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# Preserving $Z$-sets by Dranishnikov's resolution ☆

Sergey M. Ageev [a], Matija Cencelj [b,c], Dušan Repovš [d,c,*]

[a] Faculty of Mechanics and Mathematics, Belarus State University, Av. F. Skorina 4, Minsk, Belarus 220050
[b] Institute for Mathematics, Physics and Mechanics, University of Ljubljana, PO Box 2964, Ljubljana, Slovenia 1001
[c] Faculty of Education, University of Ljubljana, Kardeljeva pl. 16, Ljubljana, Slovenia 1000
[d] Faculty of Mathematics and Physics, University of Ljubljana, Jadranska 19, Ljubljana, Slovenia 1000

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ABSTRACT

We prove that Dranishnikov's $k$-dimensional resolution $d_k : \mu^k \to Q$ is a $\mathrm{UV}^{n-1}$-divider of Chigogidze's $k$-dimensional resolution $c_k$. This fact implies that $d_k^{-1}$ preserves $Z$-sets. A further development of the concept of $\mathrm{UV}^{n-1}$-dividers permits us to find sufficient conditions for $d_k^{-1}(A)$ to be homeomorphic to the Nöbeling space $\nu^k$ or the universal pseudoboundary $\sigma^k$. We also obtain some other applications.



## 1. Introduction

Dranishnikov [16] constructed for each $k \geqslant 1$ the map $d_k : \mu^k \to Q = [-1,1]^\omega$ of the $k$-dimensional Menger compactum onto the Hilbert cube $Q$ such that:

(a) $d_k$ is $(k-1)$-soft, $(k, k-2)$-soft and polyhedrally $k$-soft;
(b) $d_k^{-1}$ preserves $\mathrm{AE}(k)$- and $\mathrm{ANE}(k)$-spaces, and therefore $d_k$ is a $\mathrm{UV}^{k-1}$-map;
(c) $d_k$ is $k$-invertible (i.e. for each map $\varphi : A \to Q$, $\dim A \leqslant k$, there exists a map $\tilde{\varphi} : A \to \mu^k$ such that $d_k \circ \tilde{\varphi} = \varphi$); and
(d) $d_k$ is $k$-universal with respect to maps of compacta.

In what follows we shall call a map satisfying the properties (a)–(d), *Dranishnikov's k-resolution*. This map represents an important technique of geometric topology and permits us to demonstrate the wide analogy between Menger theory and

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* Corresponding author at: Faculty of Education, University of Ljubljana, Kardeljeva pl. 16, Ljubljana, Slovenia 1000.
*E-mail addresses:* ageev.sergei@yahoo.com (S.M. Ageev), matija.cencelj@guest.arnes.si (M. Cencelj), dusan.repovs@guest.arnes.si (D. Repovš).

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$Q$-manifold theory. For example, with its help the Triangulation and Stability Theorems in the Menger manifold theory were formulated and proved in [16].

On the other hand, Dranishnikov's resolution is by its properties the finite-dimensional analogue of the projection $d : Q \cong Q \times Q \to Q$ of the product onto factor with the exception of being $k$-soft. It is clear that the more properties of Dranishnikov's resolution will be found the more convenient instrument it will become.

In the sequel we shall call a $k$-soft map $c_k : \nu^k \to Q$, $k \geqslant 0$, of the $k$-dimensional Nöbeling space onto the Hilbert cube $k$-universal with respect to maps of Polish spaces *Chigogidze's $k$-resolution* [11]. This map, is a bridge between the Nöbeling and Hilbert ($l_2$-) manifold theories and possesses properties which are every bit as remarkable as those of Dranishnikov's resolution (the discussion on its uniqueness is put in the Epilogue). The opinion to consider $c_k$ as the finite-dimensional analogue of the projection $c : \mathrm{l}_2 \cong Q \times (-1,1)^\omega \to Q$ is justified to a greater degree than in the case of Dranishnikov's resolution.

The investigation of interconnection between these two resolutions was initiated in [7] and it was established that Chigogidze's resolution is densely contained in Dranishnikov's resolution, i.e. that there exists an embedding $i_k : \nu^k \hookrightarrow \mu^k$ such that $c_k = d_k \restriction_{\nu^k}$ and $\mathrm{Cl}\, \nu^k = \mu^k$. This result is in complete accordance with the infinite-dimensional situation: $c = d \restriction_{\mathrm{l}_2}$ and $\mathrm{Cl}\, \mathrm{l}_2 = Q$ for the natural embedding $i : \mathrm{l}_2 \cong Q \times (-1,1)^\omega \hookrightarrow Q$.

However, while most of the useful properties of these infinite-dimensional objects are evident (for instance, that $i$ is a UV-map), all new properties of its finite-dimensional analogues are established with excessive difficulties, especially since $d_k$ fails to be $k$-soft. The aim of the present paper is to make a definite progress towards the investigation of the finite-dimensional resolutions. Our main result is:

**Theorem 1.1.** *For each $k \geqslant 1$ there exists Dranishnikov's resolution $\delta_k : \mu^k \to Q$ which is a $\mathrm{UV}^{k-1}$-divider of Chigogidze's resolution $\chi_k : \nu^k \to Q$, i.e. there exists a $\mathrm{UV}^{k-1}$-embedding $i_k : \nu^k \hookrightarrow \mu^k$ such that $\chi_k = \delta_k \restriction_{\nu^k}$ and $\mathrm{Cl}\, \nu^k = \mu^k$.*

The proof of this central theorem is based on a careful analysis of the concept of the $\mathrm{UV}^{k-1}$-dividers, which may in fact, be considered as the other purpose of this paper. In particular, we find a piecewise linear version of Theorem 1.1 which is a crucial ingredient of its proof.

**Theorem 1.2.** *Let $P$ be a compact polyhedron with the triangulation $L$ and $k \geqslant 1$. Then there exist an ANE-compactum $D$ and maps $p : D \to P$ and $q : D \to P^{(k)}$ such that*:

(1) *$p$ is a $k$-conservatively soft*; *and*
(2) *$p$ is a $\mathrm{UV}^{k-1}$-divider of a $k$-soft map.*

**Remark 1.3.** It will follow directly from the proof of Theorem 1.2 that

(3) $q$ is $p^{-1}(L \circ L)$-map where $L \circ L$ is the star of $L$ with respect to $L$.

Theorem 1.1 implies several important results. Since the passing to the preimage with respect to an $n$-soft map preserves $Z_n$-sets, as it does also with respect to a $\mathrm{UV}^{k-1}$-divider of Chigogidze's resolution $\chi_k$ (see Proposition 3.4), Dranishnikov's resolution does *preserves $Z$-sets*.

**Theorem 1.4.** *For each $k \geqslant 1$ there exists Dranishnikov's resolution $\delta_k : \mu^k \to Q$ such that $(\delta_k)^{-1}(F) \subset_Z \mu^k$ as soon as $F \subset_{Z_k} Q$.*

Of course, this intriguing fact will play an important role in the theory of Menger manifolds. It should also be remarked that this fact was conjectured for a long time (see [11,17]) and it was erroneously claimed to be false [12].

Next, we find sufficient conditions for the preimage $\delta_k^{-1}(Z)$ of $Z \subset Q$ to be homeomorphic to the Nöbeling space $\nu^k$ or universal pseudoboundary $\sigma^k$.

**Theorem 1.5.** *For each $k \geqslant 1$ there exists Dranishnikov's resolution $\delta_k : \mu^k \to Q$ such that for each $Z \in \mathrm{AE}(k)$, $Z \hookrightarrow Q \in \mathrm{UV}^{k-1}$, the following holds*:

(4) *$\delta_k^{-1}(Z) \cong \nu^k$ as soon as $Z$ is topologically complete, strongly $k$-universal with respect to Polish spaces*;
(5) *$\delta_k^{-1}(Z) \cong \sigma^k$ as soon as $Z$ is $\sigma$-compact, discretely $I^k$-approximated and strongly $k$-universal with respect to compacta.*

Theorem 1.5 implies that $\delta_k^{-1}((-1,1)^\omega) \cong \nu^k$ and $\delta_k^{-1}(Q \setminus (-1,1)^\omega) \cong \sigma^k$ which is the affirmative solution of Problem 612 from [15].

## 2. Preliminaries

Throughout this paper we shall assume all spaces to be *separable complete metric* and all maps to be *continuous*, if they do not arise as a result of some constructions and their properties should be established in the process of proof. The set of all open covers of the space $X$ is denoted by $\operatorname{cov} X$. We will use $\mathrm{N}(A;\omega)$ to denote the *neighborhood* $\bigcup\{U \mid U \in \omega,\ U \cap A \neq \emptyset\}$ of $A \subset X$ with respect to $\omega \in \operatorname{cov} X$; $\omega' \circ \omega \rightleftharpoons \{\mathrm{N}(A;\omega) \mid A \in \omega'\}$ – the star of a cover $\omega'$ with respect to $\omega$ (we use the sign $\rightleftharpoons$ for introducing new objects to the left of it). The *body* $\bigcup\omega$ of a system of sets $\omega$ is the set $\bigcup\{U \mid U \in \omega\}$. We say that the embedding $A \subset B$ is *strong* and write $A \Subset B$ if $\operatorname{Cl} A \subset \operatorname{Int} B$.

The refinement of the cover $\omega$ in $\omega_1$ is denoted by $\omega \prec \omega_1$. If $f, g : X \to Y$ are maps, and $\delta$ is a family of subsets of $Y$, then *the $\delta$-closeness of $f$ to $g$* (briefly, $\operatorname{dist}(f,g) \prec \delta$ or $f \overset{\delta}{\sim} g$) means that if $f(x) \neq g(x)$, then $\{f(x), g(x)\} \subset W \in \delta$. The restriction of a map $f$ onto a subset $A$ is denoted by $f\restriction_A$ or simply $f\restriction$ if there is no ambiguity about $A$. Since $f$ is an extension of $g = f\restriction_A$, we write this as $f = \operatorname{ext}(g)$. If $\delta > 0$ is a number, then $\delta$-closeness of $f$ to $g$ is denoted by $\operatorname{dist}(f,g) < \delta$, as in the case of covers. We denote the distance $d(x,y)$ between points $x, y \in X$ of metric space $(X,d)$ as $\operatorname{dist}(x,y)$ if there is no confusion.

Let us introduce a series of notions concerned with the extension *of partial maps*, i.e. maps given on closed subspaces of a metric space [18]. If an arbitrary partial map $Z \hookleftarrow A \xrightarrow{\phi} X$, $\dim Z \leqslant k$, $k \leqslant \infty$, can be extended on the entire space $Z$ [on some neighborhood of $A$], then $X$ is called an *absolute* [*neighborhood*] *extensor* in dimension $k$, $X \in \mathrm{A[N]E}(k)$. If $k = \infty$, then the notion of absolute [neighborhood] extensor ($X \in \mathrm{A[N]E}$) arises. By the Kuratowski–Dugundji Theorem [18], the property of extendability in finite dimension correlates with the connectivity and the local connectivity of the space: $X \in \mathrm{AE}(k) \Leftrightarrow X \in \mathrm{C}^{k-1} \& \mathrm{LC}^{k-1}$.

The problem of extension of partially defined morphisms has a categorical character. In the category of maps having a fixed target $Y$ the problem of extension of morphisms is known as the problem of extension of a partial lift to the global lift. For a given map $f : X \to Y$, the *partial lift of the map* $\psi : Z \to Y$ with respect to $f$ is the map $\varphi : A \to X$ which is defined on the closed subset $A \subset Z$ and which makes the following diagram commutative:

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d, "i"'] & X \arrow[d, "f"] \\ Z \arrow[r, "\psi"'] & Y \end{tikzcd}$$

A partial lift $\varphi$ of the map $\psi$ is extended to a *global* (*local*) *lift with respect to* $f$ if there exists a global (local) extension of $\varphi : Z \to X$ which is the lift of $\psi$. Thus, the problem of global lifting consists in the splitting of the square diagram above by the map $\hat{\varphi}$ into two triangular commutative diagrams.

Recall that a map $f$ is called *soft* (*locally soft*) with respect to pair $(Z, A)$, if any partial lift $\phi : A \to X$ (with respect to $f$) of any map $\psi : Z \to Y$ can be extended to the global (local) lift. The collection $\mathfrak{S}(f)$ of all pairs $(Z, A)$ for which $f$ is soft will be called a *softness envelope* of the map $f$. Note that if $|Y| = 1$, then the problem of extension of lifts is transformed into the problem of extension of maps.

Let $\mathfrak{C}$ be a class of pairs $(Z, A)$ in which $A \subset Z$ is a closed subset. The map $f$ is called $\mathfrak{C}$-*soft* (*locally* $\mathfrak{C}$-*soft*) if it is soft (locally soft) with respect to all pairs $(Z, A)$ from $\mathfrak{C}$. Along this line, we can introduce the notions of $(n,k)$-softness ($\mathfrak{C} = \{(Z,A) \mid \dim Z \leqslant n,\ \dim A \leqslant k\}$), polyhedral softness ($\mathfrak{C} = \{(Z,A) \mid Z, A$ are polyhedra, and $\dim Z \leqslant n\}$). We denote the class of $(n,n)$-soft maps, or briefly, $n$-soft maps by $\mathfrak{S}_n$. If $\mathfrak{S}(f)$ contains all pairs $(Z \times [0,1], Z \times \{0\})$, where $\dim Z \leqslant n$, then $f$ is called a *Hurewicz $n$-fibration*. The following assertion is well known [18]:

**Proposition 2.1.** *Let $Y \in \mathrm{ANE}(n)$. Then for each $\varepsilon \in \operatorname{cov} Y$ there exists $\delta \in \operatorname{cov} Y$, $\delta \prec \varepsilon$ such that for every closed subspace $A \subset W$, $\dim W \leqslant n$, and also for all maps $\hat{\alpha} : W \to Y$ and $\beta : A \to Y$ such that* $\operatorname{dist}(\hat{\alpha}\restriction_A, \beta) \prec \delta$, *there exists an extension $\hat{\beta} : W \to Y$, $\hat{\beta}\restriction_A = \beta$, such that* $\operatorname{dist}(\hat{\alpha}, \hat{\beta}) \prec \varepsilon$.

We say that the family $\mathcal{L}$ of subsets in metric space $Y$ is an equi-$\mathrm{LC}^{n-1}$-*family* provided that $\mathcal{L}$ consists of closed subsets in the body $\bigcup\mathcal{L}$ and for any $x \in \bigcup\mathcal{L}$ and $\varepsilon > 0$ there exists $\delta > 0$ such that any map $\phi : S^k \to \mathrm{N}(x;\delta) \cap L$, $L \in \mathcal{L}$ and $k < n$, defined on the boundary of the ball $B^{k+1}$, is extended to the map $\hat{\phi} : B^{k+1} \to \mathrm{N}(x;\varepsilon) \cap L$.

The local $n$-softness of the surjective open map $f : Y \to Z$ where $Y$ is complete metric space, by the Michael Selection Theorem [20], is equivalent to $\{f^{-1}(y)\} \in \text{equi-}\mathrm{LC}^{n-1}$. The following assertion is a corollary of the filtered finite-dimensional selection theorem [20] which is in turn a far reaching generalization of the Michael Selection Theorem.

**Theorem 2.2.** *Let $f : Y \to Z$ be a complete locally $k$-soft surjective map of metric spaces, $A \subset Y$ and $B \subset Z$ closed subsets such that $f(A) = B$ and $f\restriction_A : A \to B$ is a homeomorphism. Then for each neighborhood $Y_k \subset Y$ of $A$ there exist a decreasing sequence $Y_k \supset Y_{k-1} \supset \cdots \supset Y_0$ of open neighborhoods of $A$ and a neighborhood $Z_0 \subset f(Y_0)$ of $B$ in $Z$ such that*

(a) *The embedding $Y_i \cap f^{-1}(z) \hookrightarrow Y_{i+1} \cap f^{-1}(z)$ is $(k-1)$-aspherical for each $z \in Z_0$ and for each $0 \leqslant i \leqslant k$ (i.e. this embedding induces trivial homomorphism of homotopy groups $\pi_i$, $i \leqslant k-1$);*
(b) *For each map $\psi : W \to Z_0$ of $k$-dimensional metric space $W$ there exists a lift $\tilde{\psi} : W \to \operatorname{Cl} Y_k$ of $\psi$ with respect to $f$.*

In the sequel we need the following easy consequence of Theorem 2.2 the direct proof of which is rather cumbersome.

**Proposition 2.3.** *Let $f : Y \to Z$ be a locally $k$-soft complete surjective map of metric spaces. Then for each map $\alpha : W \to Y$ of $k$-dimensional metric space $W$ and for each function $\varepsilon : W \to (0,1)$ there exists a function $\delta : W \to (0,1)$ such that for each map $\beta : W \to Z$ with $\mathrm{dist}(f(\alpha(w)), \beta(w)) < \delta(w)$, $w \in W$, there exists a lift $\tilde{\beta} : W \to Y$ of $\beta$ with respect to $f$ such that $\mathrm{dist}(\alpha(w), \tilde{\beta}(w)) < \varepsilon(w)$, $w \in W$.*

We say that a *dense map* $f : X \to Y$ (i.e. $f(X)$ is dense in $Y$) from ANE($k$)-space $X$ into $Y$ is:

(1) A $\mathrm{UV}^{k-1}$-*map* (briefly, $f \in \mathrm{UV}^{k-1}$) if for each neighborhood $\mathcal{U}(y)$, $y \in Y$, there exists a neighborhood $\mathcal{V}(y)$ such that the embedding $f^{-1}(\mathcal{V}(y)) \hookrightarrow f^{-1}(\mathcal{U}(y))$ is $(k-1)$-aspherical; and
(2) The map $f$ is *approximately polyhedrally $k$-soft* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $k$-dimensional compact polyhedral pair $(W, A)$ and for all maps $\varphi : A \to X$ and $\psi : W \to Y$ with $\mathrm{dist}(\psi, f \circ \varphi) < \delta$, there exists an extension $\hat{\varphi} : W \to X$ of $\varphi$ satisfying $\mathrm{dist}(\psi, f \circ \hat{\varphi}) < \varepsilon$.

In general, a $\mathrm{UV}^{k-1}$-preimage of ANE($k$)-space is not an ANE($k$)-space. But there exists one important exception:

(3) If $X_0 \hookrightarrow X \in \mathrm{UV}^{k-1}$ and $X \in \mathrm{ANE}(k)$, then $X_0 \in \mathrm{ANE}(k)$.

The following criterion is well known (see [19]):

**Proposition 2.4.** *If $f \in \mathrm{UV}^{k-1}$, then $f$ is an approximately polyhedrally $k$-soft map. Conversely, if $f$ is an approximately polyhedrally $k$-soft map and $Y \in \mathrm{ANE}(k)$, then $f \in \mathrm{UV}^{k-1}$.*

From Proposition 2.4 we deduce several known properties of $\mathrm{UV}^{k-1}$-maps.

**Proposition 2.5.** *If $f : X \to Y \in \mathrm{UV}^{k-1}$ and $Y$ is complete, then $Y \in \mathrm{ANE}(k)$. If additionally $X \in \mathrm{AE}(k)$, then $Y \in \mathrm{AE}(k)$.*

**Proposition 2.6.** *Let $g : X \to Y$ be a $\mathrm{UV}^{k-1}$-map, and let $f : X \to Z$ and $h : Y \to Z$ be maps between ANE($k$)-spaces such that $f = h \circ g$. Then $f \in \mathrm{UV}^{k-1}$ if and only if $h \in \mathrm{UV}^{k-1}$.*

**Proposition 2.7.** *If $f : X \to Y$ is a $\mathrm{UV}^{k-1}$-map of ANE($k$)-spaces, and $f^{-1}(Y_0) \hookrightarrow X \in \mathrm{UV}^{k-1}$, then $f\restriction : f^{-1}(Y_0) \to Y_0 \in \mathrm{UV}^{k-1}$.*

Recall that the *fiberwise product* $W = X_f \times_g Z$ of $X$ and $Z$ with respect to $f : X \to Y$ and $g : Z \to Y$ is the subset $\{(x,z) \mid f(x) = g(z)\} \subset X \times Z$. The projections of $X \times Z$ onto $Z$ and onto $X$ generate the maps $f' : W \to Z$ and $g' : W \to X$ which are called the *projections parallel $f$ and $g$*, respectively. We write it $f' \parallel f$ and $g' \parallel g$ for brevity.

Several properties of maps are inherited by parallel projections. For instance, the softness envelope $\mathfrak{S}(f)$ is contained in $\mathfrak{S}(f')$. The following is easily established:

(c) If $f$ is $n$-soft and $g \in \mathrm{UV}^{n-1}$, then $f' \parallel f$ is $n$-soft, and $g' \parallel g$ is a $\mathrm{UV}^{n-1}$-map.

In [7] we described the reasonable part of softness envelope of Dranishnikov's resolution $d_n$.

**Definition 2.8.** The pair $(Z, A)$ is called *$n$-conservative* if any partial lift $\phi : A \to S^n \times S^n$ of $\psi : Z \to S^n$ with respect to the projection $\mathrm{pr}_2 : S^n \times S^n \to S^n$ of the $n$-spheres product onto the second factor is extended to the global lift $\hat{\phi} : Z \to S^n \times S^n$ of $\psi$ such that $(\hat{\phi})^{-1}(\mathrm{Diag}) \subset A$.

The map $f : X \to Y$ which is soft with respect to all $n$-dimensional $n$-conservative pairs $(Z, A)$ is called *$n$-conservatively soft*. Dranishnikov's resolution $d_n$ is $n$-conservatively soft [7]. This, in turn, implies all known soft properties of $d_n$.

**Definition 2.9.** *The Polish space $X$* (i.e. complete and separable) is called *strongly $k$-universal with respect to Polish spaces* if any map $\phi : Z \to X$ of Polish space $Z$, $\dim Z \leqslant k$ is arbitrarily closely approximated by closed embedding.

**Definition 2.10.** Let $\{I_i^k\}$ be a countable family of $k$-dimensional disks, and $D$ – their discrete union $\coprod\{I_i^k \mid 1 \leqslant i < \infty\}$. The space $X$ is called *discretely $I^k$-approximated* if any map $\phi : D \to X$ is arbitrarily closely approximated by a map $\tilde{\phi} : D \to X$[1] with discrete $\{\tilde{\phi}(I_i^k) \mid 1 \leqslant i < \infty\}$.

[1] This means that for each function $\varepsilon : X \to (0,1)$ there exists a map $\tilde{\phi}$ such that $\mathrm{dist}(\tilde{\phi}(d), \phi(d)) < \varepsilon(\phi(d))$ for all $d \in D$.

For Polish ANE($k$)-space the property of strong $k$-universality with respect to Polish spaces and discrete $I^k$-approximation are equivalent [10, p. 127]. In [1–4] the following criterion for the Nöbeling space was established.

**Theorem 2.11.** *The Polish space $X$ of dimension $k$ is homeomorphic to the Nöbeling space $\nu^k$ if and only if $X$ is an* AE($k$)*-space which is strongly $k$-universal with respect to Polish spaces.*

Let $\mathcal{C}$ be a class of spaces. Recall [9] that a space $X$ is called *strongly $\mathcal{C}$-universal* if any map $f : D \to X$ of $D \in \mathcal{C}$, the restriction of which on a closed subspace $C \in \mathcal{C}$ is a $Z$-embedding, is arbitrarily closely approximable by a $Z$-embedding $f' : D \to X$ with $f\restriction_C = f'\restriction_C$. Under the *universal $k$-dimensional pseudoboundary* we understand a $\mathcal{C}_{c(k)}$-absorber $X$, where $\mathcal{C}_{c(k)}$ means the class of all $k$-dimensional compacta. This means that $k$-dimensional $\sigma$-compact AE($k$)-space $X$ is strongly $\mathcal{C}_{c(k)}$-universal and discretely $I^k$-approximated. The paper [22] called attention to the fact that Theorem 2.11 implies the uniqueness of the topological type of the universal $k$-dimensional pseudoboundary.

**Theorem 2.12.** *([14,1]) Any two universal $k$-dimensional pseudoboundaries are homeomorphic.*

## 3. Basic properties of UV$^{n-1}$-divider

By $\mathcal{P}$ we denote a subclass of $n$-soft maps of ANE($n$)-spaces, $\mathcal{P} \subset \mathfrak{S}_n$.[2]

**Definition 3.1.** A proper map $h : Y \to Z$ between ANE($n$)-spaces is called

(i) A UV$^{n-1}$-*divider* of $f : X \to Z$ if there exists a topological embedding $g : X \hookrightarrow Y \in \mathrm{UV}^{n-1}$ such that $f = h \circ g$ (i.e. $Y_0 \rightleftharpoons g(X)$ is dense $G_\delta$ in $Y$, $Y_0 \hookrightarrow Y \in \mathrm{UV}^{n-1}$ and $f = h\restriction_{Y_0}$); and
(ii) A UV$^{n-1}$-*divider* of $\mathcal{P}$ if $h$ is a UV$^{n-1}$-divider of some map $f : X \to Z \in \mathcal{P}$.

Our interest is basically in the UV$^{n-1}$-dividers $h : Y \to Z$ of $\mathcal{P}$ with $\dim Y = n \leqslant \dim Z$. The first nontrivial example of a UV$^{n-1}$-divider was constructed in [13]. Prior to establishing that Dranishnikov's resolution is a UV$^{n-1}$-divider of Chigogidze's resolution we present their general properties. It follows from Proposition 2.6 that a UV$^{n-1}$-divider $h$ of $f$ is UV$^{n-1}$ iff $f \in \mathrm{UV}^{n-1}$. Hereafter and also from $\mathcal{P} \subset \mathfrak{S}_n$ it easily follows that:

**Proposition 3.2.** *Any* UV$^{n-1}$*-divider of $\mathcal{P}$ is an open* UV$^{n-1}$*-map between* ANE($n$)*-spaces.*

If in the definition of UV$^{n-1}$-divider we restricted ourselves to compact spaces, then the conclusion of Proposition 3.2 can be essentially strengthened [19,7].

**Theorem 3.3** *(On division of locally $n$-soft maps of compact spaces). Let the locally $n$-soft map $f : X \to Z$ be a composition of a* UV$^{n-1}$*-map $g : X \to Y$ and a map $h : Y \to Z$. If all spaces $X$, $Y$ and $Z$ are* ANE($n$)*-compacta, then $h$ is locally $n$-soft.*

By Proposition 2.4, any UV$^{n-1}$-map is approximately polyhedrally $k$-soft. On the other hand, the passing to the preimage with respect to $n$-soft map preserves $Z_n$-sets. These facts easily imply:

**Proposition 3.4.** *If $h : Y \to Z$ is a* UV$^{n-1}$*-divider of $\mathcal{P}$, then $h^{-1}(F) \subset_{Z_n} Y$, for each $F \subset_{Z_n} Z$.*

We say that the subclass $\mathcal{P}$ is closed with respect to

(1) *Composition* if for any $f : X \to Y \in \mathcal{P}$ and $g : Y \to Z \in \mathcal{P}$, $g \circ f : X \to Z \in \mathcal{P}$; and
(2) *Passing to complete preimages* if for any $f : X \to Y \in \mathcal{P}$ and ANE($n$)-subspace $Y_0 \hookrightarrow Y$, $f\restriction_{X_0} : X_0 \rightleftharpoons f^{-1}(Y_0) \to Y_0 \in \mathcal{P}$.

It can be seen that the class of all $n$-soft strongly $n$-universal maps of Polish ANE($n$)-spaces (which are the basic interest of this paper) satisfies the conditions (1) and (2).

**Proposition 3.5.** *Let $\mathcal{P}$ be a class which is closed both with respect to composition and passing to complete preimages. If $h_1 : Y_1 \to Y_2$ is a* UV$^{n-1}$*-divider of $f_1 : X_1 \to Y_2 \in \mathcal{P}$, where $g_1 : X_1 \hookrightarrow Y_1 \in \mathrm{UV}^{n-1}$, and $h_2 : Y_2 \to Y_3$ is a* UV$^{n-1}$*-divider of $f_2 : X_2 \to Y_3 \in \mathcal{P}$, where $g_2 : X_2 \hookrightarrow Y_2 \in \mathrm{UV}^{n-1}$, then the composition $h_2 \circ h_1 : Y_1 \to Y_3$ is a* UV$^{n-1}$*-divider of the composition $X_0 \rightleftharpoons X_1 \cap (h_1)^{-1}(X_2) \xrightarrow{f_1\restriction_{X_0}} X_2 \xrightarrow{f_2} Y_3 \in \mathcal{P}$.*

[2] Recall that throughout this paper all spaces (in particular, all ANE($n$)-spaces) are assumed to be Polish.

**Proof.** Since $f_1\restriction_{X_0} \in \mathcal{P}$ (as the restriction of $f_1$ onto the complete preimage $X_0 = (f_1)^{-1}(X_2))$ and $f_2 \circ f_1\restriction_{X_0} \in \mathcal{P}$ (as the composition of maps from $\mathcal{P}$), it suffices to prove that $e: X_0 \hookrightarrow Y_1$ is a $\mathrm{UV}^{n-1}$-embedding. But the embedding $e_1: X_0 \hookrightarrow X_1 \in \mathrm{UV}^{n-1}$, being a parallel projection in the fiberwise product of $n$-soft map $f_1: X_1 \to Y_2$ and $\mathrm{UV}^{n-1}$-embedding $g_2: X_2 \hookrightarrow Y_2$ (see 2.7(a)). Then the embedding $e \in \mathrm{UV}^{n-1}$, being a composition of $\mathrm{UV}^{n-1}$-maps $e_1$ and $g_1$. $\square$

We give one more property of $\mathrm{UV}^{n-1}$-dividers.

**Proposition 3.6.** *Let $\mathcal{P}$ be closed with respect to passing to complete preimages, and $h: Y \to Z$ be a $\mathrm{UV}^{n-1}$-divider of $f: X \to Z \in \mathcal{P}$, where $g: X \hookrightarrow Y \in \mathrm{UV}^{n-1}$. Then for any $Z_0 \hookrightarrow Z \in \mathrm{UV}^{n-1}$ the following holds*:

(3) $f\restriction_{X_0}: X_0 \to Z_0 \in \mathcal{P}$ *where* $X_0 \rightleftharpoons f^{-1}(Z_0) = h^{-1}(Z_0) \cap X$;
(4) $X_0 \hookrightarrow Y \in \mathrm{UV}^{n-1}$;
(5) $h\restriction_{Y_0}: Y_0 \to Z_0$ *is a* $\mathrm{UV}^{n-1}$*-divider of* $f$ *where* $Y_0 \rightleftharpoons h^{-1}(Z_0)$; *and*
(6) $Y_0 \hookrightarrow Y \in \mathrm{UV}^{n-1}$.

**Proof.** By the $n$-softness of $f$ it follows that $X_0 \hookrightarrow X \in \mathrm{UV}^{n-1}$ and $X_0 \in \mathrm{ANE}(n)$. Since $Z \in \mathrm{ANE}(n)$ and $Z_0 \hookrightarrow Z \in \mathrm{UV}^{n-1}$, it follows by 2.3(3), that $Z_0 \in \mathrm{ANE}(n)$. Then the conditions imposed on the subclass $\mathcal{P}$ imply that $f\restriction_{X_0} \in \mathcal{P}$, hence (3) is proved. Since $X_0 \hookrightarrow X$ and $g: X \hookrightarrow Y \in \mathrm{UV}^{n-1}$, Proposition 2.6 implies $X_0 \hookrightarrow Y \in \mathrm{UV}^{n-1}$ which proves (4).

The property (5) is equivalent to the following fact.

**Lemma 3.7.** $X_0 \hookrightarrow Y_0 \in \mathrm{UV}^{n-1}$.

**Proof.** We consider a neighborhood $\mathcal{U} \subset Y$ of $y_0 \in Y_0$ and a map of pairs $\varphi: (B^n, S^{n-1}) \to (\mathcal{U} \cap Y_0, \mathcal{U} \cap X_0)$. By virtue of $g \in \mathrm{UV}^{n-1}$, the map $\varphi$ can be arbitrarily closely approximated by $\varphi': B^n \to \mathcal{U} \cap X$ with $\varphi' = \varphi$ on $S^{n-1}$. Also, by $Z_0 \hookrightarrow Z \in \mathrm{UV}^{n-1}$, the map $f \circ \varphi'$ can be arbitrarily closely approximated by $\psi: B^n \to Z_0$ with $\psi = f \circ \varphi$ on $S^{n-1}$. And finally, $n$-softness of $f$ implies the existence of a lift $\tilde{\psi}: B^n \to X_0$ of $\psi$ which coincides with $\varphi$ on $S^{n-1}$ and is arbitrarily close to $\varphi'$. $\square$

It follows by Lemma 3.7 and Proposition 2.5 that $Y_0 \in \mathrm{ANE}(n)$. Since the composition $X_0 \hookrightarrow Y_0 \hookrightarrow Y$ is a $\mathrm{UV}^{n-1}$-embedding it follows by 3.7 and 2.6 that $Y_0 \hookrightarrow Y \in \mathrm{UV}^{n-1}$. Hence (6) is proved. $\square$

Up to the end of the section we fix an $n$-dimensional space $Y$ and a $\mathrm{UV}^{n-1}$-divider $h: Y \to Z$ of an $n$-soft map $f: X \to Z$ (with $g: X \hookrightarrow Y \in \mathrm{UV}^{n-1}$).

**Proposition 3.8.** *If $Z$ is discretely $I^n$-approximated, then $Y$ is also discretely $I^n$-approximated.*

**Proof.** Consider maps $\varepsilon: Y \to (0,1)$ and $\varphi: D \to Y$, where $D$ is a countable discrete union $\coprod\{I_i^n \mid i < \infty\}$ of $n$-dimensional cubes. Since $h$ is proper, we can assume that $\varepsilon$ coincides with $\zeta \circ f$ for a sufficiently small function $\zeta: Z \to (0,1)$.

Since $g \in \mathrm{UV}^{n-1}$, $\varphi$ is approximated by a map $\varphi': D \to X$ such that $\mathrm{dist}(\varphi', \varphi) \prec \varepsilon \circ \varphi$. Next, we approximate $\psi' \rightleftharpoons f \circ \varphi'$ sufficiently closely by a map $\psi: D \to Z$ for which the family $\{\psi(I_i^n) \mid i < \infty\}$ is discrete. By Proposition 2.3, $\psi$ can be $\delta$-lifted to the map $\tilde{\psi}: D \to X$ which is arbitrarily close to $\varphi'$. It can be easily seen that the family $\{\tilde{\psi}(I_i^n)\}$ is discrete in $Y$, and $\tilde{\psi}$ is at the required distance from $\varphi$. $\square$

The proof of Theorem 1.5 will be given in the end of Section 8, and the rest of this section presents some necessary results for this.

Since the notions of strong $n$-universality with respect to Polish spaces and discrete $I^n$-approximateness are equivalent for Polish $\mathrm{ANE}(n)$-spaces, we can assert, using the criterion of the Nöbeling space $\nu^n$ (Theorem 2.11), that

(a) If a Polish space $Z \in \mathrm{AE}(n)$ is strongly $n$-universal with respect to Polish spaces and $\dim Y = n$, then $Y = h^{-1}(Z) \cong \nu^n$.

**Proposition 3.9.** *Let $Z$ be a discretely $I^n$-approximated and strongly $\mathcal{C}_{c(n)}$-universal space, where we denote by $\mathcal{C}_{c(n)}$ the class of all $n$-dimensional compacta. Then $Y = h^{-1}(Z)$ is strongly $\mathcal{C}_{c(n)}$-universal.*

**Proof.** By 3.8, $Y = h^{-1}(Z)$ is a discretely $I^n$-approximated space. Since any compactum in a discretely $I^n$-approximated $\mathrm{ANE}(n)$-space is a $Z_n$-set [9], it follows that

(iii) Any compactum in $Z$ (as well as in $Y$) is a $Z_n$-set.

Let $\varphi: D \to Y$ be a map of an $n$-dimensional compactum $D$ such that its restriction onto a closed subspace $C$ is an embedding. Since $g \in \mathrm{UV}^{n-1}$, we can assume without loss of generality that $\varphi(D \setminus C) \subset X$. It follows from (iii) that

$(h \circ \varphi)(C) \subset_Z Z$. Therefore $h \circ \varphi$ can be arbitrarily closely approximated by a map $\psi : D \to Z$ such that $\psi = h \circ \varphi$ on $C$, and $\psi\restriction_{D\setminus C}$ is an embedding whose image does not intersect $(h \circ \varphi)(C)$.

Since $f : X \to Z$ is $n$-soft, $\psi\restriction_{D\setminus C}$ can be lifted with respect to $f$, by Proposition 2.3, to the map $\tilde{\psi} : D \setminus C \to X$, arbitrarily close to $\varphi\restriction_{D\setminus C} : D \setminus C \to X$. If $\tilde{\psi}$ and $\varphi\restriction_{D\setminus C}$ are sufficiently close, the map $\varphi' : D \to Y$, defined as $\varphi' = \varphi$ on $C$ and $\varphi' = \tilde{\psi}$ on $D \setminus C$, becomes continuous. It is clear that the map $\varphi'$ is an embedding which is arbitrarily close to $\varphi$. Hence the proof is completed. □

The characterization theorem for universal pseudoboundary (Theorem 2.12) and Proposition 3.9 imply that

(b) If $\dim Y = n$ and $Z \in \mathrm{AE}(n)$ is $\sigma$-compact discretely $I^n$-approximated and strongly $n$-universal with respect to compact spaces, then $Y$ is homeomorphic to the $n$-dimensional universal pseudoboundary $\sigma^n$.

## 4. Inverse limit properties of $\mathrm{UV}^{n-1}$-dividers

**Definition 4.1.** A map $f : X \to Y$ is called *strongly n-universal with respect to Polish spaces* if for each $n$-dimensional Polish space $Q$ and also for all maps $\varepsilon : X \to (0, 1)$ and $\varphi : Q \to X$ there exists a closed embedding $\varphi' : Q \hookrightarrow X$ $\varepsilon$-close to $\varphi$ such that $f \circ \varphi' = f \circ \varphi$.

**Definition 4.2.** A map $f : X \to Y$ is called *n-filled* if for each map $\varphi : Z \to X$ of $n$-dimensional Polish space $Q$ there exists a closed embedding $\psi : Q \hookrightarrow X$ such that $f \circ \varphi = f \circ \psi$.

Note that the composition of $n$-soft and $n$-filled maps is $n$-filled.

The commutative diagrams $\mathcal{D}_t$ for $t = 1, 2, 3, \ldots$

$$\begin{tikzcd} X_{t+1} \arrow[r, "\eta_t"] \arrow[d, "f_{t+1}"'] & X_t \arrow[d, "f_t"] \\ Z_{t+1} \arrow[r, "\sigma_t"] & Z_t \end{tikzcd} \qquad (\mathcal{D}_t)$$

generate the map $f : X \rightleftharpoons \varprojlim\{X_t, \eta_t\} \to Z \rightleftharpoons \varprojlim\{Z_t, \sigma_t\}$ of inverse limit of spectra. In general, $n$-softness ($n$-conservative softness and so on) of all maps $f_t$, $\sigma_t$, $\eta_t$ does not imply that $f$ possesses the corresponding property. As care should be taken to see that the map properties are preserved by passage to the inverse limit of spectra, we introduce the following

**Definition 4.3.** The commutative diagram $\mathcal{D}_t$ *possesses a property* $\mathcal{Q}$, if *its characteristic map* $\chi_t : X_{t+1} \to W_t$ into the fiberwise product $W_t \rightleftharpoons (Z_{t+1})_{\sigma_t} \times_{f_t} X_t$, given by $\chi_t(x) = (f_{t+1}(x), \eta_t(x)) \in W_t$, possesses $\mathcal{Q}$.

Basically, we are interested in *n-soft and n-filled* commutative diagrams. The particular case of the following proposition is given in [17, 2.2.4].

**Proposition 4.4.** *Let $f : X \to Z$ be a map of inverse limit of spectra $X \rightleftharpoons \varprojlim\{X_t, \eta_t\}$ and $Z \rightleftharpoons \varprojlim\{Z_t, \sigma_t\}$, generated by commutative diagrams $\mathcal{D}_t$, $t \geqslant 1$. Let also the diagrams $\mathcal{D}_t$, $t \geqslant 1$, be n-soft and n-filled, and $f_1$ and all maps $\sigma_t$, $t \geqslant 1$, n-soft. Then the map $f$ is n-soft and strongly n-universal with respect to Polish spaces.*

**Proof.** Since $f$ is $n$-soft by [17, 3.4.7], we complete the proof of 4.4 as soon as the strong $n$-universality of $f$ will be established. For this purpose, pick any $n$-dimensional Polish space $Q$, any function $\varepsilon : X \to (0, 1)$ assessing closeness of maps, and any map $\varphi : Q \to X$. Let us construct a closed embedding $\varphi' : Q \hookrightarrow X$ which is $\varepsilon$-close to $\varphi$.

Note that the space $X = \varprojlim\{X_t, \eta_t\}$ naturally lies in $\prod\{X_i \mid i \geqslant 1\}$ and $\varphi = (\varphi_1, \varphi_2, \ldots)$, where $\varphi_i$ is a map of $Q$ into $X_i$. It is clear that $\eta_t \circ \varphi_{t+1} = \varphi_t$, for all $t \geqslant 1$.

The open cylinder $\mathcal{U} \subset \prod\{X_i \mid i \geqslant 1\}$ with the base $\mathcal{V} \subset \prod\{X_i \mid 1 \leqslant i \leqslant n\}$ and generators $\{a \times \prod_{i>n} X_i \mid a \in \mathcal{V}\}$, being intersected with $X$, generates the corresponding structure in $X$: the open cylinder $\mathcal{U}_X \subset X$, the base $\mathcal{V}_X \subset \prod\{X_i \mid 1 \leqslant i \leqslant n\}$ and the family of generators. From the definition of $\varprojlim\{X_t\}$ it easily follows that there exists a maximal subset $\tilde{\mathcal{V}} \subset \prod\{X_i \mid 1 \leqslant i \leqslant n\}$, the intersection of which with $X$ equals the chosen base $\mathcal{V}_X$:

(1) $\tilde{\mathcal{V}} = X_1 \times \cdots \times X_{n-1} \times \mathcal{W}_X$, where the set $\mathcal{W}_X \subset X_n$ is open (we further identify the base $\mathcal{V}_X$ with this set $\mathcal{W}_X$).

It is easy to establish the existence of the increasing sequence $\mathcal{U}_X(1) \Subset \mathcal{U}_X(2) \Subset \cdots \subset X$ of open cylindrical sets for which

(2) $\bigcup \mathcal{U}_X(i) = X$;
(3) $\mathcal{U}_X(2i-1) \Subset \mathcal{U}_X(2i)$ have the bases $\mathcal{W}_X(2i-1) \Subset \mathcal{W}_X(2i) \subset X_{n_i}$ for all $i \geqslant 1$ (we can assume without loss of generality that $n_i = i$); and
(4) The map $\varepsilon$ *has small oscillation on generators of* $\mathcal{U}_X(i)$, i.e. for any $x$ and $x'$ from one generator we have $|\varepsilon(x) - \varepsilon(x')| < \frac{\varepsilon(x)}{10}$.

Let $\xi_i = \chi_i \circ \varphi_{i+1} : Q \to W_i$, where $\chi_i : X_{i+1} \to W_i = (Z_{i+1})_{\sigma_i} \times_{f_i} X_i$ is a characteristic map of $\mathcal{D}_t$; $A_{2i} \rightleftharpoons \xi_i^{-1}(Z_{i+1} \times \mathcal{W}_X(2i)) \subset Q$ and $A_{2i-1} \rightleftharpoons \xi_i^{-1}(Z_{i+1} \times \mathcal{W}_Y(2i-1)) \subset Q$. Since $\sigma_i' \circ \chi_i = \eta_i$, we have

(5) $A_1 \Subset A_2 \Subset \cdots \subset Q$ and $\bigcup A_i = Q$.

Fix a refining sequence of open covers $\omega_i \in \operatorname{cov} Q$. As the characteristic map $\chi_1 : X_2 \to W_1$ is $n$-soft and $n$-filled, there exists a map $\varphi_2' : Q \to X_2$ such that $\chi_1 \circ \varphi_2'$ equals $\xi_1 = \chi_1 \circ \varphi_2$, and moreover,

(†)$_1$ $\varphi_2' \restriction_{\mathrm{Cl}\, A_1}$ is an $\omega_1$-map; and
(†)$_2$ $\varphi_2' = \varphi_2$ outside $A_2 \subset Q$.

By the same reason, there exists a map $\varphi_3' : Q \to X_3$ such that $\chi_2 \circ \varphi_3'$ equals $\xi_2 = \chi_2 \circ \varphi_3$, and moreover,

(†)$_3$ $\varphi_3' \restriction_{\mathrm{Cl}\, A_3}$ is an $\omega_2$-map; and
(†)$_4$ $\varphi_3' = \varphi_3$ outside $A_4 \subset Q$.

It should now be clear to the reader how to continue these constructions, a result of which are the maps $\{\varphi_i' \mid i \geqslant 1\}$ (for definiteness sake we suppose $\varphi_1' = \varphi_1$). Since $\eta_t \circ \varphi_{t+1}' = \varphi_t'$ for all $t \geqslant 1$, we have that $\varphi' \rightleftharpoons (\varphi_1', \varphi_2', \varphi_3', \ldots)$ is a map passing $Q$ into $X$. It is clear that

(6) $f \circ \varphi' = f \circ \varphi$.

Let $q \in A_{2m} \setminus A_{2m-2}$. For $1 \leqslant l \leqslant m-1$ it follows from (†)$_{2l}$ that

(7) $\varphi_i(q) = \varphi_i'(q)$ for all $i \leqslant m$.

Since the oscillation of $\varepsilon$ on generators of the cylinder $U_X(2m-2)$ is small, $\varphi(q)$ and $\varphi'(q)$ are $\varepsilon(\varphi(q))$-close, i.e. $\varphi' \overset{\varepsilon}{\sim} \varphi$. The straightforward check using "odd" properties $\{(\dagger)_{2i-1}\}$ permits us to assert that $\varphi'$ is a closed embedding of $Q$ into $X$. $\square$

Now we track the $\mathrm{UV}^{n-1}$-division property by a passage to inverse limit of spectra. The following auxiliary assertion permits us to make further inductive step.

**Proposition 4.5.** *Let the commutative diagram $\mathcal{E}_t$*

$$\begin{tikzcd} K_{t+1} \arrow[r, "\theta_t"] \arrow[d, "h_{t+1}"'] & K_t \arrow[d, "h_t"] \\ Z_{t+1} \arrow[r, "\sigma_t"] & Z_t \end{tikzcd} \tag{$\mathcal{E}_t$}$$

*(more precisely, its characteristic map) be a $\mathrm{UV}^{n-1}$-divider of an $n$-soft $n$-filled map. If $\sigma_t$ is $n$-soft, $h_t : K_t \to Z_t$ is a $\mathrm{UV}^{n-1}$-divider of $n$-soft map $f_t : X_t \to Z_t$ where $X_t \hookrightarrow K_t \in \mathrm{UV}^{n-1}$, then $h_{t+1}$ is a $\mathrm{UV}^{n-1}$-divider of $n$-soft map $f_{t+1} : X_{t+1} \to Z_{t+1}$ where $X_{t+1} \hookrightarrow K_{t+1} \in \mathrm{UV}^{n-1}$. Moreover $\theta_t(X_{t+1}) \subset X_t$, and the commutative diagram $\mathcal{D}_t$ in which $\eta_t \rightleftharpoons \theta_t \restriction_{X_{t+1}}$, is $n$-soft and $n$-filled.*

**Proof.** Let, for definiteness sake, the characteristic map $\chi_t : K_{t+1} \to W_t = (Z_{t+1})_{\sigma_t} \times_{h_t} K_t$ be a $\mathrm{UV}^{n-1}$-divider of $n$-soft and $n$-filled map $r : K_{t+1}' \to W_t$ where $K_{t+1}' \hookrightarrow K_{t+1} \in \mathrm{UV}^{n-1}$. We use the notation for parallel projection: $\sigma_t' \parallel \sigma_t$ and $h_t' \parallel h_t$.

From $n$-softness of $\sigma_t$ it follows that $\tilde{W}_t \rightleftharpoons (\sigma_t')^{-1}(X_t) \hookrightarrow W_t \in \mathrm{UV}^{n-1}$. From here and from Proposition 3.6 (applied to the class $\mathcal{P}$ of all $n$-soft strongly $n$-universal maps of Polish ANE($n$)-spaces) it easily follows that

(a) $X_{t+1} \rightleftharpoons \theta_t^{-1}(X_t) \cap K_{t+1}' = \chi_t^{-1}(\tilde{W}_t) \cap K_{t+1}' \hookrightarrow K_{t+1} \in \mathrm{UV}^{n-1}$; and
(b) $h_t'$ is a $\mathrm{UV}^{n-1}$-divider of $h_t' \restriction_{\tilde{W}_t}$.

It follows by Proposition 3.5 on composition for $\mathcal{P}$ that the composition $h_{t+1} = h_t' \circ \chi_t$ is a $\mathrm{UV}^{n-1}$-divider of $f_{t+1} \rightleftharpoons h_{t+1} \restriction_{X_{t+1}} : X_{t+1} \to Z_{t+1}$. $\square$

From Propositions 4.4 and 4.5 one can derive the basic technical result, the further application of which for Dranishnikov's resolution permits us to represent it as a $\mathrm{UV}^{n-1}$-divider of the corresponding Chigogidze's resolution.

**Theorem 4.6.** *Let $h: K \to Z$ be a map of inverse limits of spectra $K \rightleftharpoons \varprojlim\{K_t, \theta_t\}$ and $Z \rightleftharpoons \varprojlim\{Z_t, \sigma_t\}$, generated by commutative diagrams $\mathcal{E}_t$, which are n-conservatively soft for each $t \geqslant 1$. Let also for each t,*

(c) *The map $\sigma_t$ be n-soft*;
(d) *The diagram $\mathcal{E}_t$ be a $\mathrm{UV}^{n-1}$-divider of n-soft n-filled map*; *and*
(e) *The map $h_1: K_1 \to Z_1 \in \mathrm{ANE}(n)$ be n-conservatively soft and a $\mathrm{UV}^{n-1}$-divider of n-soft map $f_1: X_1 \to Z_1$, where $X_1 \hookrightarrow K_1 \in \mathrm{UV}^{n-1}$.*

*Then the map $f: X \to Z$ of inverse limits of spectra $X \rightleftharpoons \varprojlim\{X_t, \eta_t\}$ and Z, generated by commutative diagrams $\mathcal{D}_t$, $t \geqslant 1$, from Proposition 4.5,*[3] *satisfies the following properties*:

(f) *$f$ is n-soft strongly n-universal with respect to Polish spaces, and X, Z are Polish $\mathrm{ANE}(n)$-spaces*;
(g) *h is n-conservatively soft*; *and*
(h) *h is a $\mathrm{UV}^{n-1}$-divider of f.*

**Proof.** By Proposition 4.4, the map $f: X \to Z$ is $n$-soft strongly $n$-universal with respect to Polish spaces. From $n$-softness of maps $\sigma_t$ and $Z_1 \in \mathrm{ANE}(n)$ it follows that $Z \in \mathrm{ANE}(n)$, and hence $X$ is $\mathrm{ANE}(n)$. The property (g) follows from [17, 3.4.7].

Since the inverse spectrum $\{X_t, \eta_t\}$ consists of $n$-soft projections, and each embedding $X_t \hookrightarrow K_t$, by 4.5, is $\mathrm{UV}^{n-1}$, it follows that $X \hookrightarrow K \in \mathrm{UV}^{n-1}$ which proves (h). $\square$

## 5. Multivalued retraction of a ball onto its boundary

In the next two sections we outline (after [16]) the base of the construction of Dranishnikov's resolution: a multivalued retraction of the ball onto its boundary (going back to I.M. Kozlowski) and multivalued retraction of a polyhedron onto its $k$-dimensional skeleton.

Let $\partial B^{n+1}$ be the boundary of unit ball $B^{n+1}$, $n \geqslant 1$. By $B^{n+1}_y$, $y \in \partial B^{n+1}$, we denote the ball of radius 3/4, tangent to the sphere $\partial B^{n+1}$ in $y$. It is evident that the multivalued mappings $\mathcal{Q}_{n+1}: \partial B^{n+1} \rightsquigarrow B^{n+1}$, $\mathcal{Q}_{n+1}(y) = B^{n+1}_y$, and $\mathcal{P}_{n+1}: B^{n+1} \rightsquigarrow \partial B^{n+1}$, $\mathcal{P}_{n+1}(x) = \{y \in \partial B^{n+1} \mid B^{n+1}_y \ni x\}$, are inverse each to other. Since

(1) The restriction $\mathcal{P}_{n+1}$ on $\partial B^{n+1}$ is the identity, i.e. $\mathcal{P}_{n+1}(x) = x$, for each $x \in \partial B^{n+1}$,

$\mathcal{P}_{n+1}$ is a multivalued retraction of the ball onto its boundary. It is the base of the construction of Dranishnikov's resolution. We list several rather easy properties of $\mathcal{P}_{n+1}$ which will be used later:

**Lemma 5.1.**

(2) $\{x \in B^{n+1} \mid \mathcal{P}_{n+1}(x) = \partial B^{n+1}\} = \frac{1}{2} \cdot B^{n+1}$;
(3) $\mathcal{P}_{n+1}(x) \subsetneqq \mathcal{P}_{n+1}(a \cdot x)$ *for all* $x \in B^{n+1} \setminus \frac{1}{2} \cdot B^{n+1}$ *and* $a < 1$; *and*
(4) $\mathcal{P}_{n+1}(a \cdot y) \not\ni (-y)$, *for all* $y \in \partial B^{n+1}$ *and* $\frac{1}{2} < a \leqslant 1$.

Next, consider the graph $D_{n+1} \rightleftharpoons \{(y, x) \mid x \in B^{n+1}_y\} \subset \partial B^{n+1} \times B^{n+1}$ of the map $\mathcal{Q}_{n+1}$. It is clear that $D_{n+1}$ and the graph of $\mathcal{P}_{n+1}$ are symmetric with respect to the permutation of $x$- and $y$-coordinates. By (1), $\partial B^{n+1}$ is naturally contained in $D_{n+1}$. Concerning the natural projections $p_{n+1}: D_{n+1} \to B^{n+1}$ and $q_{n+1}: D_{n+1} \to \partial B^{n+1}$ of the graph $D_{n+1}$ onto its factors, the following is known [7]:

**Proposition 5.2.** *$p_{n+1}$ is n-conservatively soft, and $q_{n+1}$ is a soft retraction.*

Moreover, since $q^{-1}_{n+1}(y) = B^{n+1}_y$ and $\{y\} \subset_Z B^{n+1}_y$, $y \in \partial B^{n+1}$, it follows that

(5) $\partial B^{n+1} \subset D_{n+1}$ is a fiberwise $Z$-set with respect to $q_{n+1}$, i.e. for each partial map $Z \hookleftarrow A \xrightarrow{\varphi} D_{n+1}$ which is the local lift of $\psi: Z \to \partial B^{n+1}$, there exists an extension $\hat{\varphi}: Z \to D_{n+1}$ of the map $\varphi$, which is a global lift of $\psi$, such that $\hat{\varphi}(Z \setminus A)$ does not intersect $\partial B^{n+1}$.

We conclude this section by studying of the $\mathrm{UV}^n$-division of $p_{n+1}$. Let

$$T^n \rightleftharpoons \left\{\left(y, -\frac{1}{2} \cdot y\right) \,\middle|\, y \in \partial B^{n+1}\right\} \quad \text{and} \quad C_{n+1} \rightleftharpoons D_{n+1} \setminus T^n \text{ be an open subset } D_{n+1}.$$

[3] More precisely, the commutative diagram $\mathcal{E}_t$ and the map $f_t$ (from $\mathcal{E}_{t-1}$ for $t > 1$) generates, by 4.5, the commutative diagram $\mathcal{D}_t$, $t = 1, 2, \ldots$.

**Proposition 5.3.** *The map $p_{n+1}: D_{n+1} \to B^{n+1}$ is a* $\mathrm{UV}^n$*-divider of the $n$-soft map $p_{n+1}\restriction_{C_{n+1}}: C_{n+1} \to B^{n+1}$.*

**Proof.** The fact that the map $p_{n+1}\restriction_{C_{n+1}}$ of complete spaces is $n$-soft follows from $\{(p_{n+1})^{-1}(x) \cap C_{n+1} \mid x \in B^{n+1}\} \in$ equi-$\mathrm{LC}^{n-1}$.

The homotopy $h_t: B^{n+1} \to B^{n+1}$, $t \in I$, given by $h_t(x) = (1 - t(1 - \|x\|)) \cdot x$ is called *radial*. It joins Id with $h_1 = \|x\| \cdot x$. Consider also the continuous homotopy $H_t: D_{n+1} \to \partial B^{n+1} \times B^{n+1}$, $0 \leqslant t \leqslant 1$, given by $H(y, x) = (y, h_t(x))$.

Finally, the property $C_{n+1} = D_{n+1} \setminus T^n \hookrightarrow D_{n+1} \in \mathrm{UV}^{n-1}$ easily follows from the assertion given below. □

**Lemma 5.4.** *For all $t \geqslant 0$, we have $H_t(D_{n+1}) \subset D_{n+1}$, and also*

(a) *$H_0 = \mathrm{Id}$ and $H_t(D_{n+1}) \cap T^n = \emptyset$ for all $t > 0$.*

**Proof.** Since $h_t(x) = a \cdot x$, where $a \leqslant 1$, it follows by 5.1(iii) that $\mathcal{P}_{n+1}(x) \subset \mathcal{P}_{n+1}(h_t(x))$. Hence, if $y \in \mathcal{P}_{n+1}(x)$, then $y \in \mathcal{P}_{n+1}(h_t(x))$, i.e. $H_t(D_{n+1}) \subset D_{n+1}$.

Suppose that $H_t(y_0, x_0) \in T^n$, for some point $(y_0, x_0) \in D_{n+1}$, i.e. $h_t(x_0) = -\frac{1}{2} \cdot y_0$, where $y_0 \in \partial B^{n+1}$. Since $h_t(x_0) = b \cdot x_0$ where $b < 1$, $x_0 = \alpha \cdot (-y_0)$ for $\alpha > \frac{1}{2}$. In view of (4) we have $\mathcal{P}_{n+1}(x_0) \not\ni -(-y_0) = y_0$, i.e. $(y_0, x_0) \notin D_{n+1}$, a contradiction. □

Fix a point (called a center) $O$ of the relative interior $\operatorname{rint} \Delta^{n+1}$ of a simplex. Then the dilation with center $O$ generates a multiplication $a \cdot x$ for $x \in \Delta^{n+1}$ and $0 \leqslant a \leqslant 1$. By the *antipode to* $y \in \partial \Delta^{n+1}$ we understand the intersection of the ray $[y, O)$ with $\partial \Delta^{n+1}$. Then the multiplication $a \cdot x$ can be extended on all $x \in \Delta^{n+1}$ and $-1 \leqslant a \leqslant 1$. If $x = a \cdot y$, where $y \in \partial \Delta^{n+1}$ and $0 \leqslant a \leqslant 1$, then $a$ is called the *norm* $\|x\|$ of $x$.

Let $\theta: \Delta^{n+1} \to B^{n+1}$ be a *radial homeomorphism*, i.e. $\theta(a \cdot x) = a \cdot \theta(x)$, for all $x \in \Delta^{n+1}$ and $-1 \leqslant a \leqslant 1$. The conjugacy operation with respect to homeomorphism $\theta$ transforms all early obtained constructions for pair $(B^{n+1}, \partial B^{n+1})$ into constructions for pair $(\Delta^{n+1}, \partial \Delta^{n+1})$. In particular, the radial homotopy $h_t: B^{n+1} \to B^{n+1}$ (see 5.3) passes to *the radial homotopy* $\theta^{-1} \circ h_t \circ \theta: \partial \Delta^{n+1} \to \partial \Delta^{n+1}$ which we continue to denote by $h_t$. Since all results obtained earlier are valid also for simplexes, in the case of simplex we will use the previous notations for the corresponding spaces and maps.

## 6. Multivalued retraction of a polyhedron onto its skeleton

Let $k \geqslant 1$ and $P$ be a compact polyhedron of dimension $m$ given with some triangulation $L$. Represent the $(n+1)$-dimensional skeleton $P^{(n+1)}$, $n \geqslant k$, as $\bigcup\{\Delta_i^{n+1} \mid i \geqslant 1\}$. By previous section, the following objects are defined for each $i$: the multivalued map $\mathcal{P}_{n+1}(i): \Delta_i^{n+1} \rightsquigarrow \partial \Delta_i^{n+1}$, the graph $D_{n+1}(i) \subset \partial \Delta_i^{n+1} \times \Delta_i^{n+1}$ of the mapping $\mathcal{Q}_{n+1}(i) = (\mathcal{P}_{n+1}(i))^{-1}$ and the natural projections $p_{n+1}(i): D_{n+1}(i) \to \Delta_i^{n+1}$ and $q_{n+1}(i): D_{n+1}(i) \to \partial \Delta_i^{n+1}$ of $D_{n+1}(i)$ onto factors.

Since $\mathcal{P}_{n+1}(i): \Delta_i^{n+1} \rightsquigarrow \partial \Delta_i^{n+1}$ and $\mathcal{P}_{n+1}(j): \Delta_j^{n+1} \rightsquigarrow \partial \Delta_j^{n+1}$ for $\Delta_i^{n+1} \cap \Delta_j^{n+1} \neq \emptyset$ agree on the common domain (where they are identical), we have that

$$D_{n+1}^n \rightleftharpoons \left\{(a_n, a_{n+1}) \in \partial \Delta_i^{n+1} \times \Delta_i^{n+1} \;\middle|\; (a_n, a_{n+1}) \in D_{n+1}(i)\right\}$$

contains in a natural manner the union of the boundaries of all simplexes $\Delta_i^{n+1}$. Also, the natural projections $p_{n+1}^n: D_{n+1}^n \to P^{(n+1)}$ and $q_{n+1}^n: D_{n+1}^n \to P^{(n)}$ are correctly defined. The following is true:

(i) $(a_n, a_{n+1}) \in D_{n+1}^n$ and $a_{n+1} \in P^{(n)}$ imply $a_n = a_{n+1}$.

Hence $P^{(n)}$ is naturally contained in $D_{n+1}^n$, and $q_{n+1}^n$ is a retraction. It was known that $p_{n+1}^n$ is $n$-conservatively soft map [7], but $q_{n+1}^n$ fails to be soft map.

We consider the increasing sequence $P^{(s)} \subset P^{(s+1)} \subset \cdots \subset P^{(t-1)} \subset P^{(t)}$, $k \leqslant s < t \leqslant m$, of the skeleta of the $m$-dimensional polyhedron $P$ which generates the following objects: $D_t^s \rightleftharpoons \{a = (a_s, a_{s+1}, \ldots, a_{t-1}, a_t) \in P^{(s)} \times \cdots \times P^{(t-1)} \times P^{(t)} \mid (a_i, a_{i+1}) \in D_{i+1}^i,\ s \leqslant i < t\}$, the maps $p_t^s: D_t^s \to P^{(t)}$ and $q_t^s: D_t^s \to P^{(s)}$ by formulas $p_t^s(a) \rightleftharpoons a_t$ and $q_t^s(a) \rightleftharpoons a_s$, respectively.

We note that the map $p_t^s$ is $n$-conservatively soft [7] and $q_m^s \circ (p_m^s)^{-1}$ is a multivalued retraction of $P$ onto its skeleton $P^{(s)}$. Dranishnikov proved that $D_t^s \in \mathrm{ANE}$ and formulated without proof the following plausible (and, apparently, difficult) assertion [16, p. 124].

**Conjecture 6.1.** *The compactum $D_t^s$ is a polyhedron* (*and therefore it is an* ANE).

We also do not want to spend effort on the proof of this conjecture as the basic result of the present paper does not depend on its validity (in the case of the conjecture failure, one must draw on the Edwards Theorem and Chapman Theorem from $Q$-manifold theory as it was done in [16]). But for the simplicity of the text we do assume that $D_t^s$ *is a polyhedron*.

Because of this we replace Theorem 1.2 with the following assertion.

**Theorem 6.2.** *Let $P$ be a compact polyhedron with the triangulation $L$ and $k \geqslant 1$. Then there exist a compact polyhedron $D$ and maps $p : D \to P$ and $q : D \to P^{(k)}$ such that* 1.2(1)–(3) *hold.*

## 7. Synchronized Hurewich fibration and the proof of Theorem 6.2

The map $q^s_t : D^s_t \to P^{(s)}$ fails to be soft as for example it is not open. But nevertheless a weak softness property of $q^s_t$ can be detected which will be a key moment in our arguments.

**Theorem 7.1.** *The projection $q^s_t$ is a synchronized Hurewich fibration.*

Below we explain the introduced notion.

**Definition 7.2.** The homotopy $\varphi : X \times I \to P^{(s)}$ is called *synchronized* if

(1) $\varphi^{-1}(\Delta) = (\varphi_0)^{-1}(\Delta) \times I$, for each simplex $\Delta \subset P^{(s)}$.

In other words, (1) means that, if $\varphi_0(x) \in \Delta$, then $\varphi_t(x) \in \Delta$ for all $t \in I$.

**Definition 7.3.** The map $f : D^s_t \to P^{(s)}$ is called *synchronized Hurewich fibration* if for each synchronized homotopy $\varphi : X \times I \to P^{(s)}$ and for each partial lift $\theta_0 : X \times \{0\} \to D^s_t$ of $\varphi_0$ with respect to $f$ there exists a homotopy $\theta : X \times I \to D^s_t$ lifting the homotopy $\varphi$ such that $p^s_t \circ \theta : X \times I \to P^{(t)}$ is also synchronized homotopy.

First we prove the partial case of Theorem 7.1:

**Proposition 7.4.** *The projection $q^n_{n+1}$ is a synchronized Hurewich fibration.*

Then applying Proposition 7.4 several times, it can be easily proved that $q^s_t$ is also a synchronized Hurewich fibration.

**Proof.** Let $\varphi : X \times I \to P^{(n)}$ be a synchronized homotopy and $\psi_0 : X \times \{0\} \to P^{(n+1)}$ a map. It is sufficient to establish that if the map $\theta_0 \rightleftharpoons (\varphi_0, \psi_0)$ transforms $X$ into $D^n_{n+1}$, then there exists a synchronized homotopy $\psi : X \times I \to P^{(n+1)}$ extending $\psi_0$ such that $\theta \rightleftharpoons (\varphi, \psi)$ is a homotopy of $X$ into $D^n_{n+1}$.

Let $P^{(n+1)} = \{\Delta^{(n+1)}_i \mid i \geqslant 1\}$. Consider the following subsets of $X$: $X_0 \rightleftharpoons (\psi_0)^{-1}(P^{(n)})$ and $X_i \rightleftharpoons (\psi_0)^{-1}(\Delta^{(n+1)}_i)$. It is clear that $X = \bigcup X_i$ and

(2) $X_i \setminus X_0 \subset \operatorname{Int} X_i$ for all $i \geqslant 1$.

It follows from (i) (see Section 6) that $\varphi_0 = \psi_0$ on $X_0$. As $\psi_0(X_i) \subset \Delta^{(n+1)}_i$ and $\operatorname{Im}(\theta_0) \subset D^n_{n+1}$, then $\varphi_0(X_i) \subset \partial\Delta^{(n+1)}_i$. Since $\varphi$ is the synchronized homotopy, we have $\varphi_t(X_i) \subset \partial\Delta^{(n+1)}_i$, for each $t \in I$.

Given $i \geqslant 1$, consider the following commutative diagram,

$$
\begin{tikzcd}
D_{n+1}(i) \arrow[r, "q_{(n+1)}(i)"] & \partial\Delta^{(n+1)}_i \\
A_i \arrow[u, "\sigma_i"] \arrow[r, hook] & X_i \times I \arrow[u, "\varphi"']
\end{tikzcd}
$$

in which $A_i \rightleftharpoons ((X_i \cap X_0) \times I) \cup (X_i \times \{0\})$, $\sigma_i = \theta_0$ on $X_i \times \{0\}$ and $\sigma_i = (\varphi, \varphi)$ on $(X_i \cap X_0) \times I$. Since $q_{n+1}(i)$ is soft, there exists an extension $\theta_i : X_i \times I \to D_{n+1}(i)$ of $\sigma_i$ such that $q_{n+1}(i) \circ \theta_i = \varphi\restriction_{X_i \times I}$.

By Proposition 5.2, $\partial\Delta^{(n+1)}_i \subset D_{n+1}(i)$ is a fiberwise $Z$-set with respect to $q_{n+1}(i)$. Then $\theta_i$ can be chosen in a such manner that

(3) $\theta_i(X_i \times I \setminus A_i)$ is contained in $D_{n+1}(i) \setminus \partial\Delta^{(n+1)}_i$ (i.e. $p^n_{n+1} \circ \theta_i(X_i \times I \setminus A_i) \subset \operatorname{rint}\Delta^{(n+1)}_i$).

The desired homotopy $\theta : X \times I \to D^n_{n+1}$ equals $\theta_i$ on $X_i \times I$. We can check straightforwardly with help of (2) and (3) that $\theta$ is continuous, and $p^n_{n+1} \circ \theta$ is a synchronized homotopy. $\square$

The homotopy $h_t : P^{(n+1)} \to P^{(n+1)}$ is called *radial*, if its restriction on each simplex $\Delta^{n+1}_i \subset P^{(n+1)}$ is a radial homotopy. It is clear that $h_t$ is identity on $P^{(n)}$. Let $C_{n+1}(i) \subset D_{n+1}(i)$ be an open subset taken from 5.3, and $C^n_{n+1} \rightleftharpoons \bigcup\{C_{n+1}(i) \mid i\}$ an open subset of $D^n_{n+1}$. The proof of the fact that

(4) The map $p^n_{n+1}: D^n_{n+1} \to P^{(n+1)}$ is a $\mathrm{UV}^{n-1}$-divider of the $n$-soft map $p^n_{n+1}\restriction_{C^n_{n+1}}: C^n_{n+1} \to P^{(n+1)}$

is performed analogously to that of Proposition 5.3 with the help of the fact given below.

**Proposition 7.5.** *The continuous homotopy $H_t: D^n_{n+1} \to D^n_{n+1}$, $0 \leqslant t \leqslant 1$, given by $H(y,x) = (y, h_t(x))$ transforms $D^n_{n+1}$ into $C^n_{n+1}$ for each $t > 0$.*

By $C^s_t$ we denote an open subset $\{a \mid (a_i, a_{i+1}) \in C^i_{i+1}$ for all $i$, $s \leqslant i < t\} \subset D^s_t$. The essential complement of [7] where the $k$-conservative softness of $p^s_t$ was established is the following *key result* which proves Theorem 6.2:

**Theorem 7.6.** *The map $p^k_m: D^k_m \to P = P^{(m)}$ is a $\mathrm{UV}^{k-1}$-divider of the k-soft map $p^k_m\restriction_{C^k_m}: C^k_m \to P$.*

**Proof.** Consider a closed subset

$$F = D^k_m \setminus C^k_m = \{a \mid (a_i, a_{i+1}) \notin C^i_{i+1} \text{ for some } i,\ k \leqslant i < m\} \subset D^k_m$$

and its closed filtration $F_m \subset F_{m+1} \subset \cdots \subset F_{k+2} \subset F_{k+1} = F$ where

$$F_s = \{a \in F \mid (a_i, a_{i+1}) \in C^i_{i+1} \text{ for all } i,\ k \leqslant i < s-1\}.$$

**Lemma 7.7.** *The restriction $p^k_m\restriction_{C^k_m}: C^k_m \to P = P^{(m)}$ is a k-soft map.*

**Proof.** Suppose that the partial map $Z \supset A \xrightarrow{\varphi} C^k_m \xrightarrow{p^k_m} P$, $\dim Z \leqslant k$, has an extension $\psi: Z \to P$. Represent $\varphi$ in the coordinate form $(\varphi_k, \varphi_{k+1}, \ldots, \varphi_m)$ where $\varphi_i$ is the map from $A$ into $P^{(i)}$. Then $\varphi_m = \psi\restriction_A$. Since $p^i_{i+1}: C^i_{i+1} \to P^{(i+1)}$ is $k$-soft for all $i$, we can construct, by inverse induction on $m, m-1, \ldots, k+1$, the maps $\psi_m = \psi, \psi_{m-1}, \ldots, \psi_{k+1}$ from $Z$ into $P^{(i)}$ such that $\varphi_i = \psi_i\restriction_A$ and $(\psi_{i+1}, \psi_i) \in C^i_{i+1}$ for all $i$. In view of $p^k_m \circ \hat{\varphi} = \psi$ it is clear that $\hat{\varphi} \rightleftharpoons (\psi_k, \psi_{k+1}, \ldots, \psi_m): Z \to C^k_m$ is the desired extension of $\psi$. $\square$

**To complete the proof** of Theorem 7.6 and therefore **Theorem 6.2** it is sufficient to show that $C^k_m \hookrightarrow D^k_m \in \mathrm{UV}^{k-1}$, i.e. each map $\varphi: A \to D^k_m$ of a compactum $A$, $\dim A \leqslant k$, is arbitrarily closely approximable by a map which does not intersect $F$. Let $h_t: P^{(m)} \to P^{(m)}$ be a radial homotopy and $H_t: D^k_m \to D^k_m$ a homotopy given by $H_t(a_k, a_{k+1}, \ldots, a_{m-1}, a_m) = (a_k, a_{k+1}, \ldots, a_{m-1}, h_t(a_m))$. It is clear that the homotopy $H_t \circ \varphi$ removes $A$ from $F_m$.

Taking this remark into account, it is sufficient to show that if $\varphi(A) \cap F_{s+1} = \emptyset$, $k \leqslant s < m$, then there exists a map $\varphi': A \to D^k_m$ arbitrarily close to $\varphi$ such that $\varphi'(A) \cap F_s = \emptyset$. Again represent $\varphi$ in the coordinate form $(\varphi_k, \varphi_{k+1}, \ldots, \varphi_m)$, where $\varphi_i$ is the map from $A$ into $P^{(i)}$. Let $h_t: P^{(s)} \to P^{(s)}$ be a radial homotopy. Then $\Psi^s_t \rightleftharpoons h_t \circ \varphi_s: A \to P^{(s)}$, $0 \leqslant t \leqslant 1$, is a synchronized homotopy.

The final accord sounds due to Theorem 7.1: the map $q^s_m: D^s_m \to P^{(s)}$, $q^s_m(a) = a_s$, is a synchronized Hurewich fibration. Therefore there exist synchronized homotopies $\Psi^{s+1}_t, \Psi^{s+2}_t, \ldots, \Psi^m_t$ from $A$ into $P^{(s+1)}, P^{(s+2)}, \ldots, P^{(m)}$ such that the formula $\Psi_t \rightleftharpoons (\Psi^s_t, \Psi^{s+1}_t, \Psi^{s+2}_t, \ldots, \Psi^m_t)$ defines the homotopy from $A$ into $D^s_m$.

We take

$$(\varphi_k, \varphi_{k+1}, \ldots, \varphi_{s-1}, \Psi_t) = (\varphi_k, \varphi_{k+1}, \ldots, \varphi_{s-1}, \Psi^s_t, \Psi^{s+1}_t, \Psi^{s+2}_t, \ldots, \Psi^m_t)$$

as a homotopy $\Phi_t: A \to D^k_m$ removing $A$ from $F_s$. Since $(\varphi_{s-1}, \Psi^s_t)(A) \subset D^{s-1}_s$, for each $t \geqslant 0$, we easily deduce that $\Phi_t(A) \subset D^k_m$. Next, we note that, due to Proposition 7.5, $(\varphi_{s-1}, \Psi^s_t)$ maps $A$ into $C^{s-1}_s$ for each $t > 0$. Hence, we have proved that for each $\delta > 0$, $\varphi \overset{\delta}{\sim} \Phi_t$ for sufficiently small $t > 0$, and $\mathrm{Im}(\Phi_t) \cap F_s = \emptyset$. $\square$

## 8. The construction of Dranishnikov's resolution and the proof of Theorem 1.1

To construct Dranishnikov's resolution, take a cube $R$ of sufficiently high dimension, and represent the Hilbert cube $R \times Q$ as the inverse limit of the spectrum $\varprojlim\{Z_t \rightleftharpoons R \times I^t, \sigma_t\}$ where $\sigma_t: Z_{t+1} \to Z_t$ is the projection along the last factor $I$. The goal is to construct consecutively the inverse spectrum $\{K_t, \theta_t\}$ consisting of polyhedra whose limit is $K = \mu^k$, and the morphism $\{h_t: K_t \to Z_t\}$ of these spectra which will generate Dranishnikov's resolution $h = d_k: \mu^k \to Z = R \times Q \cong Q$.

For $t = 1$ we set $K_t = R \times I^1 = Z_1$ and $h_t = \mathrm{Id}$. Suppose that for some $t > 1$ there is a map $h_t: K_t \to Z_t \rightleftharpoons R \times I^t$. The cornerstone in the construction of Dranishnikov's resolution and the proof of Theorem 1.1 consists in producing of the commutative diagram,

$$\begin{tikzcd} K_{t+1} \arrow[r, "\theta_t"] \arrow[d, "h_{t+1}"'] & K_t \arrow[d, "h_t"] \\ Z_{t+1} \arrow[r, "\sigma_t"] & Z_t \end{tikzcd} \qquad (\mathcal{E}_t)$$

the characteristic map of which is

(1) $k$-conservatively soft;
(2) A $\mathrm{UV}^{k-1}$-divider of $k$-filled $k$-soft map; and such that
(3) $K_{t+1}$ admits a $\theta_t^{-1}(\omega)$-map into $k$-dimensional polyhedron with arbitrarily fine cover $\omega \in \operatorname{cov} K_t$.

To make sure that this the case, let us consider the compactum $K \rightleftharpoons \varprojlim\{K_t, \theta_t\}$ and the limit map $h = \delta_k : K \to R \times Q$, generated by commutative diagrams $\mathcal{E}_t$, $t \geqslant 1$. It was established in [16] that $K$ is a strongly $k$-universal AE($k$)-compactum of dimension $k$, and $\delta_k^{-1}$ preserves AE($k$)-spaces. From Bestvina's criterion [8] it follows that $K$ is homeomorphic to the Menger compactum $\mu^k$. By Proposition 4.4, $h = \delta_k$ is $k$-conservatively soft.

We point out that, by virtue of (1)–(3), Theorem 4.6 is applied. As a result we get that

(i) The map $\delta_k$ is a $\mathrm{UV}^{k-1}$-divider of the limit map $\chi_k : X \to Z = R \times Q$ which is $k$-soft strongly $k$-universal with respect to Polish spaces.[4]

Now, it easily follows that $X \subset K$ is a strongly $k$-universal Polish AE($k$)-space of dimension $k = \dim K$. By characterization Theorem 2.11 for Nöbeling space, it follows that $X \cong \nu^k$, which *completes the proof of Theorem* 1.1. The evident application of (i), 3.8(a) and 3.9(b) *proves Theorem* 1.5.

Now we show that Theorem 6.2 implies the proof of the (1)–(3). To this end, we consider the polyhedron $P \rightleftharpoons K_t \times I$ simultaneously with an arbitrarily fine triangulation $L$. It is clear that $P$ is a fiberwise product of $Z_{t+1} = R \times I^{t+1}$ and $K_t$ with respect to $\sigma_t$ and $h_t$. By Theorem 6.2 there exists a polyhedron $D$ and maps $p : D \to P$ and $q : D \to P^{(k)}$ such that 1.2(1)–(3) hold. It is easily seen that

(4) The projection $\pi$ of the compactum $K_{t+1} \rightleftharpoons D \times T$ onto $D$ along the cube $T$ of dimension $t \geqslant 2k+1$ is a $\mathrm{UV}^{k-1}$-divider of $k$-soft $k$-filled projection $D \times N_k^t$ onto $D$ along the standard $k$-dimensional Nöbeling space $N_k^t \hookrightarrow T \in \mathrm{UV}^{k-1}$ (see, for example, [2–4]).

Complete the definition of the diagram $\mathcal{E}_t$ as $\theta_t \rightleftharpoons \sigma_t' \circ p \circ \pi : K_{t+1} \to K_t$ and $h_{t+1} \rightleftharpoons h_t' \circ p \circ \pi : K_{t+1} \to Z_{t+1}$, where $\sigma_t' \parallel \sigma_t$ is a projection along $I$ and $h_t' \parallel h_t$. It follows from 1.2(2) and (4) that the characteristic map of $\tilde{\mathcal{E}}_t$ – the map $p \circ \pi : K_{t+1} \to K_t$, is a $\mathrm{UV}^{k-1}$-divider of a $k$-filled $k$-soft map. It follows from 1.2(3) that if the triangulation $L$ is sufficiently fine, then the composition $q \circ \pi : K_{t+1} \to P^{(k)}$ satisfies (3).

## 9. Epilogue

Here we list a selection of unsolved problems.

**Uniqueness problem of Chigogidze's resolution.** By the *$k$-dimensional Chigogidze's resolution over $Y \in$ ANE($k$)* we understand a $k$-soft map $f : X \to Y$ of $k$-dimensional space $X$ onto $Y$, which is strongly $k$-universal with respect to maps of Polish spaces. One of the central problems of the Nöbeling space theory consists in establishing of the topological uniqueness of such a resolution [17].

**Problem 9.1.** Prove that any two Chigogidze's resolution $f, g : \nu^k \to Q$ are homeomorphic, i.e. there exists a homeomorphism $h : \nu^k \to \nu^k$ such that $f = g \circ h$.

For $k = \infty$ this problem was solved in affirmative [21]. The case $k = 0$ was also settled (see, for example, [6]).

**Problem of the characterization of Dranishnikov's resolution.** This resolution no doubt represent the analogy of the Menger compactum in the category of maps. In analogy with compacta, the question of its characterization arises naturally. But prior to doing this, we should understand what is Dranishnikov's resolution. In view of the results of this paper, the *$k$-dimensional Dranishnikov's resolution over $Y \in$ ANE($k$)* is any proper map $f : X \to Y$ from a $k$-dimensional space $X$ onto $Y$ such that

(a) $f$ is $k$-conservatively soft strongly $k$-universal with respect to compacta;
(b) $f$ is a $\mathrm{UV}^{k-1}$-divider of $k$-dimensional Chigogidze's resolution over $Y$; and
(c) $f^{-1}$ preserves ANE($k$)-spaces.

These properties imply all other properties of $k$-dimensional Dranishnikov's resolution. There is a definite hope that the topological type of this resolution is unique.

[4] We leave the proof of the following strengthening of (i) to the reader: Given ANE($k$)-space $A \subset R$, $\chi_k\restriction : \chi_k^{-1}(A \times Q) \to A \times Q$ is a $k$-soft strongly $k$-universal with respect to Polish spaces, and $\delta_k\restriction : \delta_k^{-1}(A \times Q) \to A \times Q$ is a $\mathrm{UV}^{k-1}$-divider of $\chi_k$.

**Problem 9.2.** Are any two $k$-dimensional Dranishnikov's resolutions over $Q$ homeomorphic?

**Problem of geometrization of Dranishnikov's resolution.** Initially Dranishnikov's and Chigogidze's resolutions were obtained in a nonconstructive manner as the limit projections of some countable spectra. We can identify their domains lying in Hilbert cube with Menger and Nöbeling spaces only with help of corresponding characterization theorems. On the other hand, in [5] Chigogidze's resolution was constructed in a geometric manner as the orthogonal projection of the standard Nöbeling space. The fractal structure of this resolution was thereby revealed. It was interesting to realize Dranishnikov's resolution also in a geometric manner. We precede the formulation of the corresponding conjecture by the series of definitions.

*The standard Menger space* $M^m_k \subset [0,1]^m$ and *geometric pseudointerior* $I(M^m_k) \subset [0,1]^m$ can be defined as follows:

$M^m_k = \{x = (x_1, x_2, \dots, x_m) \mid$ each $x_n$ can be recorded as an infinite ternary fraction $0, \xi^n_1 \xi^n_2 \xi^n_3 \cdots$ such that for each $p \geqslant 1$ at most $k$ numbers $\xi^i_p$ is equal to $1\}$ and $I(M^m_k) = \{x = (x_1, x_2, \dots, x_m) \mid$ each record of arbitrary $x_n$ as an infinite ternary fraction $0, \xi^n_1 \xi^n_2 \xi^n_3 \cdots$ is so that for each $p \geqslant 1$ at most $k$ numbers $\xi^i_p$ equal $1\}$. The standard Menger space $M^m_k$ and geometric pseudointerior $I(M^m_k)$ for $m \geqslant (2k+1)$ are homeomorphic to $\mu^k$ and $\nu^k$, respectively.

**Conjecture 9.3.** *Let $m \geqslant (2k+1) + (k+1)^2$. Is it true that there exists an orthogonal projection $p : \mathbb{R}^m \to \Sigma$ onto $(2k+1)$-dimensional subspace $\Sigma$ such that $p\restriction : M^m_k \to p(M^m_k)$ has the same soft properties as Dranishnikov's resolution? Is it* $\mathrm{UV}^{k-1}$*? Is it true that $p\restriction : I(M^m_k) \to p(I(M^m_k))$ is a Chigogidze's resolution? Is it true that $p\restriction_{M^m_k}$ is a* $\mathrm{UV}^{k-1}$*-divider of $p\restriction_{I(M^m_k)}$?*

**Problem of the *k*-soft core.** In [7] it was proved that Dranishnikov's resolution $\delta_k$ is not homogeneous which breaks its analogy with the Menger space. This result follows from the fact that *the $k$-soft core* of $\delta_k$

$$\mathfrak{s}_k(\delta_k) \rightleftharpoons \{x \in \mu^k \mid \text{the collection of all fibers of } \delta_k \text{ is equi-locally } (k-1)\text{-connected at } x\}$$

does not coincide with $\mu^k$, but $\mathfrak{s}_k(\delta_k) \hookrightarrow \mu^k \in \mathrm{UV}^{k-1}$. With the help of additional analysis we can show that $\mathfrak{s}_k(d_k)$ contains the domain of Chigogidze's resolution which is a $\mathrm{UV}^{n-1}$-divider of Dranishnikov's resolution $\delta_k$. In this connection the series of questions arises.

**Problem 9.4.** Is it true that $k$-soft core $\mathfrak{s}_k(\delta_k)$ is homeomorphic to $\nu^k$? the restriction of $\delta_k$ on $\mathfrak{s}_k(\delta_k)$ is Chigogidze's resolution?

Since $\delta_k : \mu_k \to Q$ is $k$-invertible, there exists a section $s : P \to \mu_k$ for each polyhedron $P \subset Q$, $\dim P \leqslant k$. The following question is concerned with the possibility of constructing the section $s$ in the equi-continuous manner, in the following sense.

**Problem 9.5.** For each $\varepsilon > 0$ there exists $\delta > 0$ such that for any polyhedron $P \subset Q$, $\dim P \leqslant k$, there exists a section $s$ of $\delta_k$ such that $\operatorname{diam} s(A) < \varepsilon$, $A \subset P$, as soon as $\operatorname{diam} A < \delta$.

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