mathbb{T})^{\omega}$, such that*

(a) *the restriction of f to $\mathbb{X}$ is isovariant*;

(b) $f\big(\mathbb{Y} \setminus \bigcap\{\mathbb{V}_n \mid n \geqslant 1\}\big) \subset (\mathbb{T}_G)_{\mathscr{E}}$.

Proof. We can assume without loss of generality that X and Y contain no isolated points. Otherwise one should pass from $\mathbb{X}$ and $\mathbb{Y}$ to $\mathbb{X}\times G^{\omega}$ and $\mathbb{Y}\times G^{\omega}$, respectively.

Since $Y = \mathbb{Y}/G$ is metrizable, there exists a family $\mathscr{B} = \{W_\mu\}_{\mu\in M}$ of open subsets of Y intersecting X such that

(1) $\mathscr{B} = \bigcup_{n=1}^{\infty}\mathscr{B}_n$, $\mathscr{B}_n = \{W_\mu\}_{\mu\in M_n\subset M}$ is a discrete family, $\coprod_{n=1}^{\infty} M_n = M$;

(2) the body of $\mathscr{B}_n$ is contained in $\mathbb{V}_n$ for each n;

(3) the restriction $\mathscr{B}\restriction_X$ generates a basis of X.

We denote by $\mathbb{W}_\mu$ $\pi^{-1}W_\mu$, where $\pi\colon \mathbb{Y}\to Y$ is the orbit projection. Since the slices of $\mathbb{W}_\mu$ (for instance, trivial) exist, the quantity

$$\begin{aligned} i(\mu) \risingdotseq \inf\big\{&\operatorname{diam}(\mathbb{X}\cap\varphi^{-1}(g\cdot[H])) \mid \\ &\varphi\colon \mathbb{W}_\mu \to G/H \text{ is the slice map, } g\in G\big\} \geqslant 0 \end{aligned}$$

is well defined (here we take the diameter with respect to a compatible invariant metric ϱ, which exists on $\mathbb{Y}$ by [18]). In view of the assumption made above, $i(\mu)>0$. It follows from the invariance of σ that both $\mathbb{X}\cap\varphi^{-1}(g\cdot[H]) = \mathbb{X}\cap g\cdot\varphi^{-1}([H])$ and $\mathbb{X}\cap\varphi^{-1}([H])$ have equal diameters. Hence it is sufficient to take $g=e$ in the definition of $i(\mu)$. Of particular interest is the slice map $\varphi_\mu\colon \mathbb{W}_\mu \to G/H_\mu$, $\mu\in M_n$ for which $\operatorname{diam}\big(\mathbb{X}\cap\varphi_\mu^{-1}([H_\mu])\big) < j(\mu) \risingdotseq 2i(\mu)$. It is easy to see that φ_μ has the following important property.

Lemma 6. *If $\mu\in M_n$, then $\operatorname{diam}(\mathbb{X}\cap\varphi_\mu^{-1}([H_\mu])) < 2\operatorname{diam}(\mathbb{X}\cap\varphi^{-1}([H]))$ for any other slice map $\varphi\colon \mathbb{W}_\mu\to G/H$.*

We consider the G-map

$$\psi_\mu \risingdotseq \operatorname{Con}\varphi_\mu\colon \mathbb{Y}\to\operatorname{Con} G/H_\mu,$$

which coincides with (φ_μ,ξ_μ) on $\mathbb{W}_\mu$ and with the vertex $\{*\}$ on the complement to $\mathbb{W}_\mu$ (here $\xi_\mu\colon \mathbb{Y}\to[0,1]$ is a function constant on orbits such that $\xi_\mu^{-1}(0)=\mathbb{Y}\setminus\mathbb{W}_\mu$). It is clear that $\psi_\mu^{-1}(G/H_\mu\times(0,1]) = \mathbb{W}_\mu$ and $G_{\varphi_\mu(y)} = G_{\psi_\mu(y)}$, $y\in\mathbb{W}_\mu$.

Since the family $\mathscr{B}_n$ is discrete, it is easy to see that the formulae $\psi_n\restriction_{\mathbb{W}_\mu} = \psi_\mu$ for $\mu\in M_n$ and $\psi_n\restriction_{\mathbb{Y}\setminus\bigcup\{\mathbb{W}_\mu\mid\mu\in M_n\}} = \{*\}$ define consistently a continuous G-map $\psi_n\colon \mathbb{Y}\to\operatorname{Con}\mathbb{T}$. It is clear that $\psi_n^{-1}(*) = \mathbb{Y}\setminus\bigcup\{\mathbb{W}_\mu \mid \mu\in M_n\}$. We shall show that the diagonal product $\eta\risingdotseq\Delta\psi_n\colon \mathbb{Y}\to(\operatorname{Con}\mathbb{T})^{\omega} = \mathbb{T}_G$ is the desired G-map.

By Theorem 14 on an approximate slice, for each $x\in\mathbb{X}$ and $\alpha>0$ there exists a G-map $r\colon \mathbb{U}(x)\to G/H\in G$-ANE of some neighbourhood $\mathbb{U}(x)\subset\mathbb{Y}$ for which $r^{-1}([H])$ has diameter less than $\alpha/2$. Since by (3) the restriction $\mathscr{B}\restriction_X$ is a basis of X, there exists an index $\mu\in M_n$ such that $x\in\mathbb{W}_\mu\subset\mathbb{U}(x)$. From $\operatorname{diam}(r^{-1}([H]))<\alpha/2$ it follows that $i(\mu)<\alpha/2$, and hence by Lemma 6 we have

(c) $\operatorname{diam}(\mathbb{X}\cap\varphi_\mu^{-1}([H_\mu])) < i(\mu) < \alpha$.

Therefore, $\operatorname{diam}(\mathbb{X} \cap \varphi_\mu^{-1}(g \cdot [H_\mu])) < \alpha$ for all $g \in G$. The existence of slice maps $\{\varphi_{\mu_i} : \mathbb{W}_{\mu_i} \to G/H_{\mu_i}\}_{i \geqslant 1, \mu_i \in M_{n_i}}$ such that $x \in \varphi_{\mu_i}^{-1}(a_i) \subset \mathbb{W}_{\mu_i}$ for all $i \geqslant 1$, where $a_i \rightleftharpoons \varphi_{\mu_i}(x)$, follows from the above, and also

(d) $\{\operatorname{diam} \varphi_{\mu_i}^{-1}(a_i)\} \to 0$.

Since $G_{a_i} = G_{\varphi_{\mu_i}(x)} = G_{\psi_{\mu_i}(x)} \supset G_{\eta(x)} \supset G_x$, it follows that $G_x \subset \bigcap\{G_{a_i} \mid i \geqslant 1\}$. To prove that the stabilizers of x and $\eta(x)$ are equal, it is sufficient to establish the reverse inclusion $\bigcap\{G_{a_i} \mid i \geqslant 1\} \subset G_x$. If $g \in \bigcap\{G_{a_i} \mid i \geqslant 1\}$, then $g \cdot x \in \varphi_{\mu_i}^{-1}(a_i)$ for all $i \geqslant 1$, and by virtue of (d), x and $g \cdot x$ coincide. Hence $g \in G_x$.

Finally, we check (b). If $y \notin \mathbb{V}_n$, then $y \notin \mathbb{V}_{n+m}$. It follows from (2) that all coordinates of $\eta(y)$ apart from the first n coincide with the vertex $\{*\}$. Hence $G_{\eta(y)}$ is the intersection of finitely many extensor subgroups, that is, $G_{\eta(y)}$ is an extensor subgroup (Proposition 9).

Theorem 18 implies the following important results on the structure of solutions of the PEA, which in turn with the help of Theorem 1 enables one to deduce Theorem 2, (4).

Proposition 13. *Let $s' : \mathbb{X} \hookrightarrow \mathbb{Y}'$ be a solution of the PEA for an admissible diagram $\mathscr{D}$. Then*

(4) *if $\mathbb{X}$ is a closed subset of the metric space $\mathbb{Y}'$, then there exists a solution $s : \mathbb{X} \hookrightarrow \mathbb{Y}$ of the PEA for $\mathscr{D}$ such that $s' \geqslant s$ and $\mathbb{Y} \setminus \mathbb{X} \subset \mathbb{Y}_{\mathscr{E}}$;*

(5) *if $\mathbb{X}$ and Y are metrizable, then there exists a solution $s : \mathbb{X} \hookrightarrow \mathbb{Y} \in \mathscr{M}$ of the PEA for $\mathscr{D}$ such that $s' \geqslant s$.*

Proof. By Theorem 18 there exists a G-map $f : \mathbb{Y}' \to \mathbb{T}_G$ such that $f\restriction_{\mathbb{X}}$ is isovariant; moreover, in the case of (4), $f(\mathbb{Y}' \setminus \mathbb{X}) \subset (\mathbb{T}_G)_{\mathscr{E}}$. We consider the fibrewise product $\mathbb{Y} \rightleftharpoons Y_f \times_p \mathbb{T}_G \in \mathscr{M}$, where $p : \mathbb{T}_G \to \mathbb{T}_G/G$ is the orbit projection.

Since $f\restriction_{\mathbb{X}}$ is isovariant, it follows that $\mathbb{X} \subset \mathbb{Y}$. It is easy to check that $s : \mathbb{X} \hookrightarrow \mathbb{Y}$ covers $X \hookrightarrow Y$, and the natural G-map $h : \mathbb{Y}' \to \mathbb{Y}$, $h(y') = (p(y'), f(y'))$, makes the diagram commutative: $h \circ s' = s$ and also $\widetilde{h} = \mathrm{Id}_Y$. It is clear that in the case of (4), $\mathbb{Y} \setminus \mathbb{X} \subset \mathbb{Y}_{\mathscr{E}}$.

Let $\mathbb{Z} \hookleftarrow \mathbb{A} \xrightarrow{f} \mathbb{X}$ be a partial G-map. As an easy application of Proposition 13 to the solution $s : \mathbb{A} \hookrightarrow \mathbb{Z}$ of the closed PEA for the diagram $\mathbb{A} \xrightarrow{p} A \hookrightarrow Z$, we obtain the following.

Proposition 14. *For each partial G-map $\mathbb{Z} \hookleftarrow \mathbb{A} \xrightarrow{f} \mathbb{X} \in G$-AE there exists a G-extension $\widehat{f} : \mathbb{Z} \to \mathbb{X}$ such that $\widehat{f}(\mathbb{Z} \setminus \mathbb{A}) \subset \mathbb{X}_{\mathscr{E}}$. A similar result holds for G-ANE-spaces.*

Proof of Theorem 8. We apply Proposition 14 to the partial G-map

$$\mathbb{X} \times [0,1] \hookleftarrow \mathbb{X} \times \{0\} \xrightarrow{\mathrm{Id}} \mathbb{X} \in G\text{-AE}.$$

The case $\mathbb{X} \in G$-ANE is similarly proved.

§ 8. Problem of extension of *P*-orbit projections

We treat the problem of extending the action in the general context. Let $f : \mathbb{X} \to \mathbb{Y}$ be a P-orbit projection for the kernel P of the epimorphism $\pi : G \to H$ of compact groups, and $i : \mathbb{Y} \hookrightarrow \mathbb{Z}$ an arbitrary H-embedding of $\mathbb{Y}$ into the H-space $\mathbb{Z}$, in which each open invariant subset is paracompact. The resulting diagram $\mathbb{X} \xrightarrow{f} \mathbb{Y} \overset{i}{\hookrightarrow} \mathbb{Z}$ is called *H-admissible*. It is clear that the induced diagram $\mathbb{X} \xrightarrow{p_{\mathbb{Y}}\circ f} Y \overset{\widetilde{i}}{\hookrightarrow} Z$ is admissible in the previous sense. We say that

(1) the generalized problem on extending the action (GPEA) is solvable for the H-admissible diagram $\mathbb{X} \xrightarrow{f} \mathbb{Y} \overset{i}{\hookrightarrow} \mathbb{Z}$ if there exists a G-embedding $j\colon \mathbb{X} \hookrightarrow \mathbb{W}$ into a G-space $\mathbb{W}$ and a P-orbit projection $\widehat{f}\colon \mathbb{W} \to \mathbb{Z}$ such that $\widehat{f}\circ j = i\circ f$;

(2) the generalized problem on extending the action is locally solvable for the H-admissible diagram $\mathbb{X} \xrightarrow{f} \mathbb{Y} \overset{i}{\hookrightarrow} \mathbb{Z}$ if it is solvable for some H-admissible diagram $\mathbb{X} \xrightarrow{f} \mathbb{Y} \overset{i}{\hookrightarrow} \mathbb{R}$, where $\mathbb{Y} \subset \operatorname{Int}\mathbb{R} \subset \mathbb{Z}$;

(3) the generalized problem on extending the action is solvable for P-orbit projections if the GPEA is solvable for each H-admissible diagram $\mathbb{X} \xrightarrow{f} \mathbb{Y} \overset{i}{\hookrightarrow} \mathbb{Z}$.

As will further be shown, the main result, Theorem 1, is reduced to the following key theorem on the solvability of the generalized problem on extending the action.

Theorem 19. *The generalized problem on extending the action is solvable for P-orbit projections, provided that the kernel P is a compact Lie group.*

We observe that Theorems 1 and 19 coincide in the case of a compact Lie group $G = P$. First, we consider the simplest cases of Theorem 19 carrying over its complete proof to the next section.

Lemma 7. *The generalized problem on extending the action for each H-admissible diagram $\mathbb{X} \xrightarrow{f} \mathbb{Y} \overset{i}{\hookrightarrow} \mathbb{Z}$ is solvable, provided that f is a P-orbit projection and i is an open H-embedding.*

Hence it easily follows that

(4) if the generalized problem on extending the action is locally solvable for the H-admissible diagram $\mathbb{X} \xrightarrow{f} \mathbb{Y} \overset{i}{\hookrightarrow} \mathbb{Z}$, then the GPEA is solvable for this diagram.

Proof. We shall construct the G-space $\mathbb{W}$ in such a manner that $\mathbb{W}\setminus\mathbb{X} = \mathbb{Z}\setminus\mathbb{Y}$. With this aim in view, we set $\mathbb{W} \rightleftharpoons \mathbb{X} \sqcup (\mathbb{Z}\setminus\mathbb{Y})$. The basis of $\mathbb{W}$ is generated by all open sets in $\mathbb{X}$ and by sets $\widetilde{O} = f^{-1}(O\cap\mathbb{Y})\sqcup(O\setminus\mathbb{Y})$, where $O \subset \mathbb{Z}$ is an arbitrary open set. The action of G on $\mathbb{W}$ coincides with that of G on $\mathbb{X}$, and it is given on $\mathbb{Z}\setminus\mathbb{Y}$ by the formula $g\cdot y = \pi(g)\cdot y$, $y \in \mathbb{Z}\setminus\mathbb{Y}$. The continuity of the action is easily checked as well as the fact that the map $\widehat{f}\colon \mathbb{W} \to \mathbb{Z}$, which coincides with f on $\mathbb{X}$ and with Id on $\mathbb{Z}\setminus\mathbb{Y}$, is a P-orbit projection extending f.

Let $\mathbb{X} \xrightarrow{f} \mathbb{Y} \overset{i}{\hookrightarrow} \mathbb{Z}$ be an H-admissible diagram. We note that $\mathbb{A} = f^{-1}(f(\mathbb{A}))$ for all $\mathbb{A} \subset \mathbb{X}$, and also $G_x = P\cdot G_x = \pi^{-1}(\pi(G_x))$ if $P < G_x$. Hence it follows by the perfectness of f that

(5) $f{\restriction}_{\mathbb{X}^P} : \mathbb{X}^P \to f(\mathbb{X}^P)$ is an equimorphism of closed subsets of $\mathbb{X}$ and $\mathbb{Y}$.

It is clear that for $\mathbb{X}' \rightleftharpoons \mathbb{X}\setminus\mathbb{X}^P$, $\mathbb{Z}' \rightleftharpoons \mathbb{Z}\setminus\mathrm{Cl}_{\mathbb{Z}}(f(\mathbb{X}^P))$ and $\mathbb{Y}' \rightleftharpoons \mathbb{Y}\cap\mathbb{Z}' = \mathbb{Y}\setminus f(\mathbb{X}^P)$ the following holds:

(6) $(\mathbb{X}')^P = \varnothing$ and the map $f{\restriction}\colon \mathbb{X}' \to \mathbb{Y}'$ is a P-orbit projection.

The following assertion reduces the investigation of the generalized problem on extending the action to the case of the absence of P-fixed-point sets in $\mathbb{X}$.

Lemma 8. *If the generalized problem on extending the action for all H-admissible diagrams with empty P-fixed-point sets is solvable, then the GPEA is solvable for all H-admissible diagrams $\mathbb{X} \xrightarrow{f} \mathbb{Y} \xhookrightarrow{i} \mathbb{Z}$ with $\mathbb{X}^P \neq \varnothing$.*

Proof. Since Z' is hereditarily paracompact, the diagram $\mathbb{X}' \xrightarrow{f\restriction} \mathbb{Y}' \xhookrightarrow{i'} \mathbb{Z}'$ is admissible. Since $(\mathbb{X}')^P = \varnothing$, in view of the hypothesis of the lemma, the GPEA for this diagram is solvable. Therefore there exist a G-embedding $j' : \mathbb{X}' \hookrightarrow \mathbb{W}'$ into a G-space $\mathbb{W}'$ and a P-orbit projection $\widehat{f}\colon \mathbb{W}' \to \mathbb{Z}'$ such that $\widehat{f} \circ j' = i' \circ f\restriction$. We apply Lemma 7 to the natural H-embedding $i''\colon \mathbb{Z}' \hookrightarrow \mathbb{Z}$, which is open: the GPEA for the diagram $\mathbb{W}' \xrightarrow{\widehat{f}} \mathbb{Z}' \xhookrightarrow{i''} \mathbb{Z}$ is solvable. Therefore the GPEA is solvable for the initial diagram.

Now we explain the reduction of Theorem 1 to Theorem 19.

Proposition 15. *The validity of Theorem 19 implies the validity of Theorem 1.*

Proof. Let $\{P_\alpha \lhd G\}$ be a Lie series of G, $\mathbb{X}_\alpha \rightleftharpoons \mathbb{X}/P_\alpha$ (see the notations in Lemma 2). Since $P_1 = G$ and $\mathbb{X}_1 = X$, the embedding $i\colon X \hookrightarrow Y$ can be regarded as the G/P_1-embedding $i_1\colon \mathbb{X}_1 \hookrightarrow \mathbb{Y}_1$, where $\mathbb{Y}_1 \rightleftharpoons Y$ is regarded as a G/P_1-space. Using the line of reasoning by transfinite induction and using Theorem 19 repeatedly (since $P_\alpha/P_{\alpha+1}$ is a compact Lie group), one can construct G/P_α-embeddings $i_\alpha\colon \mathbb{X}_\alpha \hookrightarrow \mathbb{Y}_\alpha$ into the G/P_α-space $\mathbb{Y}_\alpha$ and $P_\alpha^{\alpha+1}$-orbit projections $\widehat{f}_\alpha^{\alpha+1}\colon \mathbb{Y}_{\alpha+1} \to \mathbb{Y}_\alpha$ such that $\widehat{f}_{\alpha+1} \circ i_{\alpha+1} = i_\alpha \circ f_\alpha$ for all $\alpha < \omega$. Then the required G-space $\mathbb{Y}$ is $\varprojlim\{\mathbb{Y}_\alpha, \widehat{f}_\alpha^\beta\}$ and $\mathbb{X} = \varprojlim\{\mathbb{X}_\alpha, f_\alpha^\beta\}$ lies in $\mathbb{Y}$ in a natural manner.

We conclude this section by giving a result on gluing solutions of the GPEA, which will play an important role in performing the transfinite induction in the final part of the paper.

Theorem 20. *Let $\mathbb{T}_1$ and $\mathbb{T}_2 \subset \mathbb{T}$ be H-spaces which generate the topology of the H-space $\mathbb{T}$, let $f_1\colon \mathbb{W}_1 \to \mathbb{T}_1$ and $f_2\colon \mathbb{W}_2 \to \mathbb{T}_2$ be P-orbit projections. Suppose that there exists a G-homeomorphism $h\colon (f_1)^{-1}(\mathbb{T}_0) \to (f_2)^{-1}(\mathbb{T}_0)$, where $\mathbb{T}_0 \rightleftharpoons \mathbb{T}_1 \cap \mathbb{T}_2$, such that $f_2 \circ h = f_1$. Then there exists a solution $s\colon \mathbb{W}_1 \hookrightarrow \mathbb{W}$ of the GPEA for the H-admissible diagram $\mathbb{W}_1 \xrightarrow{f_1} \mathbb{T}_1 \hookrightarrow \mathbb{T}$.*

Proof. It is clear that the equivalence relation $\approx$ on the discrete union $\mathbb{W}_1 \sqcup \mathbb{W}_2$, nontrivial classes of which are pairs $\{t_1, t_2 \mid t_2 = h(t_1), t_1 \in (f_1)^{-1}(\mathbb{T}_0)\}$, is invariant with respect to the action of G; the quotient topology on the quotient space $\mathbb{W} \rightleftharpoons (\mathbb{W}_1 \sqcup \mathbb{W}_2)/\approx$ coincides with the weak topology generated by $\mathbb{W}_1$ and $\mathbb{W}_2$, which naturally lie in $\mathbb{W}$.

Since the map presenting the action of G on $\mathbb{W}_2$ is perfect, it follows that for each compactum $K \subset G$ and for each neighbourhood $\mathscr{U}$, $K \cdot F \subset \mathscr{U}$, where $F \subset \mathbb{W}_2$, there exists a neighbourhood $\mathscr{O}(F) \subset \mathscr{U}$ such that $K \cdot \mathscr{O}(F) \subset \mathscr{U}$. Hence it easily follows that a continuous action of G on $\mathbb{W}$ is given in a consistent manner by the formulae $(g, [t_1]) \mapsto [g \cdot t_1]$ for $t_1 \in \mathbb{W}_1$ and $(g, [t_2]) \mapsto [g \cdot t_2]$ for $t_2 \in \mathbb{W}_2$. The following result obtained by a straightforward verification completes the proof of the theorem.

Lemma 9. *The orbit space $\mathbb{W}/P$ coincides with $\mathbb{T}$ equipped with the weak topology generated by $\mathbb{T}_1$ and $\mathbb{T}_2 \subset \mathbb{T}$.*

§ 9. The proof of Theorem 19

The reasoning will proceed by induction on the compact Lie group P by the use of the Palais metatheorem (Proposition 4). If $|P| = 1$, then π is an isomorphism that trivializes the situation under consideration. We suppose now that for each proper subgroup $Q < P$ the GPEA for each Q-orbit projection is solvable and show that it is solvable for each P-orbit projection $f\colon \mathbb{X} \to \mathbb{Y}$. By Lemma 8 it is sufficient to study the G-space $\mathbb{X}$ without a P-fixed-point set, $\mathbb{X}^P = \varnothing$.

First we consider the case of the P-orbit projection f having a nontrivial tube structure.

Lemma 10. *If f has a nontrivial tube structure, then the GPEA is solvable for each H-admissible diagram $\mathbb{X} \xrightarrow{f} \mathbb{Y} \overset{i}{\hookrightarrow} \mathbb{Z}$.*

Proof. Consider the slice maps $\varphi\colon \mathbb{X} \to G/K$ and $\psi\colon \mathbb{Y} \to H/L$ from the commutative diagram $\mathscr{A}$ which generate a nontrivial tube structure on f. By Corollary 1 one can assume that ψ has an H-extension $\widehat{\psi}\colon \mathbb{U} \to H/L$ in some invariant neighbourhood $\mathbb{U}, \mathbb{Y} \subset \mathbb{U} \subset \mathbb{Z}$.

By Lemma 4 the map $f{\restriction}\colon \mathbb{S} = \varphi^{-1}[K] \to \mathbb{T} = \psi^{-1}[L]$ is a Q-orbit projection for the proper compact group $Q < P$. In view of the induction hypothesis, the GPEA is solvable for Q-orbit projections, and so it is solvable for the diagram $\mathbb{S} \xrightarrow{f} \mathbb{T} \hookrightarrow \mathbb{T}' \rightleftharpoons \widehat{\psi}^{-1}[L]$, which is L-admissible since $\mathbb{T}'$ is equivariantly hereditarily paracompact. Hence there exist a K-embedding $j'\colon \mathbb{S} \hookrightarrow \mathbb{S}'$ into a K-space $\mathbb{S}'$ and a Q-orbit projection $f'\colon \mathbb{S}' \to \mathbb{T}'$ such that $f' \circ j' = i \circ f\restriction$.

Next, by Lemma 5 the map $\widehat{f} = \pi \times f'\colon G \times_K \mathbb{S}' \to H \times_L \mathbb{T}' = \mathbb{U}$ is a P-orbit projection and, as is easy to see, $\widehat{f}$ solves the GPEA for the H-admissible diagram $\mathbb{X} \xrightarrow{f} \mathbb{Y} \overset{i}{\hookrightarrow} \mathbb{U}$. The application of Lemma 7 completes the proof of the lemma for the diagram $G \times_K \mathbb{S}' \xrightarrow{\widehat{f}} H \times_L \mathbb{T}' = \mathbb{U} \hookrightarrow \mathbb{Z}$ with open embedding $\mathbb{U} \hookrightarrow \mathbb{Z}$.

The last case of the proof of Theorem 19 consists of the consideration of a P-orbit projection $f\colon \mathbb{X} \to \mathbb{Y}$ with $\mathbb{X}^P = \varnothing$ and H-embedding $\mathbb{Y} \overset{i}{\hookrightarrow} \mathbb{Z}$.

Lemma 11. *There exist a neighbourhood $\mathbb{E}$, $\mathbb{Y} \subset \mathbb{E} \subset \mathbb{Z}$, and a locally finite H-cover $\sigma \in \operatorname{cov} \mathbb{E}$ of it consisting of open subsets $\{\mathbb{F}_\gamma \subset \mathbb{E}\}_{\gamma \in \Gamma}$ such that $\mathbb{F}_\gamma \cap \mathbb{Y} \neq \varnothing$ for each $\gamma \in \Gamma$ and also*

(1) *the map $g_\gamma \rightleftharpoons f{\restriction}\colon \mathbb{V}_\gamma \rightleftharpoons f^{-1}(\mathbb{F}^\star_\gamma) \to \mathbb{F}^\star_\gamma$ has a nontrivial tube structure, where $\mathbb{F}^\star_\gamma$ is the set of points adherent to $\mathbb{F}_\gamma$ lying in $\mathbb{Y}$, that is, $\mathbb{F}^\star_\gamma = \mathrm{Cl}_{\mathbb{E}}(\mathbb{F}_\gamma) \cap \mathbb{Y}$.*

Proof. Since $\mathbb{X}^P = \varnothing$, Proposition 12 implies that for each $x \in \mathbb{X}$, the restriction of f onto the orbit $G(x)$ has a nontrivial tube structure. Hence it follows by Theorem 16 on the extension of a slice that there exists a locally finite open H-cover $\omega = \{\mathbb{U}_\alpha\} \in \operatorname{cov} \mathbb{Y}$ such that

(1) each P-orbit projection $f{\restriction}\colon f^{-1}(\mathbb{U}_\alpha) \to \mathbb{U}_\alpha$ has a nontrivial tube structure.

Let ν be a family of open H-sets of $\mathbb{Z}$ whose restriction to $\mathbb{Y}$ coincides with ω. Since Z is hereditarily paracompact, the body $\cup\nu$ of the family ν is paracompact.

Hence there exists a locally finite open H-cover σ' of $\cup\nu$ the closure of which refines ν. It is clear that $\sigma \risingdotseq \{\mathbb{F} \in \sigma' \mid \mathbb{F} \cap \mathbb{Y} \neq \varnothing\}$ and $\mathbb{E} \risingdotseq \cup\sigma$ are as required.

Assume for the moment that the GPEA is solvable for the diagram $\mathbb{X} \xrightarrow{f} \mathbb{Y} \overset{i}{\hookrightarrow} \mathbb{E}$, which is evidently H-admissible. Then the GPEA for the H-admissible diagram $\mathbb{X} \xrightarrow{f} \mathbb{Y} \overset{i}{\hookrightarrow} \mathbb{Z}$ will be solvable by Lemma 7 and the openness of $\mathbb{E} \subset \mathbb{Z}$. Hence without loss of generality we can assume that $\mathbb{E} = \mathbb{Z}$, which we do in what follows.

The further reasoning will proceed with the help of a new induction, namely, a *transfinite* induction, for which we well order the set Γ by indexing the elements of $\{\mathbb{F}_\gamma\}$. Without loss of generality we can assume that Γ has the maximal element ω, which is not a limit ordinal. We set for a limit ordinal γ, $\mathbb{Q}_\gamma \risingdotseq \bigcup\{\mathbb{F}_{\gamma'} | \gamma' < \gamma\}$, otherwise we set $\mathbb{Q}_\gamma \risingdotseq \bigcup\{\mathbb{F}_{\gamma'} | \gamma' \leqslant \gamma\}$. It is obvious that $\mathbb{Z} = \mathbb{Q}_\omega$ is the body of a locally finite increasing system of open subsets $\{\mathbb{Q}_\gamma\}$, moreover $\mathbb{Q}_{\gamma'} \cup \mathbb{F}_\gamma = \mathbb{Q}_\gamma$ for $\gamma = \gamma' + 1$. One notes an important consequence of local finiteness of the cover $\sigma = \{\mathbb{F}_\gamma\}$:

(3) if γ is a limit ordinal, then the collection $\{\mathbb{Y} \cup \mathbb{Q}_{\gamma'} \mid \gamma' < \gamma\}$ generates the topology of $\mathbb{Y} \cup \mathbb{Q}_\gamma$, that is, $U \subset \mathbb{Y} \cup \mathbb{Q}_\gamma$ is open if and only if $(\mathbb{Y} \cup \mathbb{Q}_{\gamma'}) \cap U$ is open in $\mathbb{Y} \cup \mathbb{Q}_{\gamma'}$ for all $\gamma' < \gamma$.

With the help of transfinite induction, for all $\gamma \in \Gamma$ we construct a solution $j_\gamma \colon \mathbb{X} \hookrightarrow \mathbb{W}_\gamma \xrightarrow{\widehat{f}_\gamma} \mathbb{Y} \cup \mathbb{Q}_\gamma$ of the GPEA for the H-admissible diagram $\mathbb{X} \xrightarrow{f} \mathbb{Y} \overset{i_\gamma}{\hookrightarrow} \mathbb{Y} \cup \mathbb{Q}_\gamma$ (see the footnote[6]) such that

(4) $\mathbb{W}_{\gamma_1} \subset \mathbb{W}_{\gamma_2}$ and $\widehat{f}_{\gamma_2}\restriction_{\mathbb{W}_{\gamma_1}} = \widehat{f}_{\gamma_1}$ for all $\gamma_1 < \gamma_2$, that is, $\widehat{f}_{\gamma_2}$ is a solution of the GPEA for the H-admissible diagram $\mathbb{W}_{\gamma_1} \xrightarrow{\widehat{f}_{\gamma_1}} \mathbb{Y} \cup \mathbb{Q}_{\gamma_1} \hookrightarrow \mathbb{Y} \cup \mathbb{Q}_{\gamma_2}$.

Since $\mathbb{Y} \cup \mathbb{Q}_\omega = \mathbb{Z}$, the suggested plan guarantees the solvability of the GPEA for the admissible diagram $\mathbb{X} \xrightarrow{f} \mathbb{Y} \overset{i}{\hookrightarrow} \mathbb{Z}$, which leads to the completion of the proof of Theorem 19.

The basis of the inductive argument is easily established with the help of Lemma 10.

Lemma 12. *The GPEA for the diagram* $\mathbb{X} \xrightarrow{f} \mathbb{Y} \overset{i_{\gamma_0}}{\hookrightarrow} \mathbb{Y} \cup \mathbb{Q}_{\gamma_0}$ *is solvable.*

Proof. By Lemmas 10 and 11 the GPEA for the H-admissible diagram $\mathscr{D}_1 = \{\mathbb{V}_{\gamma_0} \xrightarrow{g_{\gamma_0}} \mathbb{F}^\star_{\gamma_0} \hookrightarrow \mathbb{F}^\star_{\gamma_0} \cup \mathbb{F}_{\gamma_0}\}$ is solvable. Since the GPEA for the H-admissible diagram $\mathbb{V}_{\gamma_0} \xrightarrow{g_{\gamma_0}} \mathbb{F}^\star_{\gamma_0} \cap \mathbb{Y} \hookrightarrow \mathbb{Y}$ is evidently solvable and the H-subspaces $\mathbb{F}^\star_{\gamma_0} \cup \mathbb{F}_{\gamma_0}$ and $\mathbb{Y}$ generate the topology of the H-space $\mathbb{Y} \cup \mathbb{Q}_{\gamma_0} = \mathbb{Y} \cup \mathbb{F}_{\gamma_0}$, it follows by Theorem 20 that the GPEA for the H-admissible diagram $\mathbb{X} \xrightarrow{f} \mathbb{Y} \hookrightarrow \mathbb{Y} \cup \mathbb{F}_{\gamma_0}$ is solvable.

The inductive step is based on the following proposition.

Lemma 13. *For each* $\gamma' \in \Gamma, \gamma' < \omega$ *the GPEA for the H-admissible diagram* $\mathbb{W}_{\gamma'} \xrightarrow{\widehat{f}_{\gamma'}} \mathbb{Y} \cup \mathbb{Q}_{\gamma'} \hookrightarrow \mathbb{Y} \cup \mathbb{Q}_\gamma$, *where* $\gamma \risingdotseq \gamma' + 1 \in \Gamma$, *is solvable.*

Proof. By Lemma 7 it is sufficient to show that the GPEA for the given diagram is locally solvable, that is, there exists a neighbourhood $\mathbb{T}, \mathbb{Y} \cup \mathbb{Q}_{\gamma'} \subset \mathbb{T} \subset \mathbb{Y} \cup \mathbb{Q}_\gamma$, such

[6]That is, $\widehat{f}_\gamma \circ j_\gamma = i\restriction \circ f$, where $\widehat{f}_\gamma$ is the P-orbit projection.

that the H-admissible diagram $\mathbb{W}_{\gamma'} \xrightarrow{\widehat{f}_{\gamma'}} \mathbb{Y} \cup \mathbb{Q}_{\gamma'} \hookrightarrow \mathbb{T}$ is solvable. First of all, we draw attention to the fact that the P-orbit projection $g_\gamma\colon \mathbb{V}_\gamma \to \mathbb{F}^\star_\gamma$. has a nontrivial tube structure. Theorem 16 implies

(5) the existence of an invariant neighbourhood $\mathbb{S}$, $\mathbb{F}^\star_\gamma \subset \mathbb{S} \subset \mathbb{Y} \cup \mathbb{Q}_{\gamma'}$, such that the P-orbit projection $h \rightleftharpoons \widehat{f}_{\gamma'}\restriction\colon (\widehat{f}_{\gamma'})^{-1}(\mathbb{S}) \to \mathbb{S}$ also has a nontrivial tube structure.

Denoting $(\mathbb{F}_\gamma \cup \mathbb{F}^\star_\gamma) \cap \mathbb{S}$ by $\mathbb{A}$ for brevity we consider the H-admissible diagram $h^{-1}(\mathbb{A}) \xrightarrow{h\restriction} \mathbb{A} \xhookrightarrow{i} \mathbb{T}_2 \rightleftharpoons (\mathbb{F}_\gamma \cup \mathbb{F}^\star_\gamma) \cap \widehat{\mathbb{S}}$, where $\widehat{\mathbb{S}} \subset \mathbb{Y} \cup \mathbb{Q}_{\gamma'} \cup \mathbb{F}_\gamma$ is a neighbourhood of $\mathbb{F}^\star_\gamma$ such that $\widehat{\mathbb{S}} \cap (\mathbb{Y} \cup \mathbb{Q}_{\gamma'}) = \mathbb{S}$. Since by (5) the map $h\restriction\colon h^{-1}(\mathbb{A}) \to \mathbb{A}$ has a nontrivial tube structure, Lemma 10 implies the solvability of the GPEA for the considered H-admissible diagram, that is, there exist a G-embedding $j\colon h^{-1}(\mathbb{A}) \hookrightarrow \mathbb{W}_2$ into the G-space $\mathbb{W}_2$ and a P-orbit projection $f_2\colon \mathbb{W}_2 \to \mathbb{T}_2$ such that $f_2 \circ j = i \circ h\restriction$.

It is easily checked that $\mathbb{F}^\star_\gamma = \mathbb{F}^\star_\gamma \cap \widehat{\mathbb{S}} = (\mathbb{F}_\gamma \cap \widehat{\mathbb{S}})^\star$. Therefore, $\mathbb{T}_2 = (\mathbb{F}_\gamma \cup \mathbb{F}^\star_\gamma) \cap \widehat{\mathbb{S}}$ has the form $\mathbb{U} \cup \mathbb{U}^\star$, where $\mathbb{U} \rightleftharpoons \mathbb{F}_\gamma \cap \widehat{\mathbb{S}}$ is open in $\mathbb{T} \rightleftharpoons \mathbb{Y} \cup \mathbb{Q}_{\gamma'} \cup \mathbb{U}$. Furthermore, since $(\mathbb{Y} \cup \mathbb{Q}_\gamma) \setminus \mathbb{T} = \mathbb{F}_\gamma \setminus (\widehat{\mathbb{S}} \cup \mathbb{Q}_{\gamma'})$, $\mathbb{T}$ is a neighbourhood of $\mathbb{Y} \cup \mathbb{Q}_{\gamma'}$ lying in $\mathbb{Y} \cup \mathbb{Q}_\gamma$. In view of Theorem 13 on topology generating, the H-subspaces $\mathbb{T}_1 \rightleftharpoons \mathbb{Y} \cup \mathbb{Q}_{\gamma'}$ and $\mathbb{T}_2$ generate the topology of the H-space $\mathbb{T}$.

It is clear that $\mathbb{T}_0 \rightleftharpoons \mathbb{T}_1 \cap \mathbb{T}_2$ coincides with $\mathbb{S}$. Since the H-orbit projections f_2 and $f_1 \rightleftharpoons \widehat{f}_{\gamma'}\colon \mathbb{W}_1 \rightleftharpoons \mathbb{W}_{\gamma'} \to \mathbb{T}_1$ are identical on $(\widehat{f}_{\gamma'})^{-1}(\mathbb{S})$, Theorem 20 can be applied: there exists a solution $s\colon \mathbb{W}_1 \hookrightarrow \mathbb{W}$ of the GPEA for the H-admissible diagram $\mathbb{W}_1 \xrightarrow{f_1} \mathbb{T}_1 \hookrightarrow \mathbb{T}$.

Now let $\gamma \in \Gamma$ be a limit ordinal. We consider an increasing family of G-subspaces $\{\mathbb{W}_{\gamma'} \subset \mathbb{W}_{\gamma''}\}_{\gamma' \leqslant \gamma'' < \gamma}$ and a system of agreeing P-orbit projections

$$\{\widehat{f}_{\gamma'}\colon \mathbb{W}_{\gamma'} \to \mathbb{Y} \cup \mathbb{Q}_{\gamma'}\}_{\gamma' < \gamma}.$$

Next we take $\mathbb{W}_\gamma$ to be equal to $\bigcup\{\mathbb{W}_{\gamma'} \mid \gamma' < \gamma\}$ and $\widehat{f}_\gamma\colon \mathbb{W}_\gamma \to \mathbb{Y} \cup \mathbb{Q}_\gamma$ to be equal to $\widehat{f}_{\gamma'}$ on $\mathbb{W}_{\gamma'}$ for all $\gamma' < \gamma$. In view of condition (3) on topology generating, the map $\widehat{f}_\gamma\colon \mathbb{W}_\gamma \to \mathbb{Y} \cup \mathbb{Q}_\gamma$ is continuous. It is easily checked that $\mathbb{X} \hookrightarrow \mathbb{W}_\gamma$ and $\widehat{f}_\gamma$ solve the GPEA for the H-admissible diagram $\mathbb{X} \xrightarrow{f} \mathbb{Y} \hookrightarrow \mathbb{Y} \cup \mathbb{Q}_{\gamma'}$, so that $\mathbb{W}_{\gamma'} \subset \mathbb{W}_\gamma$ and $\widehat{f}_{\gamma'} = \widehat{f}_\gamma\restriction_{\mathbb{W}_{\gamma'}}$ for all $\gamma' < \gamma$.

§ 10. Proof of Theorem 2, (3)

By Theorem 1 there exists a solution $s'\colon \mathbb{X} \hookrightarrow \mathbb{Y}'$ covering $X \hookrightarrow Y$. Since Y is paracompact and $p\colon \mathbb{Y}' \to Y$ is perfect, $\mathbb{Y}'$ is paracompact too.

Since $\mathbb{X}$ is stratifiable, there exists a continuous one-to-one map from $\mathbb{X}$ onto a metric space [27]. Hence there exists a continuous map $\varphi\colon \mathbb{X} \to B$ into a Banach space B such that

(1) its restriction to each orbit is an embedding.

In view of (1), the map $\psi\colon \mathbb{X} \to \mathbb{Z} \rightleftharpoons \mathrm{C}(G, B)$ given by the formula $\psi(x)(g) = \varphi(g^{-1}x)$, $g \in G$, $x \in \mathbb{X}$, is isovariant. By Theorem 11, since $\mathbb{Z} = \mathrm{C}(G, B)$ is G-ANE for the class of paracompact spaces and $\mathbb{Y}'$ is paracompact, there exists a G-map $\widehat{\psi}\colon \mathbb{Y}' \to \mathbb{Z}$ extending ψ.

We consider the fibrewise product $\mathbb{Y} \rightleftharpoons Y_\theta \times_\pi \mathbb{Z} \subset Y \times \mathbb{Z}$, where $\pi\colon \mathbb{Z} \to Z$ is the orbit projection and $\theta\colon Y \to Z$ is the map induced by $\widehat{\psi}$. The G-space $\mathbb{Y}$ is stratifiable as a subset of the product of two stratifiable spaces. Since ψ is isovariant, the G-space $\mathbb{X}$ is naturally contained in the G-space $\mathbb{Y}$, $s\colon \mathbb{X} \hookrightarrow \mathbb{Y}$, moreover, s covers $X \hookrightarrow Y$. Finally, the natural G-map $h\colon \mathbb{Y}' \to \mathbb{Y}$, $h(y') = (p(y'), \widehat{\psi}(y'))$, has the properties $h \circ s' = s$ and $\widetilde{h} = \mathrm{Id}_Y$.

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Bibliography

[1] S. M. Ageev, D. Repovš and E. V. Ščepin, "The extension problem for complete UV^n-preimages", *Tsukuba J. Math.* **23**:1 (1999), 97–111.

[2] V. V. Fedorchuk, "Extension of fiberings", *Vestnik Moskov. Univ. Ser.* I *Mat. Mekh.*, 1985, no. 1, 60–62; English transl. in *Moscow Univ. Math. Bull.*, 1985, no. 40, 92–95.

[3] S. M. Ageev, "Extensor properties of orbit spaces and the problem of continuation of an action", *Vestnik Moskov. Univ. Ser.* I *Mat. Mekh.*, 1994, no. 1, 11–16; English transl. in S. M. Ageev, "Extensor properties of orbit spaces and the problem of continuation of an action", *Moscow Univ. Math. Bull.* **49**:1994, 9–12.

[4] S. M. Ageev, "The problem of extending an action and extensor properties of orbit spaces", *Vestnik Moskov. Univ. Ser.* I *Mat. Mekh.*, 1994, no. 6, 80. (Russian)

[5] S. A. Antonyan, "Retraction properties of an orbit space", *Mat. Sb.* **137(179)**:3(11) (1988), 300–318; English transl. in *Math. USSR-Sb.* **65**:2 (1990), 305–321.

[6] J. W. Jaworowski, "Symmetric products of ANR's associated with a permutation group", *Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys.* **20** (1972), 649–651.

[7] S. M. Ageev and D. Repovš, "The Jaworowski method in the problem of the preservation of extensor properties by the orbit functor", *Mat. Zametki* **71**:3 (2002), 470–473; English transl. in *Math. Notes* **71**:3–4 (2002), 428–431.

[8] S. M. Ageev, "On extending the action", *Vestnik Moskov. Univ. Ser.* I *Mat. Mekh.*, 1992, no. 5, 20–23; English transl. in *Moscow Univ. Math. Bull.* **47**:5 (1992), 17–19.

[9] S. M. Ageev, "On a problem of Zambakhidze-Smirnov", *Mat. Zametki* **58**:1 (1995), 3–11; English transl. in *Math. Notes* **58**:1 (1995), 679–684.

[10] R. Cauty, "Une généralisation du théorème de Borsuk–Whitehead–Hanner aux espaces stratifiables", *C. R. Acad. Sci. Paris Sér. A-B* **275**:4 (1972), 271–274.

[11] M. Murayama, "On G-ANR's and their G-homotopy types", *Osaka J. Math.* **20**:3 (1983), 479–512.

[12] H. Torunczyk and J. E. West, "The fine structure of S^1/S^1; a Q-manifold hyperspace localization of the integers", *Proceedings of the International Conference on Geometric Topology* (Warsaw 1978), PWN, Warsaw 1980, pp. 439–449.

[13] G. E. Bredon, *Introduction to compact transformation groups*, vol. 46, Academic Press, New York–London 1972.

[14] T. tom Dieck, *Transformation groups and representation theory*, Lecture Notes in Math., vol. 766, Springer-Verlag, Berlin 1979.

[15] T. Matumoto, "Equivariant K-theory and Fredholm operators", *J. Fac. Sci. Univ. Tokyo Sect.* I *A Math.* **18** (1971), 109–125.

[16] Ch. N. Maxwell, "Homomorphisms of topological transformation groups into function spaces", *Duke Math. J.* **33**:3 (1966), 567–574.

[17] C. H. Dowker, "On a theorem of Hanner", *Ark. Mat.* **2**:4 (1952), 307–313.

[18] R. S. Palais, "The classification of G-spaces", *Mem. Amer. Math. Soc.*, 1960, no. 36.

[19] R. Engelking, *General topology*, PWN, Warsaw 1977.

[20] L. S. Pontryagin, *Topological groups*, 4th ed., Nauka, Moscow 1984; English transl. of 2nd. ed., Gordon and Breach, New York–London–Paris 1966.

[21] E. H. Spanier, *Algebraic topology*, McGraw-Hill, New York–Toronto, ON–London 1966.

[22] J. Szenthe, "On the topological characterization of transitive Lie group actions", *Acta Sci. Math. (Szeged)* **36**:3–4 (1974), 323–344.

[23] P. A. Alexandroff and B. A. Pasynkov, *Introduction to dimension theory*, Nauka, Moscow 1973. (Russian)

[24] G. Villalobos, "Inversion of the theorem on the existence of slices", *Vestnik Moskov. Univ. Ser.* I *Mat. Mekh.* **55**:1 (2000), 19–22; English transl. in *Moscow Univ. Math. Bull.* **55**:1 (2000), 26–29.

[25] S. M Ageev, "The equivariant theorem of Dugundji", *Uspekhi Mat. Nauk* **45**:5 (1990), 179–180; English transl. in *Russian Math. Surveys* **45**:5 (1990), 219–220.

[26] H. Abels, "A universal proper G-space", *Math. Z.* **159**:2 (1978), 143–158.

[27] C. J. R. Borges, "On stratifiable spaces", *Pacific J. Math.* **17**:1 (1966), 1–16.

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