# Hereditary invertible linear surjections and splitting problems for selections

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## A R T I C L E   I N F O

## A B S T R A C T

Let $A + B$ be the pointwise (Minkowski) sum of two convex subsets $A$ and $B$ of a Banach space. Is it true that every continuous mapping $h : X \to A + B$ splits into a sum $h = f + g$ of continuous mappings $f : X \to A$ and $g : X \to B$? We study this question within a wider framework of splitting techniques of continuous selections. Existence of splittings is guaranteed by hereditary invertibility of linear surjections between Banach spaces. Some affirmative and negative results on such invertibility with respect to an appropriate class of convex compacta are presented. As a corollary, a positive answer to the above question is obtained for strictly convex finite-dimensional precompact spaces.

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## 1. Introduction

Recall that a single-valued mapping $f : X \to Y$ is said to be a *selection* of a multivalued mapping $F : X \to Y$ provided that $f(x) \in F(x)$, for every $x \in X$. Classically, selections exist in the category of topological spaces (for details see [3–6]), or in the category of measurable spaces (see [1,6]). Here we shall restrict ourselves only to the first case. A very typical and most known example of a selection theorem is the celebrated theorem of Michael. It states that every lower semicontinuous (LSC) mapping $F : X \to Y$ from a paracompact domain $X$ into a Banach range space $Y$ admits a continuous single-valued selection whenever each value $F(x)$, $x \in X$, is a nonempty convex and closed subset of $Y$.

Consider now two multivalued mappings $F_1 : X \to Y_1$, $F_2 : X \to Y_2$ and a single-valued mapping $L : Y_1 \times Y_2 \to Y$. Denote by $L(F_1; F_2)$ the composite mapping, which associates to each $x \in X$ the set

$$\{y \in Y : y = L(y_1; y_2), \ y_1 \in F_1(x), \ y_2 \in F_2(x)\}.$$

**Definition 1.1.** Let $f$ be a selection of the composite mapping $L(F_1; F_2)$. A pair $(f_1, f_2)$ is said to be a *splitting* of $f$ if $f_1$ is a selection of $F_1$, $f_2$ is a selection of $F_2$ and $f = L(f_1; f_2)$.

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In Sections 2 and 3 below we work in the category of topological spaces. Thus the *splitting problem* (see [7]) for the triple $(F_1, F_2, L)$ is the problem of finding continuous selections $f_1$ and $f_2$ which split a continuous selection $f$ of the composite mapping $L(F_1; F_2)$.

For $Y_1 = Y_2 = Y$ and $L(y_1; y_2) = y_1 + y_2$ we see the specific problem of splitting into a sum of two items. More generally, for constant multivalued mappings, the splitting problem can be interpreted as the problem of continuous dependence of solutions of the linear equation $y = L(y_1; y_2)$ on the data $y$ and with constraints $y_1 \in A$ and $y_2 \in B$.

One more example: let $Y_1 = Y_2 = \mathbb{R}$, $F_1(\cdot) \equiv [0, +\infty)$, $F_2(\cdot) \equiv (-\infty; 0]$ and again $L(y_1; y_2) = y_1 + y_2$. Then $L(F_1; F_2)(\cdot) \equiv \mathbb{R}$ and an arbitrary selection of $L(F_1; F_2)$ is simply an arbitrary mapping from the domain into $\mathbb{R}$. So in this case the solvability of the splitting problem means the existence of a decomposition $f = f^+ + f^-$, e.g. in the theory of the Lebesgue integral (see [2, Section 25]).

Within the framework of the general theory of continuous selections and due to the Banach open mapping principle it is quite natural to restrict ourselves to the case of paracompact domains $X$, Banach range spaces $Y_1$, $Y_2$, $Y$ and LSC convex-valued and closed-valued mappings $F_1$, $F_2$, and to the case of linear continuous surjections $L : Y_1 \times Y_2 \to Y$.

For a special case of the constant mappings $F_1(\cdot) \equiv A$ and $F_1(\cdot) \equiv B$, the splitting problem can be reduced (Theorem 3.1) to invertibility of a mapping $L : Y_1 \times Y_2 \to Y$ with respect to an appropriate family $\mathcal{C}$ of subsets of $Y_1 \times Y_2$.

**Definition 1.2.** A linear continuous mapping $L : Z \to Y$ between Banach spaces is said to be $\mathcal{C}$-hereditary invertible for a family $\mathcal{C}$ of subsets of $Z$ if for every $C \in \mathcal{C}$ the restriction $L|_C : C \to L(C)$ admits a right-inverse continuous mapping $s : L(C) \to C$, $L|_C \circ s = \mathrm{id}\,|_{L(C)}$.

In terms of continuous selections, $L : Z \to Y$ is $\mathcal{C}$-hereditary invertible whenever the inverse multivalued mapping $(L|_C)^{-1} : L(C) \to C$ admits a continuous selection. Clearly, for a class $\mathcal{C}$ consisting of closed and convex sets the $\mathcal{C}$-hereditary invertibility of $L : Z \to Y$ follows from $\mathcal{C}$-hereditary openess of $L$. This simply means that each restriction $L|_C : C \to L(C)$ is an open mapping. Therefore $\mathcal{C}$-hereditary openess of $L$ guarantees that the Michael selection theorem mentioned above is applicable to each mapping $(L|_C)^{-1} : L(C) \to C$, $C \in \mathcal{C}$.

Unfortunately, as a rule $\mathcal{C}$-hereditary openess (and also $\mathcal{C}$-hereditary invertibility) of an arbitrary map $L : Z \to Y$ is a very restrictive property. For example, for the class $\mathcal{C}$ of all convex compacta this means that $\dim Z \leqslant 2$ or $\dim Y = 1$ (Theorem 2.1 and Remark (1)). In Theorem 2.3 we prove that if the boundary of a convex finite-dimensional compactum $C$ is "transversal" to $\mathrm{Ker}\,L$ then $L|_C : C \to L(C)$ is an open mapping. On other hand, finite-dimensionality is here the principal point. Namely, Theorem 2.4 shows that in any infinite-dimensional Banach space $Z$ there is a subcompactum $C$ for which all assumptions of Theorem 2.3. hold, but $L|_C : C \to L(C)$ is not open, and moreover the inverse mapping $(L|_C)^{-1} : L(C) \to C$ admits no (even local) continuous selection.

In Section 3 we apply positive results of Section 2 to finding of the splittings. In particular, for a single-valued mapping $f$ to a compact space $L(A, B)$ we obtain results on splitting of $f$ into mappings to $A$ and to $B$ (cf. Theorem 3.5 and Corollary 3.6). As a corollary, we prove that for the Minkowski sum $A + B$ of finite-dimensional strictly convex bounded $A$ and $B$ the equality $c = a(c) + b(c)$, $c \in A + B$ holds for some continuous single-valued mappings $a : A + B \to A$ and $b : A + B \to B$.

Finally, recall that the lower semicontinuity of a multivalued mapping $F : X \to Y$ between topological spaces $X$ and $Y$ means that for each points $x \in X$ and $y \in F(x)$, and each open neighborhood $U(y)$, there exists an open neighborhood $V(x)$ such that $F(x') \cap U(y) \neq \emptyset$, whenever $x' \in V(x)$. If one identifies the mapping $F : X \to Y$ with its graph $\mathrm{Gr}\,F \subset X \times Y$, then the lower semicontinuity of $F$ is equivalent to the openess of the restriction $p_1|_{\mathrm{Gr}\,F} : \mathrm{Gr}\,F \to X$, where $p_1 : X \times Y \to X$ is the projector onto the first coordinate. Roughly speaking, lower semicontinuous multivalued mappings are exactly inverses of open single-valued mappings.

## 2. Hereditary openess and invertibility

**Theorem 2.1.** *For any Banach space $Y$ the following statements are equivalent*:

(a) *each linear continuous surjection $L : Z \to Y$ from a Banach space $Z$ is $\mathcal{F}_c(Z)$-hereditary invertible with respect to the family $\mathcal{F}_c(Z)$ of all convex closed subsets of $Z$*;
(b) *each linear continuous surjection $L : Z \to Y$ from a Banach space $Z$ is $\mathcal{C}_c$-hereditary invertible with respect to the family $\mathcal{C}_c$ of all convex subcompacta of $Z$*; *and*
(c) $\dim Y = 1$.

**Proof.** The implication (a) $\Rightarrow$ (b) is trivial. To check (b) $\Rightarrow$ (c) we shall need the following lemma.

**Lemma 2.2.** *In the Euclidean $3$-space $\mathbb{R}^3 = \mathbb{R}^2 \oplus \mathbb{R}^1$ there is a convex compact set $C$ such that the restriction $P|_C : C \to P(C)$ of the orthogonal projection $P : \mathbb{R}^3 \to \mathbb{R}^2$ is not an open mapping. Moreover, the inverse multivalued mapping $(P|_C)^{-1}$ admits no continuous selection.*

**Proof.** Let $K$ be one full rotation of the spiral

$$K = \{(\cos t, \sin t, t) \colon \ 0 \leqslant t \leqslant 2\pi \}$$

and $C = \operatorname{conv} K = \overline{\operatorname{conv}} K$. Suppose to the contrary that the point-preimages multivalued mapping $(P|_C)^{-1} : P(C) \to C$ admits a continuous selection, say $s : P(C) \to C$. Observe that $P(C)$ is the unit disk $D = \{(r\cos t, r\sin t, 0) \colon 0 \leqslant t \leqslant 2\pi, \ 0 \leqslant r \leqslant 1\}$ and that the mapping $(P|_C)^{-1}$ is single-valued over the whole boundary $\partial D$ of $D$ except over the initial point $(1, 0, 0)$. Hence the continuous selection $s$ coincides with $(P|_C)^{-1}$ on $\partial D \setminus \{(1, 0, 0)\}$.

Therefore $\lim_{t \to 0+0} s(\cos t, \sin t, 0) = (1, 0, 0)$ and $\lim_{t \to 2\pi - 0} s(\cos t, \sin t, 0) = (1, 0, 2\pi)$, which contradicts the continuity of $s$. Note that in fact, the mapping $(P|_C)^{-1}$ admits no selections which is continuous at the point $(1, 0, 0)$. $\quad \square$

Now, suppose that the assumption (c) does not hold, i.e. $\dim Y \geqslant 2$. Hence $Y = \mathbb{R}^2 \oplus Y'$ for some Banach space $Y'$. Let $Z = \mathbb{R}^3 \oplus Y'$. Then we can map $\mathbb{R}^3 \to \mathbb{R}^2$ as in Lemma 2.2, and map $Y'$ onto $Y'$ identically, consider the direct sum of these linear surjections and obtain a contradiction with the assumption (b) on the existence of the right inverse for the restriction $P|_C : C \to P(C)$.

In order to prove (c) $\Rightarrow$ (a), let us first check that the restriction $L|_C : C \to L(C)$ is an open mapping for every linear continuous map $L : Z \to Y$ and for every convex set $C \subset Z$. Suppose to the contrary that $L|_C$ is not open at some point $z \in C$. Then there exist a number $\varepsilon > 0$ and a sequence $\{y_n\}_{n=1}^\infty$ with $y_n \in L(C)$ such that $y_n \to L(z)$, $n \to \infty$ and

$$\operatorname{dist}\big(z; L^{-1}(y_n) \cap C\big) \geqslant \varepsilon, \quad n \in \mathbb{N}.$$

The set $L(C)$ is convex and one-dimensional. Thus one can assume that $\{y_n\}_{n=1}^\infty$ is monotone. Let $y_n \to L(z) + 0$. Then $y_n = (1 - t_n)L(z) + t_n y_1$, $t_n \to 0 + 0$. By the choice of $\{y_n\}_{n=1}^\infty$ there exists a point $z_1 \in L^{-1}(y_1) \cap C$. Hence $z_n = (1 - t_n)z + t_n z_1 \in [z, z_1] \subset C$ and $L(z_n) = y_n$. So $z_n \in L^{-1}(y_n) \cap C$ and $z_n \to z$. Thus $\operatorname{dist}(z; L^{-1}(y_n) \cap C) \to 0$ which contradicts the fact that $\operatorname{dist}(z; L^{-1}(y_n) \cap C) \geqslant \varepsilon$.

Now, let us return to the case when $C \in \mathcal{F}_c(Z)$. Since the set $L(C)$ is metrizable and hence paracompact, all values of the mapping $y \mapsto L^{-1}(y) \cap C$ are nonempty, convex and closed. Such the mapping is LSC because $L|_C : C \to L(C)$ is an open mapping. So applying the Michael selection theorem we find the continuous right inverse of $L|_C : C \to L(C)$. $\quad \square$

### Remarks.

(1) In the same way one can prove that $\mathcal{C}_c$-hereditary invertibility characterizes Banach spaces $Z$ with $\dim Z \leqslant 2$.
(2) The analog of Theorem 2.1 holds under substitution of hereditary openess instead of hereditary invertibility even without closedness assumption for convex subsets of $Z$ in (a). In fact, one can use instead of the example from Lemma 2.2 another (widely known) example of the convex hull $C \subset \mathbb{R}^3$ of the set $\{(\cos t, \sin t, 0) \colon 0 \leqslant t \leqslant 2\pi\} \cup \{(1, 0, 1)\}$ and orthogonal projection $p : \mathbb{R}^3 \to \mathbb{R}^2$, $p(x, y, z) = (x, y, 0)$. Note that $(p|_C)^{-1}$ here admits the obvious (identical) continuous selection. This is the key difference with the example from Lemma 2.2.

Theorem 2.1 shows that separate and independent assumptions on linear mapping $L$ and on a convex compact set $C \subset Z$ cannot give an essential result. So some linking properties on $L$ and $C$ are needed.

Let us recall that for a convex subset $M$ of a Banach space $Z$ there are (at least) two approaches to the notion of its relatively inner point. First, a point $m \in M$ is said to be inner (in the *metric* sense) point of $M$ provided that for some positive $\varepsilon$ the intersection $D(m; \varepsilon) \cap \operatorname{aff}(M)$ is subset of $M$. Here and below $D(m; \varepsilon)$ denotes the open ball with radius $\varepsilon$ centered at $m$. Second, a point $m \in M$ is said to be inner (in the *convex* sense) point of $M$ provided that for each $x \in M$, $x \neq m$, there exists $y \in M$ such that $m \in [x; y)$. Here, $[x; y)$ is the straight line semiinterval, i.e. the segment $[x; y]$ without the end point $y$.

A great advantage of finite-dimensional convex sets is that for them these approaches are equivalent (see [8, 2.3.6 and 2.6.10]). The Hilbert cube $Q$, lying in any Banach, or Frechet space, has no inner (in the *metric* sense) points. But $Q$ certainly has inner (in the *convex* sense) points: they constitute the so-called *pseudo-interior* of the Hilbert cube. Note that each infinite-dimensional convex compact subset of a Frechet space is homeomorphic to $Q$, due to the Keller theorem [5].

Below we shall use this equivalence without any special reference and we shall denote by $\operatorname{int}(M)$ (resp., $\partial(M)$) the set of all inner (resp., boundary) points of a finite-dimensional convex set $M$. Observe that $\operatorname{int}(A \times B) = \operatorname{int}(A) \times \operatorname{int}(B)$.

**Theorem 2.3.** *Let $L : X \to Y$ be a linear continuous surjection between Banach spaces. Let $C \subset X$ be a convex finite-dimensional bounded subset of $X$ such that the boundary $\partial(C)$ contains no segments parallel to the kernel $\operatorname{Ker}(L)$. Then the restriction $L|_C : C \to L(C)$ is an open mapping.*

**Proof.** (1) Let $x \in \operatorname{int}(C)$. Then the conclusion follows from the Banach open mapping principle, applied to the restriction $L|_{\operatorname{aff}(C)}$.

(2) Let $x \in C \cap \partial(C)$, $L(x) = y$, but suppose that $L^{-1}(y)$ intersects $\operatorname{int}(C)$. So let $x_0 \in L^{-1}(y) \cap \operatorname{int}(C)$. It is a well known and fundamental fact that the whole semiinterval $(x; x_0]$ lies in $\operatorname{int}(C)$ [8, 2.3.4]. Thus for every $\varepsilon > 0$ there is $x_\varepsilon \in L^{-1}(y) \cap \operatorname{int}(C) \cap D(x; \varepsilon)$. Choose $\delta > 0$ such that $D(x_\varepsilon; \delta) \subset D(x; \varepsilon)$. Due to the case (1) the image $L(D(x_\varepsilon; \delta) \cap C)$ contains some neighborhood, say $V(y)$ of the point $y$ in $L(C)$. This is why

$$V(y) \subset L\big(D(x_\varepsilon; \delta) \cap C\big) \subset L\big(D(x; \varepsilon) \cap C\big),$$

i.e. the restriction $L|_C : C \to L(C)$ is open at the point $x$.

(3) Let $x \in C \cap \partial(C)$, $L(x) = y$ and $L^{-1}(y) \cap \mathrm{int}(C) = \emptyset$. Hence $(L^{-1}(y) \cap C) \subset \partial(C)$. If $x_1$ and $x_2$ are two distinct points in $L^{-1}(y) \cap C$ then the segment $[x_1, x_2]$ is parallel to $\mathrm{Ker}(L)$ and lies in the boundary $\partial(C)$. This is a contradiction with the assumption of the theorem. Thus $L^{-1}(y) \cap C = \{x\}$.

Suppose to the contrary, that the mapping $L|_C$ is not open at $x$. This means that for some $\varepsilon > 0$ and for some sequence $y_n \to y$, $y_n \in L(C)$, $n \to \infty$, each distance $\mathrm{dist}(L^{-1}(y_n) \cap C, x)$ is greater than or equal to $\varepsilon$. For each $n \in \mathbb{N}$ pick $x_n \in L^{-1}(y_n) \cap C$. Due to precompactness of $C$ we can choose convergent subsequence, say $x_{n_k} \to x_0$, $x_0 \in Cl(C)$, $k \to \infty$. Then $y_{n_k} = L(x_{n_k}) \to L(x_0)$ and $L(x_0) = y$, or $x_0 \in L^{-1}(y) \cap Cl(C)$. But $L^{-1}(y) \cap \mathrm{int}(C) = \emptyset$. Therefore $x_0 \in L^{-1}(y) \cap \partial(C)$ and $x_0 = x$, due to the transversality type assumption that the boundary $\partial(C)$ contains no segments parallel to the kernel $\mathrm{Ker}(L)$. So

$$\mathrm{dist}(x_{n_k}, x) \geqslant \mathrm{dist}\big(L^{-1}(y_{n_k}) \cap C, x\big) \to 0, \quad k \to \infty,$$

hence we get a contradiction. $\quad\square$

The following theorem demonstrates that without the finite-dimensionality the restriction local invertibility can fail even at the inner points.

**Theorem 2.4.** *For every infinite-dimensional Banach space $Z$ and for every continuous linear projector $P : Z \to Z$ with $\dim \mathrm{Ker}(P) = 1$ there is an infinite-dimensional convex compact subset $C \subset Z$ and an inner point $z \in C$ such that the restriction $P|_C : C \to P(C)$ is not open at $z$. Moreover, the inverse multivalued mapping $(P|_C)^{-1} : P(C) \to C$ admits no continuous selections over an arbitrary neighborhood of the point $P(z)$.*

**Proof.** Choose any basic Schauder normalized sequence $e_1, e_2, \ldots, e_n, \ldots$, $\|e_n\| = 1$ with $e_1 \in \mathrm{Ker}\, P$ and $e_n \in \mathrm{Im}\, P$, $n > 1$. So, $P(e_1) = 0$, $P(e_n) = e_n$, $n > 1$. Define

$$C = \overline{\mathrm{conv}}\left\{\frac{e_n}{n}, 2e_1 - \frac{e_n}{n}\right\}_{n=1}^\infty, \qquad K = \mathrm{conv}\left\{\frac{e_n}{n}, 2e_1 - \frac{e_n}{n}\right\}_{n=1}^\infty.$$

The set $\{\frac{e_n}{n}, 2e_1 - \frac{e_n}{n}\}_{n=1}^\infty$ is precompact because it consists of two convergent sequences $\frac{e_n}{n} \to 0$ and $2e_1 - \frac{e_n}{n} \to 2e_1$. Hence $K$ is also precompact set and $C$ is a convex compact subset of $X$. The point $e_1$ is the center of symmetry of the set $C$ and hence is its inner point.

Suppose that we have already checked that the multivalued mapping $(P|_C)^{-1} : P(C) \to C$ is single-valued over the set $\{\frac{e_n}{n}, 2e_1 - \frac{e_n}{n}\}_{n=1}^\infty$. In other words, suppose that we have proved that

$$(P|_C)^{-1}\left(\frac{e_n}{n}\right) = \frac{e_n}{n}, \qquad (P|_C)^{-1}\left(-\frac{e_n}{n}\right) = 2e_1 - \frac{e_n}{n}.$$

In this assumption, if $s : U \cap P(C) \to C$ is a continuous selection of $(P|_C)^{-1}$ over some neighborhood $U$ of the point $P(e_1) = 0$ then

$$\lim_{n\to\infty} s\left(\frac{e_n}{n}\right) = 0, \qquad \lim_{n\to\infty} s\left(-\frac{e_n}{n}\right) = 2e_1,$$

which contradicts the continuity of $s$ at 0.

In order to complete the proof it suffices to check that $\lambda e_1 + \frac{e_n}{n} \in C$ if and only if $\lambda = 0$ and, analogously $\mu e_1 + (2e_1 - \frac{e_n}{n}) \in C$ if and only if $\mu = 0$.

**Lemma 2.5.** *For each $z = \lambda e_1 + \frac{e_n}{n}$, $\lambda > 0$, $n > 1$ there exists $d > 0$ such that $\mathrm{dist}(z, K) \geqslant d$ and hence $\mathrm{dist}(z, C) \geqslant d$.*

**Proof.** For every $N \in \mathbb{N}$, let $K_N = \mathrm{conv}\{\frac{e_k}{k}, 2e_1 - \frac{e_k}{k}\}_{k=1}^N$. Then $K = \bigcup_{n=1}^N K_N$ and $\mathrm{dist}(z, K) = \inf\{\mathrm{dist}(z, K_N) : N \in \mathbb{N}\}$. Clearly, $K_{n-1} \subset \mathrm{span}\{e_k\}_{k=1}^{n-1}$ and $z \notin \mathrm{span}\{e_k\}_{k=1}^{n-1}$. Therefore $z \notin K_{n-1}$ and $\mathrm{dist}(z, K_{n-1}) = d_1 > 0$.

Thus we must consider the case $N \geqslant n$. So let $y \in K_N$, i.e.

$$y = \alpha_1 e_1 + \sum_{k=2}^N \left(\alpha_k \frac{e_k}{k} + \beta_k\left(2e_1 - \frac{e_k}{k}\right)\right) = \left(\alpha_1 + 2\sum_{k=2}^N \beta_k\right)e_1 + \sum_{k=2}^N (\alpha_k - \beta_k)\frac{e_k}{k}$$

for some nonnegative $\alpha_1, \alpha_2, \ldots, \alpha_N, \beta_2, \ldots, \beta_N$ with $\alpha_1 + \sum_{k=2}^N (\alpha_k + \beta_k) = 1$.

Hence

$$z - y = \left(\lambda - \left(\alpha_1 + 2\sum_{k=2}^N \beta_k\right)\right)e_1 + \big(1 - (\alpha_n - \beta_n)\big)\frac{e_n}{n} + \sum_{k=2,\, k\neq n}^N (\beta_k - \alpha_k)\frac{e_k}{k}.$$

Note that $\alpha_n - \beta_n \leqslant \alpha_n \leqslant 1$ and therefore $1 - (\alpha_n - \beta_n) \geqslant 0$.

*Case* (a). Let $1 - (\alpha_n - \beta_n) \geqslant \frac{\lambda}{3}$. Then

$$\|z - y\| \geqslant \frac{1 - (\alpha_n - \beta_n)}{n\|P_n\|} \geqslant \frac{\lambda}{3n\|P_n\|},$$

where $P_n : \overline{\mathrm{span}}\{e_k\}_{k=1}^\infty \to \mathrm{span}\{e_n\}$ is the continuous linear projection onto the $n$-th coordinate. Recall that $\|P_n(u)\| \leqslant \|P_n\| \cdot \|u\|$. In our case $u = z - y$ and $P_n(u) = (1 - (\alpha_n - \beta_n))e_n$.

*Case* (b). Let $1 - (\alpha_n - \beta_n) < \frac{\lambda}{3}$. Then $1 - \frac{\lambda}{3} < \alpha_n - \beta_n \leqslant \alpha_n$ and $1 - \alpha_n < \frac{\lambda}{3}$. So

$$\beta_2 + \cdots + \beta_N \leqslant \alpha_1 + \beta_2 + \cdots + \beta_N \leqslant 1 - \alpha_n < \frac{\lambda}{3}$$

and $\alpha_1 + 2\sum_{k=2}^N \beta_k < \frac{2\lambda}{3}$. Hence $\lambda - (\alpha_1 + 2\sum_{k=2}^N \beta_k) > \frac{\lambda}{3}$ and $\|x - y\| > \frac{\lambda}{3\|P_1\|}$, where $P_1 : \overline{\mathrm{span}}\{e_k\}_{k=1}^\infty \to \mathrm{span}\{e_1\}$ is the continuous linear projection onto the first coordinate.

Thus, in any case $\|z - y\| \geqslant \min\{\frac{\lambda}{3n\|P_n\|}, \frac{\lambda}{3\|P_1\|}\} = d_2 > 0$. Finally, for each $N \in \mathbb{N}$

$$\mathrm{dist}(z, K_N) = \inf\{\|z - y\| : y \in K_N\} \geqslant \min\{d_1, d_2\} = d > 0$$

and $\mathrm{dist}(z, K) \geqslant d$. This completes the proof of the lemma and also of the theorem.  □

## 3. Splitting selections

To reduce the splitting problem for constant multivalued mappings to the hereditary invertibility property the following simple statement is useful. We state it in a rather abstract form.

**Theorem 3.1.** *Suppose that a continuous surjection $L : Y_1 \times Y_2 \to Y$ between Banach spaces is $(\mathcal{C}_1 \times \mathcal{C}_2)$-hereditary invertible with respect to some families $\mathcal{C}_1$ subsets of $Y_1$ and $\mathcal{C}_2$ subsets of $Y_2$. Let $A \in \mathcal{C}_1$ and $B \in \mathcal{C}_2$. Then the splitting problem for the triple $(F_1(\cdot) \equiv A, F_2(\cdot) \equiv B, L)$ is solvable for an arbitrary domain $X$.*

**Proof.** Under the assumptions the composite mapping $F = L(F_1, F_2)$ is the constant multivalued mapping $F(\cdot) \equiv L(A, B)$. Hence its continuous single-valued selection, say $f$, simply is an arbitrary continuous single-valued mapping $f : X \to L(A, B)$ from a domain $X$. We define an auxiliary multivalued mapping $\Phi : X \to Y_1 \times Y_2$, by setting

$$\Phi(x) = \{(y_1; y_2) \mid y_1 \in A, \ y_2 \in B, \ L(y_1; y_2) = f(x)\} = (A \times B) \cap L^{-1}(f(x)).$$

All values of $\Phi$ are nonempty because $f(X) \subset L(A, B)$. So first, to each argument $x \in X$ in the continuous fashion there corresponds the point $y = f(x) \in L(A, B) \subset Y$. Second, to this point $y$ corresponds the set $(A \times B) \cap L^{-1}(y)$. And the $(\mathcal{C}_1 \times \mathcal{C}_2)$-hereditary invertibility of $L$ means exactly that the last multivalued correspondence has a continuous selection (see Definition 1.2). Thus its composition with $f$ is a continuous selection, say $\varphi$, of $\Phi$.

So if $f_1 = p_1 \circ \varphi : X \to A$ and $f_2 = p_2 \circ \varphi : X \to B$, where $p_i(y_1, y_2) = y_i, i = 1, 2$ are "coordinate" projections $p_i : Y_1 \times Y_2 \to Y_i$, then

$$L(f_1(x), f_2(x)) = L(p_1(\varphi(x)), p_2(\varphi(x))) = L(\varphi(x)) = f(x),$$

because $\varphi(x) \in \Phi(x) \subset L^{-1}(f(x))$, $x \in X$. Thus the pair $(f_1, f_2)$ splits the mapping $f$.  □

We emphasize that Theorem 3.1 is a conditional statement which simply reduces one problem to another: the checking of $(\mathcal{C}_1 \times \mathcal{C}_2)$-hereditary invertibility is a separate and nontrivial job. Theorem 3.1 gives a way of transferring the results from the previous section to splitting of continuous selections. First we transfer the example from Lemma 2.2.

**Example 3.2.** For any 2-dimensional cell $D$ there exist:

(a) constant multivalued mappings $F_1 : D \to \mathbb{R}^3$ and $F_2 : D \to \mathbb{R}$ with convex compact values;
(b) a linear surjection $L : \mathbb{R}^3 \oplus \mathbb{R} \to \mathbb{R}^2$; and
(c) a continuous selection $f$ of the composite mapping $F = L(F_1, F_2)$, such that $f \neq L(f_1, f_2)$ for any continuous selections $f_i$ of $F_i$, $i = 1, 2$.

**Proof.** In the notations of Lemma 2.2 let

$$D = P(C), \qquad F_1(\cdot) \equiv C, \qquad F_2(\cdot) \equiv [0; 1], \qquad L = P \oplus 0|_\mathbb{R} : \ \mathbb{R}^3 \oplus \mathbb{R} \to \mathbb{R}^2.$$

Then $L(C \oplus [0; 1]) = D$, $F(\cdot) = L(F_1, F_2)(\cdot) \equiv D$ and $f = \mathrm{id}\,|_D$ is a continuous selection of $F$. Suppose to the contrary that $f = L(f_1, f_2)$ for some continuous selections $f_i$ of $F_i$, i.e. for mappings $f_1 : D \to C$ and $f_2 : D \to [0; 1]$. But the surjection $L$ "forgets" the second coordinate. Hence

$$D = f(x) = L\big(f_1(x), f_2(x)\big) = P\big(f_1(x)\big), \quad x \in D,$$

or $f_1(x) \in C \cap P^{-1}(x)$.

This means that $f_1$ is a continuous selection of multivalued mapping $x \mapsto C \cap P^{-1}(x)$, $x \in D$ which contradicts Lemma 2.2. $\square$

For application of Theorem 2.3 we need some additional smoothness-like restriction on boundaries of convex sets (compare with the notion of a strictly convex Banach space).

**Definition 3.3.** The convex subset $C$ of a Banach space is said to be *strictly convex* if the middle point of any nontrivial segment $[x, y]$, $x \in C$, $y \in C$ is an inner (in the convex sense) point of $C$.

Equivalently, the boundary of $C$ contains no straight line segment.

**Theorem 3.4.** *Let $A$ and $B$ be strictly convex finite-dimensional bounded subsets of Banach spaces $Y_1$ and $Y_2$, respectively. Let $L : Y_1 \times Y_2 \to Y$ be a linear continuous surjection with kernel $\mathrm{Ker}(L)$ transversal to $Y_1 \times \{0\}$ and $\{0\} \times Y_2$. Then the restriction $L|_{A \times B} : A \times B \to L(A \times B)$ is an open mapping.*

**Proof.** In view of Theorem 2.3, it suffices to check only that the boundary $\partial(A \times B)$ contains no segment parallel to $\mathrm{Ker}(L)$. Suppose to the contrary that $c_1 \neq c_2$, $[c_1, c_2] = [(a_1, b_1), (a_2, b_2)] \subset \partial(A \times B)$ and $[c_1, c_2]$ is parallel to $\mathrm{Ker}(L)$. This means that $(a_1 - a_2, b_1 - b_2) \in \mathrm{Ker}(L)$. So if $a_1 = a_2$ then the transversality assumption implies that $b_1 = b_2$ and hence $c_1 = c_2$. Contradiction. Hence $a_1 \neq a_2$ and analogously $b_1 \neq b_2$.

By strict convexity $a' = 0, 5(a_1 + a_2) \in \mathrm{int}(A)$ and $b' = 0, 5(b_1 + b_2) \in \mathrm{int}(B)$. But $(a', b') \in [c_1, c_2]$. So the segment $[c_1, c_2]$ intersects $\mathrm{int}(A \times B)$ which contradicts the existence of inclusion $[c_1, c_2] \subset \partial(A \times B)$. $\square$

Theorems 3.1 and 3.4 together imply:

**Theorem 3.5.** *Let $A$ and $B$ be strictly convex finite-dimensional bounded subsets of Banach spaces $Y_1$ and $Y_2$, respectively. Let $L : Y_1 \times Y_2 \to Y$ be a linear continuous surjection with kernel $\mathrm{Ker}(L)$ transversal to $Y_1 \times \{0\}$ and $\{0\} \times Y_2$. Then for every continuous single-valued mapping $f : X \to L(A, B)$ from a domain $X$ there are continuous single-valued mappings $f_1 : X \to A$ and $f_2 : X \to B$ such that*

$$L\big(f_1(x), f_2(x)\big) = f(x), \quad x \in X.$$

**Proof.** Theorem 3.4 implies that the restriction $L|_{A \times B} : A \times B \to L(A \times B)$ is an open mapping. Its image is a metric (and hence, perfectly normal) space. All its point-preimages are nonempty convex finite-dimensional subsets of a separable (finite-dimensional, in fact) Banach space $\mathrm{span}(A) \times \mathrm{span}(B)$. Hence Theorem 3.1$'''$ from [3] shows that $L|_{A \times B} : A \times B \to L(A \times B)$ is $\mathcal{C}$-hereditary invertible, where $\mathcal{C}$ is the family of all strictly convex finite-dimensional subsets of a Banach space $\mathrm{span}(A) \times \mathrm{span}(B)$. So an application of Theorem 3.1 completes the proof. $\square$

Remark that for a convex closed-valued LSC mappings $F_1 : X \to Y_1$ and $F_2 : X \to Y_2$ and for a linear continuous surjection $L : Y_1 \times Y_2 \to Y$, the splitting problem has an affirmative solution in the case of one-dimensional $Y_1$ and $Y_2$ and arbitrary paracompact domains (see [7, Theorem 3.1]). But in general, splittings of continuous selections exist only if members of the triple $(F_1, F_2, L)$ properly agree. See [7, Example 4.2] for a counterexample even for the case $\dim Y_1 = 2$, $\dim Y_2 = 1$ and for a countable domain.

We conclude the section by showing the partial case of Theorem 3.5 applying it for the Minkowski sum of convex sets.

**Corollary 3.6.** *Let $A$ and $B$ be strictly convex finite-dimensional bounded subsets of Banach spaces $Y$. Then there are continuous single-valued mappings $a : A + B \to A$ and $b : A + B \to B$ such that $c = a(c) + b(c)$ for all $c \in A + B$.*

**Proof.** In assumptions of Theorem 3.5 we choose the very special linear continuous surjection $L : Y_1 \times Y_2 \to Y$ and special perfectly normal (in fact, metric) domain $C$. Namely, $Y_1 = Y_2 = Y$, $L(y_1, y_2) = y_1 + y_2$ and $C = A + B$.

Clearly $(y_1, 0) \in \mathrm{Ker}(L) \Leftrightarrow y_1 = 0$, i.e. the kernel $\mathrm{Ker}(L)$ is transversal to $Y \times \{0\}$ and to $\{0\} \times Y$. So Theorem 3.5 implies that the identity mapping $\mathrm{id} : C \to C$ admits a splitting $\mathrm{id} = L(f_1, f_2)$ for some continuous single-valued $a : C \to A$ and $b : C \to B$.

In other words, if $c \in C$ and $c \mapsto \{(a,b): c = a + b\}$ then we can always assume that $a = a(c)$ and $b = b(c)$ are continuous items with respect to the data $c \in C$. $\square$

Analogously, the another version of Theorem 3.5 states that the continuous mapping $f$ from $X$ to the Minkowski sum $A + B$ splits into a sum of two continuous mappings $f_1 : X \to A$ and $f_2 : X \to B$, whenever $A$ and $B$ are strictly convex finite-dimensional bounded subsets of a Banach spaces $Y$.

Finally, we guess that the strict convexity assumption can be weakened in some ways, but that in general, Corollary 3.6 does not hold for an arbitrary convex finite-dimensional compacta.

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