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On metric spaces with the properties of de Groot and Nagata in dimension one ☆

Taras Banakh [a,b], Dušan Repovš [c,*], Ihor Zarichnyi [a]

[a] Department of Mathematics, Ivan Franko National University of Lviv, Ukraine
[b] Instytut Matematyki, Uniwersytet Humanistyczno Przyrodniczy im. Jana Kochanowskiego w Kielcach, Poland
[c] Faculty of Mathematics and Physics, and Faculty of Education, University of Ljubljana, PO Box 2964, Ljubljana, Slovenia

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ABSTRACT

A metric space (X,d) has the de Groot property GP_n if for any points $x_0, x_1, \dots, x_{n+2} \in X$ there are positive indices $i, j, k \leqslant n+2$ such that $i \neq j$ and $d(x_i, x_j) \leqslant d(x_0, x_k)$. If, in addition, $k \in \{i, j\}$ then X is said to have the Nagata property NP_n. It is known that a compact metrizable space X has dimension $\dim(X) \leqslant n$ iff X has an admissible GP_n-metric iff X has an admissible NP_n-metric.

We prove that an embedding $f : (0,1) \to X$ of the interval $(0,1) \subset \mathbb{R}$ into a locally connected metric space X with property GP_1 (resp. NP_1) is open, provided f is an isometric embedding (resp. f has distortion $\mathrm{Dist}(f) = \|f\|_{\mathrm{Lip}} \cdot \|f^{-1}\|_{\mathrm{Lip}} < 2$). This implies that the Euclidean metric cannot be extended from the interval $[-1,1]$ to an admissible GP_1-metric on the triode $T = [-1,1] \cup [0,i]$. Another corollary says that a topologically homogeneous GP_1-space cannot contain an isometric copy of the interval $(0,1)$ and a topological copy of the triode T simultaneously. Also we prove that a GP_1-metric space X containing an isometric copy of each compact NP_1-metric space has density $\geqslant \mathfrak{c}$.

1. Introduction

In this paper we shall be interested in structural properties of metric spaces possessing the properties introduced by J. de Groot [5] and J. Nagata [10].

Let n be a non-negative integer. A metric d on X is said to have the *de Groot property* GP_n if for any $n+3$ points $x_0, x_1, \dots, x_{n+2} \in X$ there is a triplet of indices $i, j, k \in \{1, \dots, n+2\}$ such that

$$d(x_i, x_j) \leqslant d(x_0, x_k) \quad \text{and} \quad i \neq j.$$

If, in addition, $k \in \{i, j\}$, then we say that the metric d has the *Nagata property* NP_n or that d is an NP_n-metric. It is clear that each NP_n-metric is also a GP_n-metric. In the Engelking's monograph [4] the properties of Nagata and de Groot are denoted by (μ_4) and (μ_5'), respectively. Those properties also are discussed in the Nagata's book [11, V.3].

According to [5] and [10], for a separable metrizable space X the following conditions are equivalent:

- X has the covering dimension $\dim(X) \leqslant n$;
- the topology of X is generated by an NP_n-metric on X;
- the topology of X is generated by a totally bounded GP_n-metric on X.

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* Corresponding author.
E-mail addresses: tbanakh@yahoo.com (T. Banakh), dusan.repovs@guest.arnes.si (D. Repovš), ihor.zarichnyj@gmail.com (I. Zarichnyi).

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In fact, the equivalence of the first two conditions holds for any metrizable space X. On the other hand, it is an open problem due to de Groot [5] if the existence of an admissible GP_n-metric on a (separable) space X implies $\dim(X) \leqslant n$, see [4, p. 231]. We recall that a metric d on a topological space X is said to be *admissible* if it generates the topology of X.

By [4, 4.2.D], a metric d has the GP_0-property if and only if it has the NP_0-property if and only if the metric d satisfies the strong triangle inequality

$$d(x_1, x_2) \leqslant \max\{d(x_0, x_1), d(x_0, x_2)\}$$

for all points $x_0, x_1, x_2 \in X$. The latter means that d is an *ultrametric*. Thus both NP_n-metric and GP_n-metric are higher dimensional analogs of ultrametric.

Due to efforts of many mathematicians the structure of ultrametric spaces is quite well understood. We shall recall two results: an Extension Theorem and a Universality Theorem.

Extension Theorem 1.1. *Each admissible ultrametric defined on a closed subspace A of a zero-dimensional compact metrizable space X extends to an admissible ultrametric on X.*

This theorem follows from its uniform version proved by Ellis in [2] or its "simultaneous" version proved by Tymchatyn and Zarichnyi [12]. The other theorem is due to A. Lemin and V. Lemin [6] and concerns universal ultrametric spaces. We define a (topological) metric space X to be (*topologically*) *homogeneous* if for any two points $x, y \in X$ there is an isometry (a homeomorphism) $h : X \to X$ such that $h(x) = y$.

Universality Theorem 1.2. *For each cardinal κ there is a (homogeneous) ultrametric space LM_κ of weight κ^ω containing an isometric copy of each ultrametric space of weight $\leqslant \kappa$.*

The universal space LM_κ in Theorem 1.2 can be constructed as follows: take any Abelian group G of size $|G| = \kappa$, let $\mathbb{Q}_+$ be the set of all positive rational numbers, and let LM_κ be the space of all maps $f : \mathbb{Q}_+ \to G$ which are eventually zero, in the sense that $f(x)$ is zero for all sufficiently large rational numbers $x \in \mathbb{Q}_+$. The space LM_κ endowed with the ultrametric $d(f, g) = \sup\{x \in \mathbb{Q}_+ \colon\ f(x) \neq g(x)\}$ (where $\sup \emptyset = 0$) has the structure of an Abelian group and therefore is metrically homogeneous.

It is natural to ask if these two theorems have analogues for GP_n- or NP_n-metrics. As we shall see later, the answer is negative already for $n = 1$. To construct a suitable counterexample we shall first study the structure of GP_1-spaces X in a neighborhood of an isometrically embedded interval $(0, 1) \subset X$.

Theorem 1.3. *If a GP_1-metric space X is locally connected, then each subset $I \subset X$, isometric to an interval $(a, b) \subset \mathbb{R}$, is open in X.*

This theorem will be proved in Section 2. Now we discuss some of its corollaries.

By the *triode* we understand the subspace

$$T = [-1, 1] \cup [0, i]$$

of the complex plane $\mathbb{C}$. By Nagata's Theorem [10], the triode T carries an admissible NP_1-metric. Nonetheless, such a metric cannot restrict to the Euclidean metric on the interval $[-1, 1] \subset T$ because the interval $(-1, 1)$ is not open in the triode. Thus we obtain:

Corollary 1.4. *The Euclidean metric on the interval $[-1, 1]$ has the Nagata property NP_1 but cannot be extended to an admissible GP_1-metric on the triode T.*

Therefore, Extension Theorem 1.1 cannot be generalized to metric spaces with the property NP_n or GP_n for $n \geqslant 1$. Next, we show that the same concerns Universality Theorem 1.2: its homogeneous version cannot be generalized to higher dimensions.

Corollary 1.5. *If a GP_1-metric space X contains both an isometric copy of the interval $[0, 1]$ and a topological copy of the triode T, then X is not topologically homogeneous.*

Proof. Let $[0, 1] \subset X$ be an isometric copy of the interval $[0, 1]$. Assuming that X is topologically homogeneous and X contains a topological copy of the triode T, we can find a topological embedding $f : T \to X$ such that $f(0) = \frac{1}{2} \in [0, 1] \subset X$. Since the triode does not embed into the interval $[0, 1]$, the point $1/2$ is not an interior point of the interval $(0, 1)$ in the locally connected subspace $Y = [0, 1] \cup f(T)$ of the GP_1-space X. This contradicts Theorem 1.3. $\square$

In spite of the negative result in Corollary 1.5, we do not know the answer to the following

Problem 1.6. Is it true that for each infinite cardinal κ there is a GP_1-metric space U of weight κ^ω that contains an isometric copy of each NP_1-metric space X of weight $\leqslant \kappa$?

The weight κ^ω in Problem 1.6 cannot be replaced by κ because of the following theorem that will be proved in Section 3.

Theorem 1.7. *If a* GP_1*-metric space X contains an isometric copy of each compact* NP_1*-metric space, then X has density* $\mathrm{dens}(X) \geqslant \mathfrak{c}$.

Now let us return to Theorem 1.3. It implies that no non-open arc I in a locally connected GP_1-metric space (X,d) is isometric to an interval $(a,b) \subset \mathbb{R}$. We can ask how much the metric d restricted to I differs from the Euclidean metric on I. We can measure this distance using the notion of the distortion.

By the *distortion* of an injective map $f : X \to Y$ between metric spaces (X, d_X) and (Y, d_Y) we understand the (finite or infinite) number

$$\mathrm{Dist}(f) = \|f\|_{\mathrm{Lip}} \cdot \left\|f^{-1}\right\|_{\mathrm{Lip}}$$

where

$$\|f\|_{\mathrm{Lip}} = \sup_{x \neq x'} \frac{d_Y(f(x), f(x'))}{d_X(x,x')}$$

is the Lipschitz constant of f (if $|X| \leqslant 1$, then $\|f\|_{\mathrm{Lip}}$ is not defined, so we put $\mathrm{Dist}(f) = 1$). The notion of distortion is widely used in studying the embeddability problems of metric spaces, see [1,7–9].

It can be shown that an embedding $f : X \to Y$ of a metric space X into a metric space Y has distortion $\mathrm{Dist}(f) = 1$ if and only if f is a *similarity*, which means that $d_Y(f(x), f(x')) = \|f\|_{\mathrm{Lip}} \cdot d_X(x,x')$ for all $x, x' \in X$.

In terms on the distortion, Theorem 1.3 can be written as follows.

Corollary 1.8. *Let X be a locally connected metric space with property* GP_1. *Each embedding $f : (0,1) \to X$ with distortion* $\mathrm{Dist}(f) = 1$ *is open.*

Proof. Let $f : (0,1) \to X$ be an embedding with distortion $\mathrm{Dist}(f) = 1$. Let $C = \|f\|_{\mathrm{Lip}}$ and

$$g : (0,C) \to (0,1), \quad g : t \mapsto t/C,$$

be the similarity mapping having the Lipschitz constant $\|g\|_{\mathrm{Lip}} = 1/C$. It follows that the composition $f \circ g : (0,C) \to X$ has distortion

$$1 = \mathrm{Dist}(f \circ g) = \|f \circ g\|_{\mathrm{Lip}} \cdot \left\|(f \circ g)^{-1}\right\|_{\mathrm{Lip}} = 1.$$

Since $\|f \circ g\|_{\mathrm{Lip}} = 1$, we conclude that $\|(f \circ g)^{-1}\|_{\mathrm{Lip}} = 1$ and hence $f \circ g : (0,C) \to X$ is an isometric embedding. By Theorem 1.3, the image $f \circ g((0,C)) = f((0,1))$ is open in X. $\square$

Problem 1.9. Can the equality $\mathrm{Dist}(f) = 1$ in Corollary 1.8 be replaced by the inequality $\mathrm{Dist}(f) < 2$?

This problem has an affirmative solution for metric spaces with the Nagata property NP_1. The following theorem can be easily derived from Proposition 4.1 and Corollary 5.2 proved at the end of the paper.

Theorem 1.10. *Let X be a locally connected metric space with property* NP_1. *Each embedding $f : (0,1) \to X$ with distortion* $\mathrm{Dist}(f) < 2$ *is open.*

The inequality $\mathrm{Dist}(f) < 2$ in this theorem is best possible because of the following simple example.

Example 1.11. On the triode $T = [-1,1] \cup [0,i]$ consider the NP_1-metric

$$\rho(z,z') = \begin{cases} |z - z'| & \text{if } \mathrm{sign}(\Re(z)) = \mathrm{sign}(\Re(z')), \\ \max\{|\Re(z)|, |\Re(z')|, \Im(z), \Im(z')\} & \text{otherwise.} \end{cases}$$

It is easy to check that the identity embedding $f : [-1,1] \to (T,\rho)$ has distortion $\mathrm{Dist}(f) = 2$ but is not open.

In spite of Corollary 1.4 there is a hope that the following problem (related to an approximative extension of NP_1-metrics) has an affirmative solution.

Problem 1.12. Let A be a closed subspace of a one-dimensional space X. Is it true that for any admissible NP_1-metric d_A on A there is an admissible NP_1-metric d_X on X such that the identity embedding $f:(A,d_A)\to(X,d_X)$ has distortion $\mathrm{Dist}(f)\leqslant 2$?

2. Isometric arcs in GP_1-metric spaces

In this section we shall prove Theorem 1.3. A map $f:X\to Y$ between metric spaces is called *non-expanding* if its Lipschitz constant $\|f\|_{\mathrm{Lip}}\leqslant 1$. For a point x of a metric space (X,d) and a subset $A\subset X$ we put $d(x,A)=\inf_{a\in A}d(x,a)$.

Lemma 2.1. *Let (X,d) be a GP_1-metric space containing an isometric copy of the closed interval $[0,1]$ and let $V=\{x\in X\colon d(x,[0,1])<\frac{1}{3}d(x,\{0,1\})\}$.*

(1) *There is a non-expanding retraction $r:V\to(0,1)$ such that*

$$d(x,t)=\max\bigl\{\bigl|t-r(x)\bigr|,d\bigl(x,[0,1]\bigr)\bigr\}\quad \text{for any } x\in V,\ t\in(0,1).$$

(2) *For any points $x,y\in V$ with $d(x,[0,1])\neq d(y,[0,1])$ we get*

$$d(x,y)\geqslant\max\bigl\{d\bigl(x,[0,1]\bigr),d\bigl(y,[0,1]\bigr)\bigr\}.$$

Proof. (1) Given any $x\in V$, let $D=d(x,[0,1])$ and consider the compact subset $D(x)=\{t\in[0,1]\colon d(x,t)=D\}$. We claim that $D(x)$ is a closed subinterval of $(0,1)$ of length $2D$. Let $a=\min D(x)$ and $b=\max D(x)$.

The triangle inequality implies that $d(a,b)\leqslant d(a,x)+d(x,b)\leqslant 2D$. It follows from $D<\frac{1}{3}d(x,\{0,1\})$ that $d(0,a)\geqslant d(0,x)-d(x,a)>3D-D>D$ and similarly, $d(b,1)>D$. Let us show that $[a,a+D]\subset D(x)$. Assuming the converse, we could find a point $x_1\in(a,a+D]\setminus D(x)$. Then for the points

$$x_0=a,\quad x_1,\quad x_2=x\quad\text{and}\quad x_3=a-D$$

we would get

$$d(x_1,x_2)>D,\qquad d(x_1,x_3)=D+(x_1-a)>D,\qquad d(x_2,x_3)>D\quad\text{and}$$
$$d(x_0,x_3)=d(a,a-D)=D,\qquad d(x_0,x_2)=d(a,x)=D,\qquad d(x_0,x_1)=d(a,x_1)\leqslant D,$$

which contradicts the GP_1-property of the metric d.

Thus $[a,a+D]\subset D(x)$. By analogy we can prove that $[b-D,b]\subset D(x)$. Combined with $b-a\leqslant 2D$, this implies that $[a,b]=[a,a+D]\cup[b-D,b]=D(x)$. Assuming that $b-a<2D$, we could take x_0 be the midpoint of the interval $[a,b]$ and put $x_1=x$, $x_2=x_0-D$, $x_3=x_0+D$. Then

$$\min\bigl\{d(x_1,x_2),d(x_1,x_3),d(x_2,x_3)\bigr\}>D=\max\bigl\{d(x_0,x_1),d(x_0,x_2),d(x_0,x_3)\bigr\},$$

which contradicts the GP_1-property of the metric d.

Therefore, $D(x)$ is a closed interval of length $2D$. Let $r(x)$ be the midpoint of this interval. Let us show that $d(x,t)=\max\{|t-r(x)|,D\}$ for all $t\in[0,1]$. This is obvious if $t\in D(x)=[a,b]$. So assume that $t\notin D(x)$. If $t<a$, then $d(t,x)\leqslant d(t,a)+d(a,x)\leqslant a-t+D=r(x)-t$. On the other hand, $b-t=d(t,b)\leqslant d(t,x)+d(x,b)=d(t,x)+D$ implies $d(t,x)\geqslant b-t-D=r(x)-t$. Therefore $d(x,t)=r(x)-t=\max\{|r(x)-t|,D\}$. The case $t>b$ can be treated by analogy.

Finally, we show that the map $r:V\to(0,1)$, $r:x\mapsto r(x)$ is a non-expanding retraction. It is clear that $r(t)=t$ for any $t\in(0,1)$. Take any two points $x,y\in V$. Without loss of generality, $r(y)\geqslant r(x)$. Let $D_x=d(x,[0,1])$ and $D_y=d(y,[0,1])$. For the point $t=r(x)-D_x=\min D(x)$ let us observe that

$$r(y)-r(x)+D_x=r(y)-t\leqslant\max\bigl\{\bigl|r(y)-t\bigr|,D_y\bigr\}=d(t,y)\leqslant d(t,x)+d(x,y)=D_x+d(x,y)$$

and hence $|r(y)-r(x)|=r(y)-r(x)\leqslant d(x,y)$.

(2) Take any two points $x,y\in V$ with $D_x=d(x,[0,1])\neq d(y,[0,1])=D_y$. We need to prove that $d(x,y)\geqslant\max\{D_x,D_y\}$. Without loss of generality, $D_x<D_y$. Assume conversely that $d(x,y)<\max\{D_x,D_y\}=D_y$. Observe that

$$d\bigl(r(x),0\bigr)\geqslant d(x,0)-D_x\geqslant d(y,0)-d(x,y)-D_x>d(y,0)-2D_y>3D_y-2D_y=D_y$$

and hence for any real a with $\max\{D_x,d(x,y)\}<a<D_y$ the point $x_1=r(x)-a\in(0,1)$ is well defined. By analogy we can prove that $x_2=r(x)+a\in(0,1)$ is well defined.

So we can consider the 4 points: $x_0=x$, $x_1=r(x)-a$, $x_2=r(x)+a$, $x_3=y$, and derive a contradiction with the GP_1-property of the metric d because:

$$\begin{aligned}\min\bigl\{d(x_1,x_2),d(x_1,x_3),d(x_2,x_3)\bigr\}&\geqslant\min\{2a,D_y,D_y\}>\max\bigl\{a,a,d(x,y)\bigr\}\\&\geqslant\max\bigl\{d(x_0,x_1),d(x_0,x_2),d(x_0,x_3)\bigr\}.\qquad\square\end{aligned}$$

Proof of Theorem 1.3. Let X be locally connected GP_1-metric space and $I \subset X$ a subset isometric to an open interval $(a,b) \subset \mathbb{R}$. We need to check that each point $x_0 \in I$ is an interior point of I in X. For a sufficiently small $\varepsilon > 0$ we can find an isometry $f : [0, 2\varepsilon] \to I \subset X$ such that $f(\varepsilon) = x_0$. Scaling the GP_1-metric d of X by a suitable constant, we can assume that $\varepsilon = \frac{1}{2}$. We shall identify the interval $[0,1]$ with a subinterval of I and $1/2$ with the point x_0. Consider the neighborhood

$$V = \{x \in X\colon\ d(x, [0,1]) < d(x, \{0,1\})/3\}$$

of $(0,1)$ in X. By the local connectedness of X at x_0, find a connected neighborhood $C(x_0) \subset V$ of the point $x_0 = 1/2$. We claim that $C(x_0) \subset I$. Otherwise there would exist a point $x_1 \in C(x_0) \setminus I$. Lemma 2.1(2) guarantees that the subset

$$D = \{x \in C(x_0)\colon\ d(x, [0,1]) = d(x_1, [0,1])\}$$

is open-and-closed in $C(x_0)$, which implies that the neighborhood $C(x_0)$ is not connected and this is a contradiction. □

3. Universal GP_1-spaces

In this section we study universal GP_1-spaces and prove Lemma 3.2 which implies Theorem 1.7 announced in the introduction.

We shall need the following (probably known)

Lemma 3.1. *Let (X, d_X) be a NP_1-metric space and (Y, d_Y) be an NP_0-metric space. Then the* max*-metric*

$$d((x,y),(x',y')) = \max\{d_X(x,x'), d_Y(y,y')\}$$

on the product $X \times Y$ has the Nagata property NP_1.

Proof. Given any 4 points $(x_0,y_0),(x_1,y_1),(x_2,y_2),(x_3,y_3) \in X \times Y$, we need to find two distinct indices $i, j \in \{1,2,3\}$ such that

$$d((x_i,y_i),(x_j,y_j)) \leqslant \max\{d((x_0,y_0),(x_i,y_i)), d((x_0,x_j),(y_0,y_j))\}.$$

Since the metric on X has the property NP_1, there are two distinct numbers $i, j \in \{1,2,3\}$ such that

$$d_X(x_i,x_j) \leqslant \max\{d_X(x_0,x_i), d_X(x_0,x_j)\}.$$

The NP_0-property of the metric space Y ensures that

$$d_Y(y_i,y_j) \leqslant \max\{d_Y(y_0,y_i), d_Y(y_0,y_j)\}.$$

Combining these two inequalities, we conclude that

$$\begin{aligned} d((x_i,y_i),(x_j,y_j)) &= \max\{d_X(x_i,x_j), d_Y(y_i,y_j)\} \\ &\leqslant \max\{d_X(x_0,x_i), d_X(x_0,x_j), d_Y(y_0,y_i), d_Y(y_0,y_j)\} \\ &= \max\{d((x_0,y_0),(x_i,y_i)), d((x_0,x_j),(y_0,y_j))\}. \qquad \square \end{aligned}$$

Lemma 3.1 implies that for a positive real number a the metric

$$d((x,y),(x',y')) = \max\{|x-x'|, |y-y'|\}$$

on the product $\mathbb{I}_a = [-1,1] \times \{0,a\} \subset \mathbb{R} \times \mathbb{R}$ has the Nagata property NP_1.

For a metric space X we shall write $\mathbb{I}_a \hookrightarrow X$ if X contains an isometric copy of the space $\mathbb{I}_a$.

Lemma 3.2. *For any GP_1-metric space X the set $A = \{a \in (\frac{1}{16}, \frac{1}{8}) : \mathbb{I}_a \hookrightarrow X\}$ has cardinality $|A| \leqslant \mathrm{dens}(X)$.*

Proof. For every $a \in A$ fix an isometric embedding $h_a : \mathbb{I}_a \to X$ and define a map $f_a : \mathbb{I}_1 \to X$ by letting $f_a : (x,t) \mapsto h_a(x, at)$ for $(x,t) \in \mathbb{I}_1$. The map f_a can be considered as an element of the function space $C(\mathbb{I}_1, X)$ endowed with the sup-metric

$$d(f,g) = \sup_{t \in \mathbb{I}_1} d(f(t), g(t)).$$

By [3, 3.4.16], the density of the function space $C(\mathbb{I}_1, X)$ is equal to the density of X. Now the assertion of the theorem will follow as soon as we check that the set $\mathcal{F}_A = \{f_a\colon\ a \in A\}$ is discrete in $C(\mathbb{I}_1, X)$. This will follow as soon as we show that $d(f_a, f_b) \geqslant \frac{1}{32}$ for any numbers $a \neq b$ in A.

To this end we first introduce some notation. For $a \in A$ and $i \in \{0,1\}$ let

$$I_a^i = f_a([-1,1] \times \{i\}), \qquad \partial I_a^i = f_a(\{-1,1\} \times \{i\}), \qquad J_a^i = I_a^i \setminus \partial I_a^i, \qquad c_a^i = f_a\left(\frac{1}{2}, i\right),$$

and

$$V_a^i = \left\{x \in X\colon d(x, I_a^i) < \frac{1}{3} d(x, \partial I_a^i)\right\}.$$

By Lemma 2.1, there is a non-expanding retraction $r_a^i : V_a^i \to J_a^i$ such that for every $x \in V_a^i$ and $t \in J_a^i$ we get

$$d(x,t) = \max\{d(r_a^i(x), t), d(x, I_a^i)\}. \tag{1}$$

Moreover, for any points $x, y \in V_a^i$ with $d(x, I_a^i) \neq d(y, I_a^i)$ we get

$$d(x,y) \geqslant \max\{d(x, I_a^i), d(y, I_a^i)\}. \tag{2}$$

To derive a contradiction, assume that $d(f_a, f_b) < \varepsilon = \frac{1}{32}$ for some distinct numbers $a, b \in A$. Observe that

$$d(c_b^0, I_a^1) \leqslant d(c_b^0, c_a^0) + d(c_a^0, I_a^1) < \varepsilon + a < \frac{1}{32} + \frac{1}{8} = \frac{5}{32}$$

while

$$d(c_b^0, \partial I_a^1) \geqslant d(c_a^0, \partial I_a^1) - d(c_a^0, c_b^0) = \frac{1}{2} - \varepsilon = \frac{1}{2} - \frac{1}{32} = \frac{15}{32}.$$

Consequently, $d(c_b^0, I_a^1) < \frac{1}{3} d(c_b^0, \partial I_a^1)$ and hence $c_b^0 \in V_a^1$. We claim that $d(c_b^0, I_a^1) = d(c_a^0, I_a^1) = a$. Otherwise, we may apply the formula (2) to derive a contradiction:

$$d(c_b^0, c_a^0) \geqslant \max\{d(c_b^0, I_a^1), d(c_a^0, I_a^1)\} \geqslant d(c_a^0, I_a^1) = a > \varepsilon > d(f_a, f_b).$$

Since the retraction $r_a^1 : V_a^1 \to J_a^1$ is non-expanding, we get

$$d(r_a^1(c_b^0), c_a^1) = d(r_a^1(c_b^0), r_a^1(c_a^0)) \leqslant d(c_b^0, c_a^0) < \varepsilon < a.$$

Now the formula (1) yields

$$d(c_b^0, c_a^1) = \max\{d(r_a^1(c_b^0), c_a^1), d(c_b^0, I_a^1)\} = d(c_b^0, I_a^1) = a.$$

By analogy we can prove that $d(c_a^1, c_b^0) = b$, which contradicts $d(c_b^0, c_a^1) = a$. □

4. Obtuse arcs and embeddings with small distortion

In this section we shall introduce the notion of an obtuse arc and show that for each embedding $f : [0,1] \to X$ with $\mathrm{Dist}(f) < 2$ the arc $f([0,1])$ is obtuse. By a *metric arc* we understand a metric space that is homeomorphic to the unit interval $\mathbb{I} = [0,1]$.

A metric arc (I, d) is called *obuse* if

- for any subarc $J \subset I$ with end-points a, b and any point $z \in J \setminus \{a, b\}$ there are points $x, y \in J$ with $d(x,y) > \max\{d(z,x), d(z,y)\}$; and
- for any subarc $J \subset I$ with end-points a, b there is a point $z \in J$ with $d(a,b) > \max\{d(z,a), d(z,b)\}$.

In this case the metric d on I is called *obtuse*.

It is easy to see that each subinterval $[a,b] \subset \mathbb{R}$ endowed with the Euclidean metric is an obtuse arc. It can be shown that each continuously differentiable curve can be covered by finitely many obtuse subarcs.

Proposition 4.1. *If an embedding $f : \mathbb{I} \to X$ of the unit interval $\mathbb{I} = [0,1]$ into a metric space (X, d_X) has distortion* $\mathrm{Dist}(f) < 2$, *then the image $I = f(\mathbb{I})$ is an obtuse arc in X.*

Proof. We need to show that the metric

$$\rho(t, t') = d_X(f(t), f(t'))$$

on $\mathbb{I}$, induced by the embedding f, is obtuse. It follows that

$$\left(\|f^{-1}\|_{\mathrm{Lip}}\right)^{-1}\cdot|x-y|\leqslant\rho(x,y)\leqslant\|f\|_{\mathrm{Lip}}\cdot|x-y|.$$

Now we establish the two conditions of the definition of an obtuse arc.

1. Take any subinterval $[a,b]\subset\mathbb{I}$ and a point $z\in(a,b)$. Let $x,y\in(a,b)$ be any two points such that z is the midpoint of the interval (x,y). Then

$$\begin{aligned}\max\{\rho(x,z),\rho(y,z)\}&\leqslant\|f\|_{\mathrm{Lip}}\cdot\max\{|x-z|,|y-z|\}=\|f\|_{\mathrm{Lip}}\cdot|x-y|/2\\&\leqslant\frac{1}{2}\|f\|_{\mathrm{Lip}}\cdot\|f^{-1}\|_{\mathrm{Lip}}\cdot\rho(x,y)<\frac{1}{2}\cdot 2\cdot\rho(x,y)<\rho(x,y).\end{aligned}$$

2. By analogy we can prove that for any subinterval $[a,b]\subset\mathbb{I}$ the midpoint z of $[a,b]$ satisfies the inequality $\max\{\rho(x,z),\rho(y,z)\}<\rho(x,y)$. □

5. Obtuse arcs in NP_1-metric spaces

In this section we study the structure of an NP_1-metric space X in a neighborhood of an obtuse arc $I\subset X$.

Proposition 5.1. *Let (X,d) be an NP_1-metric space, $I\subset X$ be an obtuse arc with endpoints a,b in X and let $V=\{x\in X\colon d(x,I)<d(x,\{a,b\})\}$.*

(1) *For every point $x\in V\setminus I$ the set $D(x)=\{t\in I\colon d(x,t)=d(x,I)\}$ is the finite union of closed subintervals of I each of which has diameter $>d(x,I)$.*
(2) *For any points $x,y\in V$ with $d(x,I)\neq d(y,I)$ we get*

$$d(x,y)\geqslant\max\{d(x,I),d(y,I)\}.$$

Proof. (1) Given a point $x\in V\setminus I$ put $D=d(x,I)$ and consider the family $\mathcal{I}$ of maximal non-generate subintervals in the closed subset

$$D(x)=\{t\in I\colon d(t,x)=D\}\subset(a,b)=I\setminus\{a,b\}.$$

We claim that each maximal subinterval $[a_1,b_1]\in\mathcal{I}$ has diameter $\operatorname{diam}[a_1,b_1]>D$. Assuming conversely that $\operatorname{diam}([a_1,b_1])\leqslant D$, and using the second condition of the definition of an obtuse metric, we can find a point $x_0\in(a_1,b_1)$ such that $D\geqslant d(a_1,b_1)>\max\{d(a_1,x_0),d(b_1,x_0)\}$. The maximality of the subinterval $[a_1,b_1]\subset D(x)\subset(a,b)$ implies the existence of points $x_1\in(a,a_1)\setminus D(x)$ and $x_2\in(b_1,b)\setminus D(x)$ such that $\max\{d(x_1,x_0),d(x_2,x_0)\}<\min\{D,d(x_1,x_2)\}$. Now we see that the quadruple of points $x_0,x_1,x_2,x_3=x$ witnesses that the metric d on X fails to have the Nagata property NP_1 because

$$\begin{aligned}&d(x_1,x_2)>\max\{d(x_1,x_0),d(x_2,x_0)\},\\&d(x_1,x_3)>D\geqslant\max\{d(x_0,x_1),d(x_0,x_3)\}\quad\text{and}\\&d(x_2,x_3)>D\geqslant\max\{d(x_0,x_2),d(x_0,x_3)\}.\end{aligned}$$

Taking into account that any two distinct maximal subintervals in the family $\mathcal{I}$ are disjoint and have diameter $>D$, we conclude that the family $\mathcal{I}$ is finite. It remains to show that $D(x)=\bigcup\mathcal{I}$. Assuming the converse, we could find a point $x_0\in D(x)\setminus\bigcup\mathcal{I}$ and a neighborhood $(a_1,b_1)\subset I\setminus\bigcup\mathcal{I}$ of the point x_0 in $I\setminus\{a,b\}$ such that $\operatorname{diam}(a_1,b_1)<D$. The intersection $(a_1,b_1)\cap D(x)$ contains no non-degenerate subinterval and hence is nowhere dense in (a_1,b_1). The obtuse property of the metric d guarantees the existence of two points $x_1,x_2\in(a_1,b_1)$ such that $d(x_1,x_2)>\max\{d(x_1,x_0),d(x_2,x_0)\}$. Since $D(x)\cap(a_1,b_1)$ is nowhere dense we can additionally assume that $x_1,x_2\notin D(x)$. Then for the quadruple of the points $x_0,x_1,x_2,x_3=x$ we get

$$\begin{aligned}&d(x_1,x_2)>\max\{d(x_1,x_0),d(x_2,x_0)\},\\&d(x_1,x_3)=d(x_1,x)>D=\max\{d(x_1,x_0),d(x_3,x_0)\}\quad\text{and}\\&d(x_2,x_3)=d(x_2,x)>D=\max\{d(x_3,x_0),d(x_2,x_0)\},\end{aligned}$$

witnessing the failure of the Nagata property NP_1 for the metric d.

(2) Given two points $x,y\in V$ with $d(x,I)\neq d(y,I)$ we should prove that $d(x,y)\geqslant\max\{d(x,I),d(y,I)\}$. Assume conversely, that $d(x,y)<\max\{d(x,I),d(y,I)\}$. Without loss of generality $d(x,I)<d(y,I)$. By the preceding item, the set

$$D(x)=\{z\in I\colon d(x,z)=d(x,I)\}$$

contains two points x_1, x_2 with $d(x_1, x_2) > d(x, I)$. Now we see that the quadruple of the points $x_0 = x, x_1, x_2, x_3 = y$ satisfies the inequalities

$$d(x_1, x_2) > d(x, I) = \max\{d(x_0, x_1), d(x_0, x_2)\},$$
$$d(x_1, x_3) \geqslant d(y, I) > \max\{d(x_0, x_1), d(x_0, x_3)\},$$
$$d(x_2, x_3) \geqslant d(y, I) > \max\{d(x_0, x_2), d(x_0, x_3)\},$$

witnessing that the metric d fails to have the Nagata property NP_1. □

By an argument similar to that from Theorem 1.3, we apply Proposition 5.1 to prove the following

Corollary 5.2. *Let X be a locally connected NP_1-metric space X and $I \subset X$ is an obtuse arc with endpoints a, b. Then the set $I \setminus \{a, b\}$ is open in X.*

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