



# Perturbation effects in nonlinear eigenvalue problems

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## ABSTRACT

We establish the complete bifurcation diagram for a class of nonlinear problems on the whole space. Our model corresponds to a class of semilinear elliptic equations with logistic type nonlinearity and absorption. Since this problem arises in population dynamics or in fishery or hunting management, we are interested only in situations allowing the existence of positive solutions. The proofs combine elliptic estimates with the method of sub- and super-solutions.

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## 1. Introduction and the main results

Bifurcation problems have a long history and their treatment dates back to the XVIIIth century. One of the first bifurcation problems is related to the buckling of a thin rod under thrust and was investigated by Daniel Bernoulli and Euler around 1744. In the case in which the rod is free to rotate at both end points, this yields the one-dimensional bifurcation problem

$$\begin{cases} u'' + \lambda \sin u = 0 & \text{in } (0, L) \\ 0 \leq u \leq \pi \\ u'(0) = u'(L) = 0. \end{cases}$$

In this paper we are concerned with the existence, uniqueness or the non-existence of positive solutions of the eigenvalue logistic problem with absorption

$$-\Delta u = \lambda (V(x)u - f(u)) \quad \text{in } \mathbb{R}^N, N \geq 3, \quad (1)$$

where  $V$  is a smooth sign-changing potential and  $f : [0, \infty) \rightarrow [0, \infty)$  is a smooth function. Equations of this type arise in the study of population dynamics. In this case, the unknown  $u$  corresponds to the density of a population, the potential  $V$  describes the birth rate of the population, while the term  $-f(u)$  in (1) signifies the fact that the population is self-limiting. In the region where  $V$  is positive (resp., negative) the population has positive (resp., negative) birth rate. Since  $u$  describes a population density, we are interested in investigating only positive solutions of problem (1). Bifurcation

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problems of this type, as well as the associated evolution equations, are naturally related to certain physical phenomena. For example, super-diffusivity equations of this type have been proposed by de Gennes [1] as a model for long-range van der Waals interactions in thin films spreading on solid surfaces. Such equations appear in the study of cellular automata and interacting particle systems with self-organized criticality, as well as to describe the flow over an impermeable plate. We also mention Brusselator type reactions, the combustion theory, dynamics of population, the Fitzhugh–Nagumo system, morphogenesis, superconductivity, superfluids, the buckling of the Euler rod, the appearance of Taylor vortices, and the onset of oscillations in an electric circuit.

Problem (1) may be treated as a linear perturbation of the logistic equation on the whole space

$$\Delta u = f(u) \quad \text{in } \mathbb{R}^N,$$

but also as a nonlinear perturbation of a related linear eigenvalue problem with anisotropic potential on the whole space. We recall in what follows the results that we need later. Let  $\Omega$  be an arbitrary open set in  $\mathbb{R}^N$ ,  $N \geq 3$ . Consider the eigenvalue problem

$$\begin{cases} -\Delta u = \lambda V(x)u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2}$$

Problems of this type have a long history. If  $\Omega$  is bounded and  $V \equiv 1$ , problem (2) is related to the Riesz–Fredholm theory of self-adjoint and compact operators. The case of a non-constant potential  $V$  has been first considered in the pioneering papers of Bocher [2], Hess and Kato [3], Minakshisundaran and Pleijel [4,5]. For instance, Minakshisundaran and Pleijel [4, 5] studied the case where  $\Omega$  is bounded,  $V \in L^\infty(\Omega)$ ,  $V \geq 0$  in  $\Omega$  and  $V > 0$  in  $\Omega_0 \subset \Omega$  with  $|\Omega_0| > 0$ . An important contribution in the study of the anisotropic eigenvalue problem (2) if  $\Omega$  is bounded has been given by Cuesta [6] under the assumption that  $V$  is a given potential which may change sign and satisfying

$$(H) \quad V^+ \neq 0, \quad \text{and} \quad V \in L^s(\Omega) \text{ for some } s > N/2.$$

We have denoted  $V^+(x) = \max\{V(x), 0\}$ . Obviously,  $V = V^+ - V^-$ , where  $V^-(x) = \max\{-V(x), 0\}$ .

In order to find the principal eigenvalue of (2), Cuesta [6] proved that the minimization problem

$$\min \left\{ \int_{\Omega} |\nabla u|^2 dx; u \in H_0^1(\Omega), \int_{\Omega} V(x)u^2 dx = 1 \right\}$$

has a solution  $\varphi_1 = \varphi_1(\Omega) \geq 0$  which is an eigenfunction of (2) corresponding to the eigenvalue  $\lambda_1(\Omega) = \int_{\Omega} |\nabla \varphi_1|^2 dx$ . Moreover, the least positive eigenvalue  $\lambda_1(\Omega)$  is simple, isolated in the spectrum and it is the unique eigenvalue associated to a nonnegative eigenfunction.

Throughout this paper the sign-changing potential  $V : \mathbb{R}^N \rightarrow \mathbb{R}$  is assumed to be a Hölder function that satisfies

$$(V) \quad V \in L^\infty(\mathbb{R}^N), \quad V^+ = V_1 + V_2 \neq 0, \quad V_1 \in L^{N/2}(\mathbb{R}^N), \quad \lim_{|x| \rightarrow \infty} |x|^2 V_2(x) = 0.$$

We suppose that the nonlinear absorption term  $f : [0, \infty) \rightarrow [0, \infty)$  is a  $C^1$ -function such that

- (f1)  $f(0) = f'(0) = 0$  and  $\liminf_{u \searrow 0} \frac{f(u)}{u} > 0$ ;
- (f2) the mapping  $f(u)/u$  is increasing in  $(0, +\infty)$ .

This assumption implies  $\lim_{u \rightarrow +\infty} f(u) = +\infty$ . We impose that  $f$  has a superlinear growth at infinity, in the sense that

$$(f3) \quad \lim_{u \rightarrow +\infty} \frac{f(u)}{u} > \|V\|_{L^\infty}.$$

Our framework includes the following important particular cases:

- (i)  $f(u) = u^2$  that corresponds to the Fisher equation [7,8] and the Kolmogoroff–Petrovsky–Piscounoff equation [9] (see also [10] for a comprehensive treatment of these equations);
- (ii)  $f(u) = u^{(N+2)/(N-2)}$  (for  $N \geq 6$ ) which is related to the conform scalar curvature equation, cf. [11].

For any  $R > 0$ , denote  $B_R = \{x \in \mathbb{R}^N; |x| < R\}$  and set

$$\lambda_1(R) = \min \left\{ \int_{B_R} |\nabla u|^2 dx; u \in H_0^1(B_R), \int_{B_R} V(x)u^2 dx = 1 \right\}. \tag{3}$$

Consequently, the mapping  $R \mapsto \lambda_1(R)$  is decreasing and so, there exists

$$\lambda^* := \lim_{R \rightarrow \infty} \lambda_1(R) \geq 0.$$

We first state a sufficient condition so that  $\lambda^*$  is positive. For this aim we impose the additional assumptions

$$\text{there exists } A, \alpha > 0 \text{ such that } V^+(x) \leq A|x|^{-2-\alpha}, \quad \text{for all } x \in \mathbb{R}^N \tag{4}$$

and

$$\lim_{x \rightarrow 0} |x|^{2(N-1)/N} V_2(x) = 0. \tag{5}$$

**Theorem 1.1.** Assume that  $V$  satisfies conditions (V), (4) and (5).

Then  $\lambda^* > 0$ .

Our main result asserts that  $\lambda^*$  plays a crucial role for the nonlinear eigenvalue logistic problem

$$\begin{cases} -\Delta u = \lambda (V(x)u - f(u)) & \text{in } \mathbb{R}^N, \\ u > 0 & \text{in } \mathbb{R}^N, \\ \lim_{|x| \rightarrow \infty} u(x) = 0. \end{cases} \quad (6)$$

The following existence and non-existence results show that  $\lambda^*$  serves as a bifurcation point in our problem (6).

**Theorem 1.2.** Assume that  $V$  and  $f$  satisfy the assumptions (V), (4), (f1), (f2) and (f3).

Then the following hold:

- (i) problem (6) has a unique solution for any  $\lambda > \lambda^*$ ;
- (ii) problem (6) does not have any solution for all  $\lambda \leq \lambda^*$ .

The additional condition (4) implies that  $V^+ \in L^{N/2}(\mathbb{R}^N)$ , which does not follow from the basic hypothesis (V). As we shall see in the next section, this growth assumption is essential in order to establish the existence of positive solutions of (1) decaying to zero at infinity.

In particular, Theorem 1.2 shows that if  $V(x) < 0$  for sufficiently large  $|x|$  (that is, if the population has negative birth rate) then any positive solution (that is, the population density) of (1) tends to zero as  $|x| \rightarrow \infty$ .

## 2. Existence of solutions for $\lambda$ large

We show in this section that  $\lambda^*$  plays a crucial role in our analysis, in the sense that the logistic equation (1) has entire positive solutions if  $\lambda$  is sufficiently large. However, at this stage, we are not able to establish that this solution decays to zero at infinity.

**Proposition 2.1.** Assume that the functions  $V$  and  $f$  satisfy conditions (V), (f1), (f2) and (f3). Then the problem

$$\begin{cases} -\Delta u = \lambda (V(x)u - f(u)) & \text{in } \mathbb{R}^N, \\ u > 0 & \text{in } \mathbb{R}^N \end{cases} \quad (7)$$

has at least one solution, for any  $\lambda > \lambda^*$ .

**Proof.** For any  $R > 0$ , consider the boundary value problem

$$\begin{cases} -\Delta u = \lambda (V(x)u - f(u)) & \text{in } B_R, \\ u > 0 & \text{in } B_R, \\ u = 0 & \text{on } \partial B_R. \end{cases} \quad (8)$$

We first prove that problem (8) has at least one solution, for any  $\lambda > \lambda_1(R)$ . Indeed, the function  $\bar{u}(x) = M$  is a super-solution of (8), for any  $M$  large enough. This follows from (f3) and the boundedness of  $V$ . Next, in order to find a positive sub-solution, let us consider the problem

$$\min_{u \in H_0^1(B_R)} \int_{B_R} (|\nabla u|^2 - \lambda V(x)u^2) dx.$$

Since  $\lambda > \lambda_1(R)$ , it follows that the least eigenvalue  $\mu_1$  is negative. Moreover, the corresponding eigenfunction  $e_1$  satisfies

$$\begin{cases} -\Delta e_1 - \lambda V(x)e_1 = \mu_1 e_1 & \text{in } B_R, \\ e_1 > 0 & \text{in } B_R, \\ e_1 = 0 & \text{on } \partial B_R. \end{cases} \quad (9)$$

Then the function  $\underline{u}(x) = \varepsilon e_1(x)$  is a sub-solution of the problem (8). Indeed, it is enough to check that

$$-\Delta(\varepsilon e_1) - \lambda \varepsilon V e_1 + \lambda f(\varepsilon e_1) \leq 0 \quad \text{in } B_R,$$

that is, by (9),

$$\varepsilon \mu_1 e_1 + \lambda f(\varepsilon e_1) \leq 0 \quad \text{in } B_R. \quad (10)$$

But

$$f(\varepsilon e_1) = \varepsilon f'(0)e_1 + \varepsilon e_1 o(1), \quad \text{as } \varepsilon \rightarrow 0.$$

So, since  $f'(0) = 0$ , relation (10) becomes

$$\varepsilon e_1 (\mu_1 + o(1)) \leq 0$$

which is true, provided  $\varepsilon > 0$  is small enough, due to the fact that  $\mu_1 < 0$ .

Fix  $\lambda > \lambda^*$  and an arbitrary sequence  $R_1 < R_2 < \dots < R_n < \dots$  of positive numbers such that  $R_n \rightarrow \infty$  and  $\lambda_1(R_1) < \lambda$ . Let  $u_n$  be the solution of (8) on  $B_{R_n}$ . Fix a positive number  $M$  such that  $f(M)/M > \|V\|_{L^\infty(\mathbb{R}^N)}$ . The above arguments show that we can assume that  $u_n \leq M$  in  $B_{R_n}$ , for any  $n \geq 1$ . Since  $u_{n+1}$  is a super-solution of (8) for  $R = R_n$ , we can also assume that  $u_n \leq u_{n+1}$  in  $B_{R_n}$ . Thus the function  $u(x) := \lim_{n \rightarrow \infty} u_n(x)$  exists and is well defined and positive in  $\mathbb{R}^N$ . Standard elliptic regularity arguments imply that  $u$  is a solution of problem (7).  $\square$

The above result shows the importance of assumption (4) in the statement of Theorem 1.2. Indeed, assuming that  $V$  satisfies only the hypothesis (V), it is not clear whether or not the solution constructed in the proof of Proposition 2.1 tends to 0 as  $|x| \rightarrow \infty$ . However, it is easy to observe that if  $\lambda > \lambda^*$  and  $V$  satisfies (4) then problem (6) has at least one solution. Indeed, we first observe that

$$\underline{u}(x) = \begin{cases} \varepsilon e_1(x), & \text{if } x \in B_R \\ 0, & \text{if } x \notin B_R \end{cases} \tag{11}$$

is a sub-solution of problem (6), for some fixed  $R > 0$ , where  $e_1$  satisfies (9). Next, we observe that  $\bar{u}(x) = n/(1 + |x|^2)$  is a super-solution of (6). Indeed,  $\bar{u}$  satisfies

$$-\Delta \bar{u}(x) = \frac{2[n(1 + |x|^2) - 4|x|^2]}{(1 + |x|^2)^2} u(x), \quad x \in \mathbb{R}^N.$$

It follows that  $\bar{u}$  is a super-solution of (6) provided

$$\frac{2[n(1 + |x|^2) - 4|x|^2]}{(1 + |x|^2)^2} \geq \lambda V(x) - \lambda f\left(\frac{n}{1 + |x|^2}\right), \quad x \in \mathbb{R}^N.$$

This inequality follows from (f3) and (4), provided that  $n$  is large enough.

### 3. Proof of Theorem 1.1

For any  $R > 0$ , fix arbitrarily  $u \in H^1_0(B_R)$  such that  $\int_{B_R} V(x)u^2 dx = 1$ . We have

$$1 = \int_{B_R} V(x)u^2 dx \leq \int_{B_R} V^+(x)u^2 dx = \int_{B_R} V_1(x)u^2 dx + \int_{B_R} V_2(x)u^2 dx.$$

Since  $V_1 \in L^{N/2}(\mathbb{R}^N)$ , using the Cauchy–Schwarz inequality and Sobolev embeddings we obtain

$$\int_{B_R} V_1(x)u^2 dx \leq \|V_1\|_{L^{N/2}(B_R)} \|u\|_{L^{2^*}(B_R)}^2 \leq C_1 \|V_1\|_{L^{N/2}(\mathbb{R}^N)} \int_{B_R} |\nabla u|^2 dx, \tag{12}$$

where  $2^* = 2N/(N - 2)$ .

Fix  $\epsilon > 0$ . By our assumption (V), there exists positive numbers  $\delta, R_1$  and  $R$  such that  $R^{-1} < \delta < R_1 < R$  such that for all  $x \in B_R$  satisfying  $|x| \geq R_1$  we have

$$|x|^2 V_2(x) \leq \epsilon. \tag{13}$$

On the other hand, by (V), for any  $x \in B_R$  with  $|x| \leq \delta$  we have

$$|x|^{2(N-1)/N} V_2(x) \leq \epsilon. \tag{14}$$

Define  $\Omega := \omega_1 \cup \omega_2$ , where  $\omega_1 := B_R \setminus \bar{B}_{R_1}$ ,  $\omega_2 := B_\delta \setminus \bar{B}_{1/R}$ , and  $\omega := B_{R_1} \setminus \bar{B}_\delta$ .

By (13) and Hardy’s inequality we find

$$\int_{\omega_1} V_2(x)u^2 dx \leq \epsilon \int_{\omega_1} \frac{u^2}{|x|^2} dx \leq C_2 \epsilon \int_{B_R} |\nabla u|^2 dx. \tag{15}$$

Now using (14) and Hölder’s inequality we obtain

$$\begin{aligned} \int_{\omega_2} V_2(x)u^2 dx &\leq \epsilon \int_{\omega_2} \frac{u^2}{|x|^{2(N-1)/N}} dx \\ &\leq \epsilon \left[ \int_{\omega_2} \left( \frac{1}{|x|^{2(N-1)/N}} dx \right)^{N/2} dx \right]^{2/N} \|u\|_{L^{2^*}(B_R)}^2 \\ &\leq C \epsilon \left( \int_{1/R}^\delta \frac{1}{s^{N-1}} s^{N-1} \omega_N ds \right)^{2/N} \int_{B_R} |\nabla u|^2 dx \\ &\leq C_3 \left( \delta - \frac{1}{R} \right)^{2/N} \int_{B_R} |\nabla u|^2 dx. \end{aligned} \tag{16}$$

By compactness and our assumption (V), there exists a finite covering of  $\bar{\omega}$  by the closed balls  $\bar{B}_{r_1}(x_1), \dots, \bar{B}_{r_k}(x_k)$  such that, for all  $1 \leq j \leq k$

$$\text{if } |x - x_j| \leq r_j \text{ then } |x - x_j|^{2(N-1)/N} V_2(x) \leq \epsilon. \tag{17}$$

There exists  $r > 0$  such that, for any  $1 \leq j \leq k$

$$\text{if } |x - x_j| \leq r \text{ then } |x - x_j|^{2(N-1)/N} V_2(x) \leq \frac{\epsilon}{k}.$$

Define  $A := \cup_{j=1}^k B_r(x_j)$ . The above estimate, Hölder’s inequality and Sobolev embeddings yield

$$\begin{aligned} \int_{B_r(x_j)} V_2(x) u^2 dx &\leq \frac{\epsilon}{k} \int_{B_r(x_j)} \frac{u^2}{|x - x_j|^{2(N-1)/N}} dx \\ &\leq \frac{\epsilon}{k} \left[ \int_{B_r(x_j)} (|x - x_j|^{-2(N-1)/N})^{N/2} dx \right]^{2/N} \|u\|_{L^{2^*}(B_R)}^2 \\ &\leq C \frac{\epsilon}{k} \left( \int_{B_r} \frac{1}{|x|^{N-1}} dx \right)^{2/N} \int_{B_R} |\nabla u|^2 dx \\ &= C \frac{\epsilon}{k} \left( \int_0^r \frac{1}{s^{N-1}} s^{N-1} \omega_N ds \right)^{2/N} \int_{B_R} |\nabla u|^2 dx \\ &= C' \int_{B_R} |\nabla u|^2 dx, \end{aligned}$$

for any  $j = 1, \dots, k$ . By addition we find

$$\int_A V_2(x) u^2 dx \leq C_4 \int_{B_R} |\nabla u|^2 dx. \tag{18}$$

It follows from (17) that  $V_2 \in L^\infty(\omega \setminus A)$ . Actually, if  $x \in \omega \setminus A$  it follows that there exists  $j \in \{1, \dots, k\}$  such that  $r_j > |x - x_j| > r > 0$ . Thus,

$$V_2(x) \leq r^{-2(N-1)/N} \epsilon.$$

Hence

$$\int_{\omega \setminus A} V_2(x) u^2 dx \leq \epsilon r^{-2(N-1)/N} \int_{\omega \setminus A} u^2 dx \leq C_5 \int_{B_R} |\nabla u|^2 dx. \tag{19}$$

Now from inequalities (12), (15), (16), (18) and (19) we have

$$\lambda_1(R) \geq \left\{ C_1 \|V_1\|_{L^{N/2}(\mathbb{R}^N)} + C_2 \epsilon + C_3 (\delta - R^{-1})^{2/N} + C_4 + C_5 \right\}^{-1}$$

and passing to the limit as  $R \rightarrow \infty$  we conclude that

$$\lambda^* \geq (C_1 \|V_1\|_{L^{N/2}(\mathbb{R}^N)} + C_2 \epsilon + C_3 \delta^{2/N} + C_4 + C_5)^{-1} > 0.$$

This completes the proof of Theorem 1.1.  $\square$

#### 4. Proof of Theorem 1.2

We split the proof of our main result into several steps. Throughout this section we assume that the hypotheses of Theorem 1.2 are fulfilled.

**Proposition 4.1.** *Let  $u$  be an arbitrary solution of problem (6). Then there exists  $C > 0$  such that  $|u(x)| \leq C|x|^{2-N}$  for all  $x \in \mathbb{R}^N$ .*

**Proof.** Let  $\omega_N$  be the surface area of the unit sphere in  $\mathbb{R}^N$ . Consider the function  $V^+u$  as a Newtonian potential and define

$$v(x) = \frac{1}{(N-2)\omega_N} \int_{\mathbb{R}^N} \frac{V^+(y)u(y)}{|x-y|^{N-2}} dy.$$

A straightforward computation shows that

$$-\Delta v = V^+(x)u \text{ in } \mathbb{R}^N. \tag{20}$$

But, by (4) and since  $u$  is bounded,

$$V^+(y)u(y) \leq C|y|^{-2-\alpha}, \text{ for all } y \in \mathbb{R}^N.$$

So, by Lemma 2.3 in Li and Ni [11],

$$v(x) \leq C|x|^{-\alpha}, \quad \text{for all } x \in \mathbb{R}^N,$$

provided that  $\alpha < N - 2$ . Set  $w(x) = Cv(x) - u(x)$ . Hence  $w(x) \rightarrow 0$  as  $|x| \rightarrow \infty$ . Let us choose  $C$  sufficiently large so that  $w(0) > 0$ . We claim that this implies

$$w(x) > 0, \quad \text{for all } x \in \mathbb{R}^N. \tag{21}$$

Indeed, if not, let  $x_0 \in \mathbb{R}^N$  be a local minimum point of  $w$ . This means that  $w(x_0) < 0$ ,  $\nabla w(x_0) = 0$  and  $\Delta w(x_0) \geq 0$ . But

$$\Delta w(x_0) = -CV^+(x_0)u(x_0) + \lambda(V(x_0)u(x_0) - f(u(x_0))) < 0,$$

provided that  $C > \lambda$ . This contradiction implies (21). Consequently,

$$u(x) \leq Cv(x) \leq C|x|^{-\alpha}, \quad \text{for any } x \in \mathbb{R}^N.$$

So, using (4) again,

$$V^+(x)u(x) \leq C|x|^{-2-2\alpha}, \quad \text{for all } x \in \mathbb{R}^N.$$

Lemma 2.3 in [11] yields the improved estimate

$$v(x) \leq C|x|^{-2\alpha}, \quad \text{for all } x \in \mathbb{R}^N,$$

provided that  $2\alpha < N - 2$ , and so on. Let  $n_\alpha$  be the largest integer such that  $n_\alpha\alpha < N - 2$ . Repeating  $n_\alpha + 1$  times the above argument based on Lemma 2.3 (i) and (iii) in [11] we obtain

$$u(x) \leq C|x|^{2-N}, \quad \text{for all } x \in \mathbb{R}^N. \quad \square$$

**Proposition 4.2.** *Let  $u$  be a solution of problem (6). Then  $V^+u, V^-u, f(u) \in L^1(\mathbb{R}^N)$ , and  $u \in H^1(\mathbb{R}^N)$ .*

**Proof.** For any  $R > 0$  consider the average function

$$\bar{u}(R) = \frac{1}{\omega_N R^{N-1}} \int_{\partial B_R} u(x) d\sigma = \frac{1}{\omega_N} \int_{\partial B_1} u(rx) d\sigma,$$

where  $\omega_N$  denotes the surface area of  $S^{N-1}$ . Then

$$\bar{u}'(R) = \frac{1}{\omega_N} \int_{\partial B_1} \frac{\partial u}{\partial \nu}(rx) d\sigma = \frac{1}{\omega_N R^{N-1}} \int_{\partial B_R} \frac{\partial u}{\partial \nu}(x) d\sigma = \frac{1}{\omega_N R^{N-1}} \int_{B_R} \Delta u(x) dx.$$

Hence

$$\begin{aligned} \omega_N R^{N-1} \bar{u}'(R) &= -\lambda \int_{B_R} (V(x)u - f(u)) dx \\ &= -\lambda \int_{B_R} V^+(x)u dx + \lambda \int_{B_R} (V^-(x)u + f(u)) dx. \end{aligned} \tag{22}$$

By Proposition 4.1, there exists  $C > 0$  such that  $|\bar{u}(r)| \leq Cr^{-N+2}$ , for any  $r > 0$ . So, by (4),

$$\int_{1 \leq |x| \leq r} V^+(x)u dx \leq CA \int_{1 \leq |x| \leq r} |x|^{-N-\alpha} dx \leq C,$$

where  $C$  does not depend on  $r$ . This implies  $V^+u \in L^1(\mathbb{R}^N)$ .

By contradiction, assume that  $V^-u + f(u) \notin L^1(\mathbb{R}^N)$ . So, by (22),  $\bar{u}'(r) > 0$  if  $r$  is sufficiently large. It follows that  $\bar{u}(r)$  does not converge to 0 as  $r \rightarrow \infty$ , which contradicts Proposition 4.1. So,  $V^-u + f(u) \in L^1(\mathbb{R}^N)$ . Next, in order to establish that  $u \in L^2(\mathbb{R}^N)$ , we observe that our assumption (f1) implies the existence of some positive numbers  $a$  and  $\delta$  such that  $f'(t) > at$ , for any  $0 < t < \delta$ . This implies  $f(t) > at^2/2$ , for any  $0 < t < \delta$ . Since  $u$  decays to 0 at infinity, it follows that the set  $\{x \in \mathbb{R}^N; u(x) \geq \delta\}$  is compact. Hence

$$\begin{aligned} \int_{\mathbb{R}^N} u^2 dx &= \int_{[u \geq \delta]} u^2 dx + \int_{[u < \delta]} u^2 dx \\ &\leq \int_{[u \geq \delta]} u^2 dx + \frac{2}{a} \int_{[u < \delta]} f(u) dx \\ &< +\infty, \end{aligned}$$

since  $f(u) \in L^1(\mathbb{R}^N)$ .

It remains to prove that  $\nabla u \in L^2(\mathbb{R}^N)^N$ . We first observe that after multiplication by  $u$  in (1) and integration we find

$$\int_{B_R} |\nabla u|^2 dx - \int_{\partial B_R} u(x) \frac{\partial u}{\partial \nu}(x) d\sigma = \lambda \int_{B_R} (V(x)u - f(u)) dx,$$

for any  $r > 0$ . Since  $Vu - f(u) \in L^1(\mathbb{R}^N)$ , it follows that the left-hand side has a finite limit as  $r \rightarrow \infty$ . Arguing by contradiction and assuming that  $\nabla u \notin L^2(\mathbb{R}^N)^N$ , it follows that there exists  $R_0 > 0$  such that

$$\int_{\partial B_R} u(x) \frac{\partial u}{\partial \nu}(x) d\sigma \geq \frac{1}{2} \int_{B_R} |\nabla u|^2 dx, \quad \text{for any } R \geq R_0. \quad (23)$$

Define the functions

$$A(R) := \int_{\partial B_R} u(x) \frac{\partial u}{\partial \nu}(x) d\sigma;$$

$$B(R) := \int_{\partial B_R} u^2(x) d\sigma;$$

$$C(R) = \int_{B_R} |\nabla u(x)|^2 dx.$$

The integro-differential inequality (23) can be rewritten as

$$A(R) \geq \frac{1}{2} C(R), \quad \text{for any } R \geq R_0. \quad (24)$$

On the other hand, by the Cauchy–Schwarz inequality,

$$A^2(R) \leq \left( \int_{\partial B_R} u^2 d\sigma \right) \left( \int_{\partial B_R} \left| \frac{\partial u}{\partial \nu} \right|^2 d\sigma \right) \leq B(R)C'(R).$$

Now using (24) we obtain

$$C'(R) \geq \frac{C^2(R)}{4B(R)}, \quad \text{for any } R \geq R_0.$$

Hence

$$\frac{d}{dr} \left[ \frac{4}{C(r)} + \int_0^r \frac{dt}{B(t)} \right]_{r=R} \leq 0, \quad \text{for any } R \geq R_0. \quad (25)$$

But, since  $u \in L^2(\mathbb{R}^N)$ , it follows that  $\int_0^\infty B(t) dt$  converges, so

$$\lim_{R \rightarrow \infty} \int_0^R \frac{dt}{B(t)} = +\infty. \quad (26)$$

On the other hand, our assumption  $|\nabla u| \notin L^2(\mathbb{R}^N)$  implies

$$\lim_{R \rightarrow \infty} \frac{1}{C(R)} = 0. \quad (27)$$

Relations (25)–(27) yield a contradiction, so our proof is complete.  $\square$

**Proposition 4.3.** *Let  $u$  and  $v$  be two distinct solutions of problem (6). Then*

$$\lim_{R \rightarrow \infty} \int_{\partial B_R} u(x) \frac{\partial v}{\partial \nu}(x) d\sigma = 0.$$

**Proof.** By multiplication with  $v$  in (6) and integration on  $B_R$  we find

$$\int_{B_R} \nabla u \cdot \nabla v dx - \int_{\partial B_R} u \frac{\partial v}{\partial \nu} d\sigma = \lambda \int_{B_R} (V(x)uv - f(u)v) dx.$$

So, by Proposition 4.2, there exists a finite  $\lim_{R \rightarrow \infty} \int_{\partial B_R} u \frac{\partial v}{\partial \nu} d\sigma$ . But, by the Cauchy–Schwarz inequality,

$$\left| \int_{\partial B_R} u \frac{\partial v}{\partial \nu} d\sigma \right| \leq \left( \int_{\partial B_R} u^2 d\sigma \right)^{1/2} \left( \int_{\partial B_R} |\nabla v|^2 d\sigma \right)^{1/2}. \quad (28)$$

Since  $u, |\nabla v| \in L^2(\mathbb{R}^N)$ , it follows that the integral

$$\int_0^\infty \left( \int_{\partial B_R} (u^2 + |\nabla v|^2) d\sigma \right) dx$$

is convergent. Hence

$$\lim_{R \rightarrow \infty} \int_{\partial B_R} (u^2 + |\nabla v|^2) d\sigma = 0. \tag{29}$$

Our conclusion now follows by relations (28) and (29).  $\square$

**Proof of Theorem 1.2 Concluded.** (i) The existence of a solution follows with the arguments given in the preceding section. In order to establish the uniqueness, let  $u$  and  $v$  be two solutions of (6). We can assume without loss of generality that  $u \leq v$ . This follows from the fact that  $\bar{u} = \min\{u, v\}$  is a super-solution of (6) and  $\underline{u}$  defined in (11) is an arbitrary small sub-solution. So, it is sufficient to consider the ordered pair consisting of the corresponding solution and  $v$ .

Since  $u$  and  $v$  are solutions we have, by Green's formula,

$$\int_{\partial B_R} \left( u \frac{\partial v}{\partial \nu} - v \frac{\partial u}{\partial \nu} \right) d\sigma = \lambda \int_{B_R} uv \left( \frac{f(v)}{v} - \frac{f(u)}{u} \right) dx.$$

By Proposition 4.3, the left-hand side converges to 0 as  $R \rightarrow \infty$ . So, (f1) and our assumption  $u \leq v$  force  $u = v$  in  $\mathbb{R}^N$ .

(ii) Arguing by contradiction, let  $\lambda \leq \lambda^*$  be such that problem (6) has a solution for this  $\lambda$ . Therefore

$$\int_{B_R} |\nabla u|^2 dx - \int_{\partial B_R} u \frac{\partial u}{\partial \nu} d\sigma = \lambda \int_{B_R} (V(x)u^2 - f(u)u) dx.$$

By Propositions 4.2 and 4.3 and letting  $R \rightarrow \infty$  we find

$$\int_{\mathbb{R}^N} |\nabla u|^2 dx < \lambda \int_{\mathbb{R}^N} V(x)u^2 dx. \tag{30}$$

On the other hand, using the definition of  $\lambda^*$  and (3) we obtain

$$\lambda^* \int_{\mathbb{R}^N} V\zeta^2 dx \leq \int_{\mathbb{R}^N} |\nabla \zeta|^2 dx, \tag{31}$$

for any  $\zeta \in C_0^2(\mathbb{R}^N)$  such that  $\int_{\mathbb{R}^N} V\zeta^2 dx > 0$ .

Fix  $\zeta \in C_0^2(\mathbb{R}^N)$  such that  $0 \leq \zeta \leq 1$ ,  $\zeta(x) = 1$  if  $|x| \leq 1$ , and  $\zeta(x) = 0$  if  $|x| \geq 2$ . For any  $n \geq 1$  define  $\Psi_n(x) = \zeta_n(x)u(x)$ , where  $\zeta_n(x) = \zeta(|x|/n)$ . Thus  $\Psi_n(x) \rightarrow u(x)$  as  $n \rightarrow \infty$ , for any  $x \in \mathbb{R}^N$ . Since  $u \in H^1(\mathbb{R}^N)$ , the Sobolev embedding theorem implies that  $u \in L^{2N/(N-2)}(\mathbb{R}^N)$ . So, the Lebesgue dominated convergence theorem yields

$$\Psi_n \rightarrow u \quad \text{in } L^{2N/(N-2)}(\mathbb{R}^N).$$

We claim that

$$\nabla \Psi_n \rightarrow \nabla u \quad \text{in } L^2(\mathbb{R}^N)^N. \tag{32}$$

Indeed, let  $\Omega_n := \{x \in \mathbb{R}^N; n < |x| < 2n\}$ . Applying Hölder's inequality we find

$$\begin{aligned} \|\nabla \Psi_n - \nabla u\|_{L^2(\mathbb{R}^N)} &\leq \|(\zeta_n - 1)\nabla u\|_{L^2(\mathbb{R}^N)} + \|u\nabla \zeta_n\|_{L^2(\Omega_n)} \\ &\leq \|(\zeta_n - 1)\nabla u\|_{L^2(\mathbb{R}^N)} + \|u\|_{L^{2N/(N-2)}(\Omega_n)} \cdot \|\nabla \zeta_n\|_{L^N(\mathbb{R}^N)}. \end{aligned} \tag{33}$$

But, since  $|\nabla u| \in L^2(\mathbb{R}^N)$ , it follows by Lebesgue's dominated convergence theorem that

$$\lim_{n \rightarrow \infty} \|(\zeta_n - 1)\nabla u\|_{L^2(\mathbb{R}^N)} = 0. \tag{34}$$

Next, we observe that

$$\|\nabla \zeta_n\|_{L^N(\mathbb{R}^N)} = \|\nabla \zeta\|_{L^N(\mathbb{R}^N)}. \tag{35}$$

Since  $u \in L^{2N/(N-2)}(\mathbb{R}^N)$  then

$$\lim_{n \rightarrow \infty} \|u\|_{L^{2N/(N-2)}(\Omega_n)} = 0. \tag{36}$$

Relations (33)–(36) imply our claim (32).



Since  $V^\pm u^2 \in L^1(\mathbb{R}^N)$  and  $V^\pm \psi_n^2 \leq V^\pm u^2$ , it follows by Lebesgue's dominated convergence theorem that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} V^\pm \psi_n^2 dx = \int_{\mathbb{R}^N} V^\pm u^2 dx.$$

Consequently

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} V \psi_n^2 dx = \int_{\mathbb{R}^N} V u^2 dx. \quad (37)$$

So, by (30) and (37), it follows that there exists  $n_0 \geq 1$  such that

$$\int_{\mathbb{R}^N} V \psi_n^2 dx > 0, \quad \text{for any } n \geq n_0.$$

This means that we can write (31) for  $\zeta$  replaced by  $\psi_n \in C_0^2(\mathbb{R}^N)$ . Then using (32) and (37) we find

$$\int_{\mathbb{R}^N} |\nabla u|^2 dx \geq \lambda^* \int_{\mathbb{R}^N} V u^2 dx. \quad (38)$$

Relations (30) and (38) yield a contradiction, so problem (6) has no solution if  $\lambda \leq \lambda^*$ .  $\square$

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## References

- [1] P.G. de Gennes, Wetting: Statics and dynamics, *Rev. Modern Phys.* 57 (1985) 827–863.
- [2] M. Bocher, The smallest characteristic numbers in a certain exceptional case, *Bull. Amer. Math. Soc.* 21 (1914) 6–9.
- [3] P. Hess, T. Kato, On some linear and nonlinear eigenvalue problems with indefinite weight function, *Comm. Partial Differential Equations* 5 (1980) 999–1030.
- [4] S. Minakshisundaran, A. Pleijel, Some properties of the eigenfunctions of the Laplace operator on Riemann manifolds, *Canad. J. Math.* 1 (1949) 242–256.
- [5] A. Pleijel, On the eigenvalues and eigenfunctions of elastic plates, *Comm. Pure Appl. Math.* 3 (1950) 1–10.
- [6] M. Cuesta, Eigenvalue problems for the  $p$ -Laplacian with indefinite weights, *Electronic J. Differential Equations* 33 (2001) 1–9.
- [7] D. Costa, P. Drabek, H. Tehrani, Positive solutions to semilinear elliptic equations with logistic type nonlinearities and constant yield harvesting in  $\mathbb{R}^N$ , *Comm. Partial Differential Equations* 33 (2008) 1597–1610.
- [8] R.A. Fisher, The wave of advance of advantageous genes, *Ann. Eugenics* 7 (1937) 353–369.
- [9] A. Kolmogoroff, I. Petrovsky, N. Piscounoff, Study of the diffusion equation with growth of the quantity of matter and its application to a biological problem, *Moscow Univ. Math. Bull.* 1 (1937) 1–25 (in French).
- [10] J.L. Kazdan, F.W. Warner, Scalar curvature and conformal deformation of Riemannian structure, *J. Differential Geom.* 10 (1975) 113–134.
- [11] Y. Li, W.-M. Ni, On conformal scalar curvature equations in  $\mathbb{R}^N$ , *Duke Math. J.* 57 (1988) 895–924.