On continua with homotopically fixed boundary

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Abstract

The paper presents two subcontinua of $\mathbb{R}^n$, one Peano-continuum, and one cellular continuum with trivial fundamental group. Both of them have the remarkable property that neither the entire spaces nor (roughly speaking) any part of them is homotopy equivalent to a lower-dimensional space. This extends work of the last three authors and of Karimov from the planar case to the higher-dimensional case, but it also contains in the cellular case the first example with all these properties in dimension two.
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1. Introduction

It is a natural property of finite simplicial complexes in $\mathbb{R}^n$ that, by systematically collapsing n-dimensional simplices with free faces, the entire complex can be homotopy equivalently transformed into an at most $(n-1)$-dimensional complex that is usually called "spine". Also, open subsets of $\mathbb{R}^n$ have this property of being homotopy equivalent to an at most $(n-1)$-dimensional set; apparently this has been known and used as an exercise in the Bing-School. However, we are not aware of any written version apart from [14]. Hence it came to some people as a kind of surprise when [16, A.4.13] contained an example of a planar Peano-continuum, which apparently did not offer any possibility for a collapse. In [4, §5] it was then rigorously proven for an analogously constructed space that this space is definitely not homotopy equivalent to any lower-dimensional space. The examples of [16] and of [4] had large (i.e. uncountable) fundamental groups. In [9, Example 1] it was shown that for a planar set, in order to achieve the property of being not homotopy equivalent to any one-dimensional space, it is not necessary to have a non-trivial fundamental group. By extending the use of the term "homotopy dimension" from complexes to arbitrary topological spaces, this property was called "of homotopy dimension two" (cf. our Definition 3.1(i)). Ref. [9, Definition 2.1(ii)] also introduced the term "everywhere homotopically two-dimensional" as a precise mathematical term that mimics the intuitive feeling that every part of a space has homotopy dimension two. We will be also using this phrase here (cf. Definition 3.1(ii)). The codiscrete subsets of the two-sphere of homotopy dimension two have meanwhile been characterized

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by Cannon and Conner in [3, Theorem 1.1]. Ref. [9] already contains, apart from the cellular planar homotopically two-dimensional continuum with trivial fundamental group (that was Example 1 that we mentioned before), an everywhere homotopically two-dimensional planar Peano-continuum [9, Example 3]. By the way, as was also shown in [9, Proposition 1.1(i)], no planar continuum can have all three properties: being homotopically two-dimensional, with trivial π_1, and a Peano-continuum altogether.

In this paper now we are giving two examples of subcontinua of $\mathbb{R}^n$ which both also have (and have everywhere) homotopy dimension n. One of them will be a Peano-continuum (Example B), the other a cellular continuum with trivial fundamental group (Example A). We construct these spaces by using the same ideas as in [9, Example 1–3]. However since some theorems in topology do only hold in the plane, here at some places we do have to work harder than it was necessary in [9, §2–3] (cf. 1.5 below). The idea for our Example B is using a fractal-like iterated gluing of thickened Hawaiian earrings. I.e. the construction starts with a space H that is pictured in Fig. 1, and it uses the property that the accumulation point of this space is homotopically fixed (cf. Definition 3.2). Then we will glue infinitely many copies of H to a dense subset of the boundary of H, in this way making all attaching points again homotopically fixed. Next, we will glue a second collection of copies of H to a dense subset of the boundary of the previous set, and after infinitely many such steps we will take the closure of this construction. It will be necessary to precisely understand how this space changes during this process for being able to prove that even after having taken the closure of this space it still has the desired properties.

Example A is a similar fractal-like iterated gluing of thickened triangular doublecomb-spaces. Hence for $n = 2$ our Example A is a planar, simply connected, homotopically and everywhere homotopically two-dimensional cellular continuum, and such an example was not contained in [9]. We will construct these examples in Section 2 of this paper, and in Sections 3–6 we will prove that they have the following properties:

Properties 1.1. Our Example A *is a pathwise connected subcontinuum of* $\mathbb{R}^n$ ($n \geqslant 2$) *which satisfies*:

(a) *It is cellular.*
(b) *It is simply connected.*
(c) *The local dimension at each of its points is n.*
(d) *It is homotopically n-dimensional.*
(e) *It is everywhere homotopically n-dimensional.*
(f) *Each of its boundary points is homotopically fixed* (*cf. Definition* 3.2).
(g) *There does not exist any deformation retraction onto a proper subset.*
(h) *Any map of our space into itself which is homotopic to the identity, is onto.*

Properties 1.2. Our Example B *is a subcontinuum of* $\mathbb{R}^n$ ($n \geqslant 2$) *which satisfies*:

(a) *It is a Peano-continuum.*
(b) *It is simply connected for n > 2.*
(c) *The local dimension at each of its points is n.*
(d) *It is homotopically n-dimensional.*
(e) *It is everywhere homotopically n-dimensional.*
(f) *Each of its boundary points is homotopically fixed* (*cf. Definition* 3.2).
(g) *There does not exist any deformation retraction onto a proper subset.*
(h) *Any map of our space into itself which is homotopic to identity, is onto.*

In the following X might denote either one of our two examples. In both possible cases X has a collection Γ of tame closed subspaces whose interiors are n-cells with the following remarkable

Properties 1.3.

 (i) *The interiors of the elements of Γ are pairwise disjoint and cover the interior of X in $\mathbb{R}^n$.*
 (ii) *For every $\gamma \in \Gamma$ there is a retraction $r : X \to \gamma$ with $r(X \setminus \gamma) \subset \partial \gamma$.*
(iii) *Every point of the boundary of every $\gamma \in \Gamma$ is homotopically fixed in X.*

(iv) *Every neighbourhood of every boundary point of X in $\mathbb{R}^n$ contains some $\gamma \in \Gamma$.*

(v) *Each γ is a closed cell in case of Example A, whereas in case of Example B it is homeomorphic to the closed complementary domain of a standard wedge of two n-cells* (*identified at some boundary points*) *in S^n.*

Remark 1.4. Roughly speaking, the last three sections are devoted to proving separately those properties which require lengthy proofs. Therefore we will in Sections 4–6, respectively, prove that Example A is cellular (Property 1.1(a)), that Example B is locally path-connected (the non-trivial claim from Property 1.2(a)), and that both of our examples are in their appropriate dimensions simply connected (Properties 1.1(b) and 1.2(b)). The remaining proofs are shorter and gathered together in Section 3 (including the path-connectivity claim for Example A, cf. Proposition 3.7). Partially they are immediate consequences of how our spaces are built (cf. Remark 3.8), and partially they follow as immediate corollaries from each other according to the following diagram:
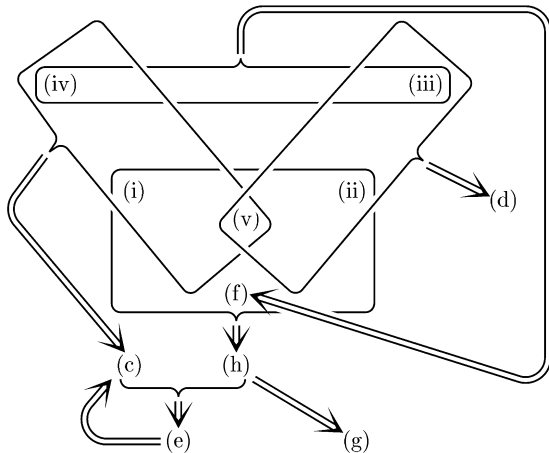
Thus, roughly speaking, Properties 1.3(i)–(v) imply Properties (c)–(h) (cf. Remark 3.9). Proving these implications will be analogous for both of our examples, and accordingly the entries (c), (d), . . . in the above diagram refer simultaneously to Properties 1.1(c) & 1.2(c), 1.1(d) & . . . , and so on, while (i), (ii), . . . are just abbreviations for Properties 1.3(i), 1.3(ii), ...

Remark 1.5. Cell-like sets can be interpreted as sets which have the shape of a point. In the planar case these sets can be characterized by their Vietoris-homology (cf. [1, Chapter VII, Theorem 7.1]) and also by their Čech-homology (cf. [11, Chapter VII, §6, Theorem 26.1]). However, this easy characterization is only available in the planar case. Also Proposition 1.1(b) would in the planar case directly follow from known results, cf. [5, Corollary 6]. For higher dimensions this is wrong, since Griffiths' space (this is Example 0.12 from [15], but with the arc C removed, cf. also [2, p. 315]) is a cellular continuum in $\mathbb{R}^3$ with non-trivial fundamental group—as either follows from Griffiths' original proof for his Example C in [6, [3.4]], [6, Theorem 4], [8, Theorem 6.3] (corrections in [7] and [13]) or by using the elementary proof that was sketched in [15, 3.11(ii)/0.12] for the space $\widehat{A} \cup \widehat{B}$. Therefore the proof of Propositions 1.1 will require some concrete constructions.

2. Construction of our examples

We start with

Example B.

Step B1: Let $B_1 \supset B_2 \supset B_3 \supset \cdots$ be the sequence of closed round n-balls in $\mathbb{R}^n$, having radii $1/k$ and centerpoints $(0, 1/k, 0, 0, \ldots, 0)$, $k = 1, 2, 3, \ldots$. The *thickened Hawaiian earring* is the space $H \subset \mathbb{R}^n$ defined as (see Fig. 1):

$$H := (B_2 \setminus \operatorname{Int} B_3) \cup (B_4 \setminus \operatorname{Int} B_5) \cup (B_6 \setminus \operatorname{Int} B_7) \cup \cdots.$$
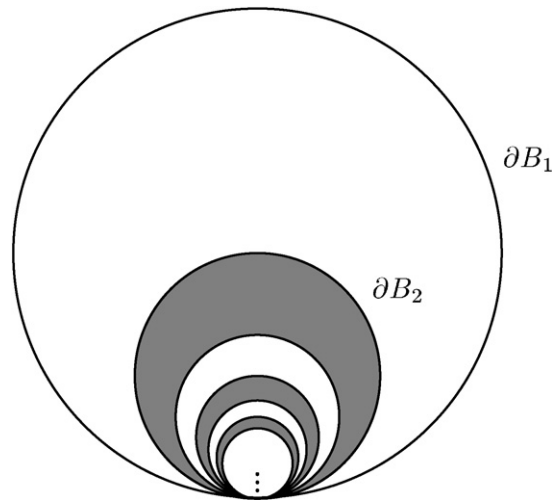
Fig. 1. The space H ("Thickened Hawaiian earring").

As already described before stating Properties 1.1, we will as a next step construct the space H_∞ which is a fractal-like iterated gluing of this space. By adding limit-points we will obtain the desired space $\tilde{H}_\infty$. Spaces H_i and $\tilde{H}_i$ will occur as approximating spaces from inside and from outside. All these spaces are already infinite gluings of copies of H. Since for some purposes we need also finite polyhedra as approximating spaces, it will also be necessary to construct the spaces H'_i and $H'_{i,j}$ in the forthcoming steps:

Step B2: Now construct the iterated Hawaiian earring H_∞ as follows.

The space H is embedded in the n-ball $C := B_1$. On the boundary ∂H choose a countable dense set of points $V := \{v_i \mid i \in \mathbb{N}\} \not\ni (0, 0, \dots, 0)$ with the additional demand that $v_i \notin B_{2i+1}$.

We will use the enumeration of points v_i to inductively define H'_i, $H'_{i,j}$ and U_i.

In the *zero-step* we let $H'_0 := H$, $H'_{0,j} = (B_2 \setminus \text{Int}\, B_3) \cup (B_4 \setminus \text{Int}\, B_5) \cup \cdots \cup (B_{2j-2} \setminus \text{Int}\, B_{2j-1}) \cup B_{2j}$ and $U_0 = \mathbb{R}^n$. The induction will be exclusively over the index i, the index j will in any step mean that $B_{2j} \cup \bigcup_{v=2}^{j}(B_{2v-2} \setminus \text{Int}\, B_{2v-1})$ is used as a base-space.

As *inductive hypothesis* we assume that $\forall_{l \leqslant k-1} U_l$, H'_l and $H'_{l,j}$ have been chosen.

In the *inductive step* we first choose U_k as a regular open neighbourhood of $H'_{k-1,k}$ with the additional conditions that $\overline{U_k} \subset U_{k-1}$ and that U_k is contained in the metric neighbourhood $U(H'_{k-1,k}, (\frac{1}{2})^k)$. For all index tuples $(i_1, \dots, i_k) \in \{1, 2, 3, \dots, k\}^k$ and for all such shorter index tuples $(i_1, \dots, i_m) \in \{1, 2, 3, \dots, k\}^m$ which satisfy that at least one of those indices i_l equals k we then choose similarities $S_\emptyset = \text{id}_C$, $S_{i_1,\dots,i_m} : C \to S_{i_1,\dots,i_{m-1}}(C)$ $(1 \leqslant m \leqslant k)$ which satisfy that

(1) If $S_{i_1,\dots,i_m}$ was chosen in the previous inductive steps then it remains unchanged.
(2) $S_{i_1,\dots,i_m}$ maps $v := (0, 0, \dots, 0)$ to $S_{i_1,\dots,i_{m-1}}(v_{i_m})$.
(3) For all $m \leqslant k$ and for all indices $i_l \leqslant k$ all $S_{i_1,\dots,i_m}(H)$ are disjoint apart from that $(S_{i_1,\dots,i_m}(H) \cap S_{i_1,\dots,i_{m-1}}(H)) = \{S_{i_1,\dots,i_{m-1}}(v_{i_m})\}$.
(4) For each m and all indices $i_l \leqslant k$ all $S_{i_1,\dots,i_m}(C)$ are disjoint and are contained in U_m; if not chosen in the previous inductive steps, then $S_{i_1,\dots,i_m}(C)$ is even contained in U_k.
(5) For the same m and i_l, $\text{diam}(S_{i_1,\dots,i_m}(C)) \leqslant (\frac{1}{2})^m$.

Observe that (5) is not a consequence of (4), since the metric neighbourhood of a non-convex set $H'_{k-1,k}$ might allow the placement of sets with a diameter bigger than the defining radius of the metric neighbourhood. Similarity maps as demanded by Items (1)–(5) can be inductively chosen, based on the idea of taking their images in the neighbourhoods of the attaching points $S_{i_1,\dots,i_{m-1}}(v_{i_m})$ of the previously chosen steps. The disjointness demands can be satisfied
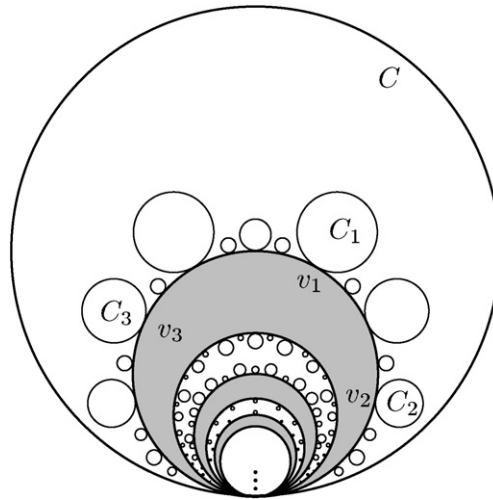
Fig. 2. The construction of $\tilde{H}_\infty$ in the two-dimensional case ($C_v := S_v(C)$).

by making the ratio of dilatation of each map $S_{i_1,\ldots,i_m}$ accordingly small. Based on such choices of similarities we finally let:

$$H'_k := H'_{k-1} \cup S_k(H) \cup \cdots \cup \bigcup_{\substack{i_1,\ldots,i_m \in \{1,2,\ldots,k\} \\ \text{but at least one } i_l \text{ equals } k}} S_{i_1,\ldots,i_m}(H) \cup \cdots$$

$$\cup \bigcup_{\substack{i_1,\ldots,i_{k-1} \in \{1,2,\ldots,k\} \\ \text{but at least one } i_l \text{ equals } k}} S_{i_1,\ldots,i_{k-1}}(H) \cup \bigcup_{i_1,\ldots,i_k \in \{1,2,\ldots,k\}} S_{i_1,\ldots,i_k}(H). \tag{$*$}$$

Analogously $H'_{k,j} := H'_{k-1,j} \cup \cdots$, where in the remainder of ($*$) the basic space H is each time to be replaced by $H'_{0,j}$.

These primed spaces H'_i have been constructed as finite unions. Since the associated spaces $H'_{i,j}$ are simplicial complexes, regular neighbourhoods must exist.

Step B3: However, in the remainder of the paper we will also need the following spaces, which are based on infinite unions using the same set of similarities:

$$H_0 := H, \qquad H_1 := H_0 \cup \bigcup_{i \in \mathbb{N}} S_i(H), \qquad H_2 := H_1 \cup \bigcup_{i,j \in \mathbb{N}} S_{i,j}(H), \quad \ldots \tag{1}$$

$$\tilde{H}_0 := C, \qquad \tilde{H}_1 := H_0 \cup \bigcup_{i \in \mathbb{N}} S_i(C), \qquad \tilde{H}_2 := H_1 \cup \bigcup_{i,j \in \mathbb{N}} S_{i,j}(C), \quad \ldots \tag{2}$$

Put $H_\infty := H_0 \cup H_1 \cup H_2 \cup \cdots$ and $\tilde{H}_\infty := \tilde{H}_0 \cap \tilde{H}_1 \cap \tilde{H}_2 \cap \cdots$. $\tag{3}$

In this way we have

$$H = H_0 \subset H_1 \subset H_2 \subset \cdots \subset H_\infty \subset \tilde{H}_\infty \subset \cdots \subset \tilde{H}_2 \subset \tilde{H}_1 \subset \tilde{H}_0 = C.$$

We will call the points of $\tilde{H}_\infty \setminus H_\infty$ "limit points"; this notion will be justified by Proposition 3.5. Note that H_∞ is by construction just an infinite gluing of copies of thickened Hawaiian earring spaces H, hence

$$H'_0 \cup H'_1 \cup H'_2 \cup \cdots = H_\infty.$$

We will associate to each of these copies a "generation index" and a "degree". The generation index is the first index k of a space H'_k which contains this copy, while the degree of a copy $S_{i_1,\ldots,i_k}(H)$ is the index k. H_0 has degree and generation index zero, and is the only copy with one of these indices being zero. Since being defined as properties of the similarity copies of H, both types of indices are unique apart from the points where two different Hawaiian

earrings are glued together. These points have precisely two of both types of indices. However, while for each number k our space H_∞ contains only finitely many copies of H which have generation index k, apart from index zero it always contains infinitely many copies with the same finite degree. Vice versa, an earring of degree k must be glued to a Hawaiian earring of degree $k-1$, while an earring of generation index k might be glued to a Hawaiian earring of any generation index $\leqslant k$.

$\tilde{H}_\infty$ is our Example B, see Sections 3, 5 and 6 for proofs that it has the above claimed properties.

Example A.

Step A1: Take the point $a_1 = (1, 0)$ and the following sequences of points in the plane $\mathbb{R}^2$:

$$a_0 = (0, 0), \qquad a_2 = (1/2, 0), \qquad a_3 = (1/3, 0), \quad \dots, \quad a_k = (1/k, 0), \quad \dots,$$
$$b_0 = (0, 1), \qquad b_2 = (1/2, 1/2), \qquad b_3 = (1/3, 2/3), \quad \dots, \quad b_k = (1/k, 1 - 1/k), \quad \dots.$$

The *triangular comb space* is the following union of line segments:

$$T' := \overline{a_0 a_1} \cup \overline{a_0 b_0} \cup \bigcup_{k=2}^{\infty} \overline{a_k b_k}$$

To construct the thickened triangular comb space $T \subset \mathbb{R}^n$ proceed as follows. Stick a sufficiently small polygonal $(n-1)$-disk orthogonally to each line segment except of the segment $\overline{a_0 b_0}$ ("accumulation line"), e.g. at its midpoint, and then construct the suspensions of that disks, with the endpoints of the segments as suspension points (Fig. 3). Those disks should be chosen so small that the suspensions along different segments $\overline{a_i b_i}$ ("teeth") are disjoint and that a_i is the only intersection point of the suspensions along $\overline{a_i b_i}$ and along $\overline{a_0 a_1}$ ("stem"). Apart from this, each thickened tooth is contained in the convex hull of each predecessor and of the accumulation line.

The *thickened double comb space* $D \subset \mathbb{R}^n$ is a wedge of two copies of T with common point b_0.

In principle the forthcoming steps are analogous as in the Example B described above, with the comb-space D taking over the role of the Hawaiian earring.

Step A2: Embed D into a circular n-cone C with $v := b_0$ going to the vertex of the cone such that $D \setminus \{b_0\}$ lies in $\text{Int}(C)$. Analogously as for Example B we choose a countable dense set $V = \{v_i \mid i \in \mathbb{N}\} \subset \partial D$, $V \cap \{b_0, a_i \mid i \geqslant 2\} = \emptyset$, and this time we require that v_i does not lie in the convex hull of all teeth with index $> i$. Then choose pairwise disjoint circular n-cones C_i $(i \geqslant 1)$ with $\partial D \cap C_i = \{v_i\}$ and let $S_i : C \to C$ with $\text{Im}(S_i) = C_i$ be similarity maps such that $S_i(v) = v_i$. These S_i and similarities $S_{i_1, \dots, i_k}$ are in an analogous inductive process chosen as for Step B2,
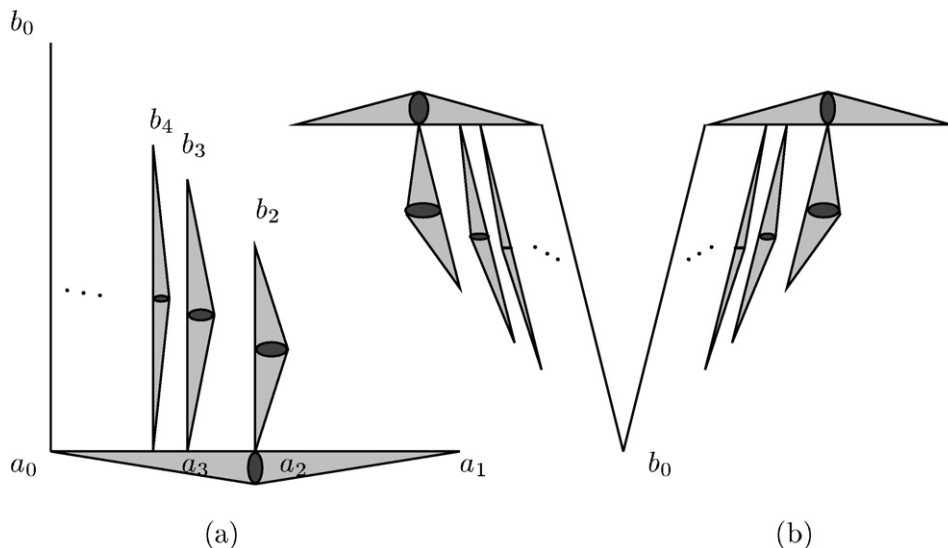

Fig. 3. The spaces (Fig. 3(a)) and $D = T \vee T$ (Fig. 3(b)) in case of three dimensions.
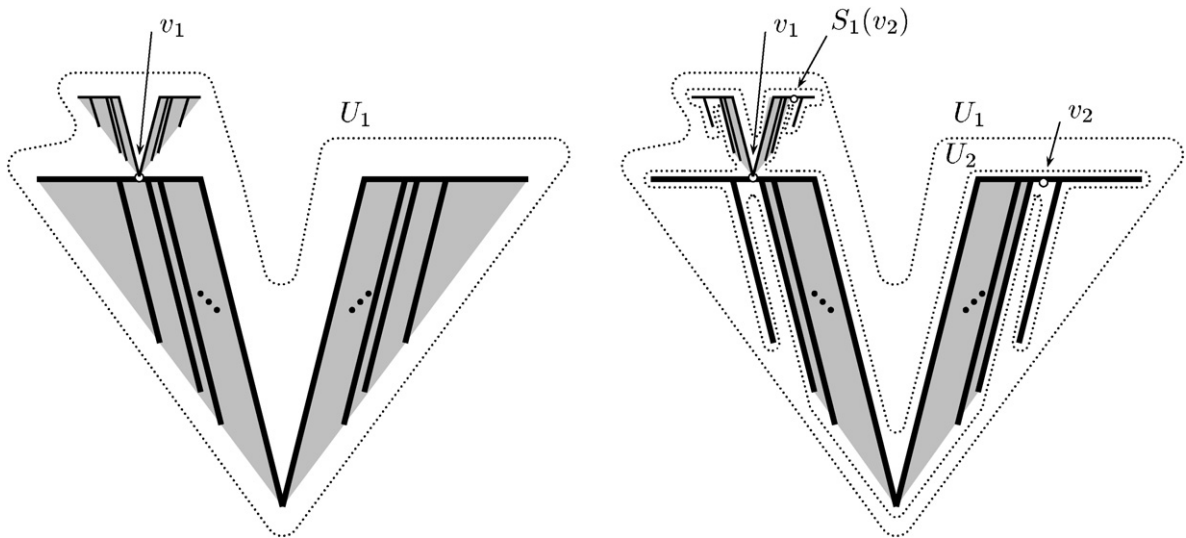
Fig. 4. The two figures above show the first two steps of the construction of $\tilde{D}_\infty$. The top left subfigure is D'_1, the top right subfigure shows D'_1 as a subspace of D'_2: In order to obtain the full D'_2 it would be necessary to attach further combs at the points v_2, $S_1(v_2)$, $S_1(v_1)$, $S_2(v_1)$ and $S_2(v_2)$, but these additional combs could not have been drawn for reasons of scale. For similar reasons we also refrained from drawing the space D in a thickened way. The subfigure below shows a more advanced state in the construction of $\tilde{D}_\infty$. Here all triangles in light grey replace analogous iterated gluings of comb spaces as is the entire triangle C.

where instead of B_1 we are now using the cone C, instead of B_2 we are now using the wedge of two topological balls each of which is defined as convex hull of each of our two thickened comb-spaces T of D, and instead of $B_{2j} \cup \bigcup_{\nu=2}^{j}(B_{2\nu-2} \setminus \operatorname{Int} B_{2\nu-1})$ we are using a space which is obtained by applying such a convex-hull operation only to the $(j+1)$st, $(j+2)$nd, $(j+3)$rd, ... teeth of each of the two subspaces T of D, i.e. to those teeth which originally have been defined as suspensions over the lines $\overline{a_\nu, b_\nu}$ with $\nu > j$ in the definition of T. As in Step B2 we obtain that such spaces are finite simplicial complexes, but in difference to Step B2 this time these complexes all are collapsible. Therefore the regular neighbourhoods U_i that we define in this case will have all the topological type of a tame cell. This property will have to be essentially used in the proof that this example is cellular (cf. Properties 1.1(a)).

With the above described substitutions, v_i and $S_{i_1,\ldots,i_m}$ now with the new meaning, we will define spaces D'_k and $D'_{k,j}$ according to the scheme as described in Step B2. Note that in practice this means that in B2(∗) only the letter "H" has to be substituted by "D".

Step A3: With symbols v and $S_{i_1,\ldots,i_k}$ in their new meanings, and by replacing the letter "H" by "D" in formulae (1), (2) and (3) from Step B3, we obtain the following chain of sets:

$$D = D_0 \subset D_1 \subset D_2 \subset \cdots \subset D_\infty \subset \tilde{D}_\infty \subset \cdots \subset \tilde{D}_2 \subset \tilde{D}_1 \subset \tilde{D}_0 = C$$

Analogously as for Example B we regard the points of $\tilde{D}_\infty \setminus D_\infty$ as "limit points"
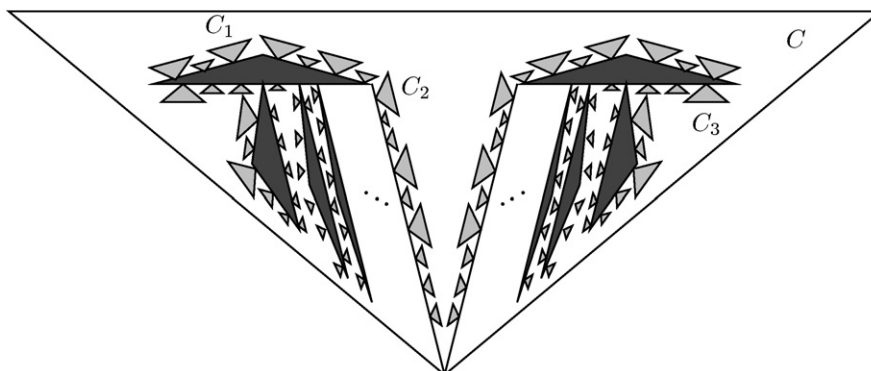
and introduce the generation index and the degree on D_∞ by calling k the degree of the copy $S_{i_1,\ldots,i_k}(D)$, and by calling such a copy "of generation index m", if m is the lowest index of a space D'_m that already contains this copy of D.

The forthcoming sections of this paper are devoted to showing that both of our examples have Properties 1.1–1.3.

Convention 2.1. In the remainder of this paper the notation $\tilde{Y}_\infty$ will address both of our examples simultaneously, i.e. it will mean either $\tilde{H}_\infty$ or $\tilde{D}_\infty$ as defined before in this section. Analogously, notations like Y_∞, Y_i and Y'_i are to be understood, where $Y = D$ or H will be considered as "building blocks".

3. Analogous proofs for both of our examples

We shall need some preliminary definitions.

Definition 3.1. (*Cf.* [9, Definition 2.1].)

(i) A space X is said to be *homotopically n-dimensional* (or to have *homotopy dimension n*), if it is homotopy equivalent to some n-dimensional space and is not homotopy equivalent to any $(n-1)$-dimensional space. Here by dimension of space we mean covering dimension.
(ii) We say that a space X is *everywhere homotopically n-dimensional* if for every non-empty open subset U of X and every id-homotopy $h: X \times I \to X$ which is *stable* on $X \setminus U$, i.e. $h(x,t) = x$ for every $x \in X \setminus U$ and for every t, the intersection $U \cap h(X,1)$ is n-dimensional. By an *id-homotopy* we mean any mapping $h: X \times I \to X$ such that $h(x,0) = x$ for all $x \in X$.

Definition 3.2. A point x_0 of a space X is called a *homotopically fixed point* (*hf-point*, cf. [9, Definition 2.2]), if for any id-homotopy $h: X \times I \to X$ and for every $t \in I$ the point x_0 is a fixed point of the mapping $h(\cdot, t)$, i.e. $h(x_0, t) = x_0$ for all $t \in I$.

Proposition 3.3. *The point* $(0, 0, \ldots, 0)$ *is homotopically fixed in H.*

Proof. An id-homotopy is a map which induces the identity on all homotopy and homology groups. Since therefore it cannot map any element of these algebraic invariants in a degenerate way, it can only map points which contain in any of their neighbourhoods non-trivial representatives of algebraic invariants to points which again satisfy this condition. Since for the thickened Hawaiian earring the accumulation point is the only point with this property, it has to be a homotopically fixed point. □

Remark 3.4. The ratio of dilatation of the unique similarity map that maps Y to some building block of the degree k is at most $(\frac{1}{2})^k$. This is an immediate consequence of our condition (5) of Step A2/B2 and of the fact that according to its definition in Step A1/B1 Y is embedded into a cell C that has a height $\geqslant 1$.

Proposition 3.5. $\overline{Y}_\infty = \tilde{Y}_\infty$.

Proof. By the fixed chosen embedding of Step A2/B2 a maximum is defined how much apart any point of C can lie from the nearest point of Y_0. With the geometric regression from Remark 3.4, it follows that any point from $\tilde{Y}_k$ can lie at most

$$\left(\frac{1}{2}\right)^k \cdot \text{ that distance}$$

apart from the nearest point of Y_k. Since $\frac{1}{2} < 1$, in the limit $\tilde{Y}_\infty$ can be at most the closure of Y_∞. Since, on the other hand, $\tilde{Y}_\infty$ is closed as an infinite intersection of closed sets, it is the closure. □

Proposition 3.6. *Each limit point of $\tilde{Y}_\infty$ is the limit point of exactly one sequence $(y_0, y_1, y_2, \ldots)$, which is constructed by one sequence of naturals $i_1, i_2, \ldots$ according to*

$$y_0 = Y_0, \qquad y_m = S_{i_1, i_2, \ldots, i_m}(Y_0) \quad \text{for } m > 0.$$

Proof. The fact that each limit point can be approximated like this only in one way, follows directly from the construction of $\tilde{Y}_\infty$. Since our point as a limit point lies in $\tilde{Y}_\infty$ but not in Y_∞, it also lies in $\tilde{Y}_k$, but not in Y_k, for all $k \in \mathbb{N}$. However,

$$\tilde{Y}_k \setminus Y_k \subset \bigcup_{(j_1, \ldots, j_k)} S_{j_1, \ldots, j_k}(C \setminus Y_0). \tag{$*$}$$

In particular $\tilde{Y}_k$ and Y_k do not differ in any part that has a degree smaller than k.

Since this argument holds for any k, and since all the attached building blocks of any fixed degree are disjoint, Formula $(*)$ defines a unique sequence $i_1, i_2, i_3, \ldots$ such that $S_{i_1, \ldots, i_k}(C \setminus Y_0)$ is for any k that copy of degree k that contains our limit point. By Remark 3.4 this implies that $S_{i_1, \ldots, i_k}(Y_0)$ converges to our limit point. Conversely, for any other sequence $\nu_1, \nu_2, \nu_3, \ldots$, if $(\nu_1, \ldots, \nu_k) \neq (i_1, \ldots, i_k)$, any continuation of the ν-sequence could only yield a limit point inside $S_{\nu_1, \ldots, \nu_k}(C)$, where $S_{\nu_1, \ldots, \nu_k}(C) \cap S_{i_1, \ldots, i_k}(C) = \emptyset$. $\quad\square$

Proposition 3.7. $\tilde{Y}_\infty$ *is pathwise connected.*

Proof. By the iterative gluing construction, it is evident that Y_∞ is pathwise connected. Hence the remaining question is, whether the limit points of $\tilde{Y}_\infty$ can be somehow connected by paths to the main body Y_∞. Here is a method how to construct such a path: Choose a limit point P and associate according to Proposition 3.6 a sequence $(y_0, y_1, y_2, \ldots)$ to this point. Recall that these sequences can be uniquely constructed so that each y_k denotes a building block of degree k. Choose a path $u : [0, 1] \to \tilde{Y}_\infty$ which between the parameter values 0 and $\frac{1}{2}$ runs from its start point in y_0 to some point in y_1, between parameter values $\frac{1}{2}$ and $\frac{3}{4}$ from this point in y_1 via the unique attaching point to some point in y_2, and so on. Finally define that $u(1) = P$. Observe that the compactness of our two possible building blocks gives that the length ℓ of a geodesic path connecting v to any possible v_i is bounded. Since by Remark 3.4 the geometric regression $(\frac{1}{2})^k \cdot \sup_i(\ell(\overline{v\,v_i})) \cdot (1 + \frac{1}{2}) \cdot \text{diam}(Y_0)$ is an upper bound for the length of the above constructed path between its parameter values $\frac{1}{k}$ and $\frac{1}{k+1}$, the above construction gives a continuous path which hence connects P to the main body Y_∞. $\quad\square$

Remark 3.8. (*On Properties* 1.3.) The subspaces γ in the sense of Properties 1.3 are the thickened teeth and the thickened stem of the double-comb D_0 and all their similarity copies as subspaces of $S_{i_1, \ldots, i_n}(D_0)$ in case of Example A; in case of Example B these are the sets $S_{i_1, \ldots, i_m}(B_{2j} \setminus \text{Int } B_{2j+1})$, i.e. the thickened rings of the Hawaiian earring H_0 and the similarity copies of these rings. Therefore 1.3(v). Properties 1.3(iv), 1.3(i) and 1.3(ii) follow for points of Y_∞ from the mere gluing structure. Regarding 1.3(ii) this is so, because this gluing structure was tree-like: In order to connect a point P from and element $\gamma \subset Y_\infty$ with a point Q in another element γ', a connecting path can only leave γ through one of its boundary points, and the retraction onto γ must then project Q to precisely this boundary point. Due to Propositions 3.5 and 3.6 this statement extends to all points of $\tilde{Y}_\infty$. Also, the extension of 1.3(i) and 1.3(iv) to limit points follows from 3.6: Since in Proposition 3.6 each limit point is proven to be the limit of (homeomorphic copies of) building blocks, it is the limit of boundary points of building blocks, and therefore it cannot have become an interior point of $\tilde{Y}_\infty$, but conversely it also contains elements of Γ in any of its neighbourhoods. In difference to the above arguments, the proof of 1.3(iii) heavily relies on individual properties of the spaces D and H:

Note that all arguments that were given in Lemma 2.3 of [9] to prove that the wedge point of the ordinary non-thickened double comb is a homotopically fixed point, are valid for this thickened version as well, and in case of Example B we can use Proposition 3.3 above. Then analogously with the arguments from the list of Remarks 2.4 of [9] it follows also in our case that all boundary points for which we attach such double points (and thus their limit points also) remain homotopically fixed points also in the space that is composed by these attaching processes. This includes in particular all boundary points of the sets from our collection Γ, hence 1.3(iii).

Remark 3.9. (*On the implications from Diagram* 1.4(∗).)

On the implication "(iii,iv) $\Longrightarrow$ (f)": This is immediate, since accumulation points of homotopically fixed points are homotopically fixed.

On the implication "(i,iv,v) $\Longrightarrow$ (c)": This is also almost immediate, since by 1.3(i) and 1.3(v) every neighbourhood from every interior point of $\tilde{Y}_\infty$ contains an open part of a Euclidean n-dimensional ball, and the same conclusion holds for boundary points by 1.3(iv) and 1.3(v).

On the implication "(ii,iii,v) $\wedge$ (f) $\Longrightarrow$ (h)": If any point of $\tilde{Y}_\infty$ is missing at the end of an id-homotopy h, then it cannot be a boundary point of $\tilde{Y}_\infty$, since those are homotopically fixed by Property (f). Hence a point x interior to some γ must be missing (cf. 1.3). Let $r : \tilde{Y}_\infty \to \gamma$ be a retraction with $r(\tilde{Y}_\infty \setminus \gamma) \subset \partial\gamma$. Then the composition $\gamma \times I \hookrightarrow \tilde{Y}_\infty \times I \xrightarrow{h} \tilde{Y}_\infty \xrightarrow{r} \gamma$ is a homotopy of γ which fixes the boundary of γ such that $x \notin h(\gamma, 1)$. This allows for the construction of a retraction of γ onto its boundary. This contradicts the fact that, if $\tilde{Y}_\infty = \tilde{D}_\infty$, the homology group $H_{n-1}(\gamma) = 0$ does not map onto $H_{n-1}(\partial\gamma) = \mathbb{Z}$. The argument for $\tilde{Y}_\infty = \tilde{H}_\infty$ is identical, because in this case $H_{n-1}(\gamma) = \mathbb{Z}$ and $H_{n-1}(\partial\gamma) = \mathbb{Z} \oplus \mathbb{Z}$.

On the implications "(h) $\Longrightarrow$ (g)" and "(h) $\wedge$ (c) $\Longrightarrow$ (e)": Both (g) and (e) follow as immediate consequences of (h): A deformation retraction is a special kind of id-homotopy, and so is the map that qualifies a space for being not everywhere homotopically n-dimensional: It is by Definition 3.1(ii) an id-homotopy with the additional demand of keeping the exterior of some neighbourhood fixed. Now, by (c) the neighbourhood of each point in $\tilde{Y}_\infty$ is n-dimensional.

On the implication "(ii,iii,v) $\Longrightarrow$ (d)": Here the same arguments as given in Remarks 2.4(i) and (ii) of [9] work, of course, using n-dimensional cohomology instead of two-dimensional. Observe, that all connected components of inner points of our spaces are just cells, namely the interiors of the sets from our collection Γ (cf. 1.3).

On the implication "(e) $\Longrightarrow$ (c)": This arrow is due to the fact that according to Definition 3.1(ii) for any open subset U and the appropriate homotopies h the intersection $U \cap h(X, 1)$ is n-dimensional—with the case $h = \mathrm{id}$ in any case being included.

As Diagram 1.4(∗) shows, the above proven implications together with the arguments from Remark 3.8 prove Properties (c)–(h) for both of our examples. The last ones of our properties from our Lists 1.1–1.3, namely Properties (a) and (b), will be proven separately in the sections below, namely 1.1(a) in Section 4, 1.2(a) in Section 5, and 1.1(b) & 1.2(b) simultaneously in Section 6.

4. Proof that Example A is cellular

Remark 4.1. The proof of cellularity for Example A requires the construction of approximating n-dimensional cells in the spirit of Fig. 4(b) of [9]. However, somewhat the construction here is more complicated than in [9, Proof 3.2], since here we do not only have two degrees of spaces which need to be respected by the positions of our approximating disks. However, the construction of our approximating cells has been prepared by choosing ball-neighbourhoods U_i in Section 2, Step A2.

Proposition 4.2. *Example A is cellular.*

Proof. By construction, D_∞ consists of copies $S_{i_1,\dots,i_k}(D)$, and in B2(4)/A2 each such copy has been placed in some ball-neighbourhood U_m, where $m \geqslant k$ is the generation index of the copy $S_{i_1,\dots,i_k}(D)$. Therefore it is contained in all neighbourhoods U_i with $i < m$, and by construction as part of the space $D'_m \subset D'_i$ it lies also in all U_i with $i > m$. Hence $D_\infty \subset \bigcap_{i=1}^\infty U_i$. Since we have $\overline{U_m} \subset U_{m-1}$, we also have that $\bigcap_{i=1}^\infty U_i = \bigcap_{i=1}^\infty \overline{U_i}$ and hence is closed, so that we have $\tilde{D}_\infty \overset{3.5}{=} \overline{D_\infty} \subset \bigcap_{i=1}^\infty U_i$. Proving the opposite inclusion will complete the proof:

By closedness of $\tilde{D}_\infty$, any point not belonging to $\tilde{D}_\infty$ would lie at positive distance ε apart. Hence it cannot be contained in a metric neighbourhood of radius $(\frac{1}{2})^m$ of $\tilde{D}_\infty$ with sufficiently high index m, and therefore also not in the neighbourhood U_m which is by construction contained in the metric neighbourhood of the same diameter of the smaller space D'_m. □

5. Proof that Example B is a Peano-continuum

Our space $\tilde{H}_\infty$ is by construction bounded, and by 3.5 closed, hence compact. By 3.7 we have also already pathwise connectivity. Hence the only missing property that needs to be shown below is the local pathwise connectivity of $\tilde{H}_\infty$.

Proposition 5.1. *$\tilde{H}_\infty$ is locally pathwise connected.*

Proof. Thanks to all the self-symmetries of the fractal-like iteration in building H_∞, the essential step has been done with having proven Proposition 3.7 already. We have to construct an arbitrary small pathwise connected neighbourhood for each point in $\tilde{H}_\infty$. Therefore we distinguish cases according to the various types of points of $\tilde{H}_\infty$.

(1) *Interior points of one of the domains $B_{2i} \setminus \operatorname{Int} B_{2i+1}$* (in H_0 or in its similarity images): As interior points of an open domain these points clearly have appropriate neighbourhoods.
(2) *Attaching points*: The set of all attaching points is $\tilde{V} = \{S_{i_1,\dots,i_k}(v_j) \mid k \in \mathbb{N}_0, \, j, i_1, \dots, i_k \in \mathbb{N}\}$. Observe that each $w \in \tilde{V}$ has precisely two representations of that type according to $w = S_{i_1,\dots,i_k}(v_j) = S_{i_1,\dots,i_k,j}(v)$. By our construction in Step B2 of Section 2, for such a w the domain $S_{i_1,\dots,i_k,j}(C)$ gives a ball in $\mathbb{R}^n$ such that $\partial(S_{i_1,\dots,i_k,j}(C)) \cap \tilde{H}_\infty = \{w\}$. The part of $\tilde{H}_\infty$ that sits inside this ball we call the "associated subspace of w". This space is analogously built as $\tilde{H}_\infty$.
A path-connected neighbourhood of w with a diameter smaller than ε can be constructed according to

$$\big((w, \delta) \cap \big(S_{i_1,\dots,i_k}(H) \cup S_{i_1,\dots,i_k,j}\big((B_2 \setminus \operatorname{Int} B_3) \cup \cdots \cup (B_{2l-2} \setminus \operatorname{Int} B_{2l-1})\big)\big)\big)$$
$$\cup \, S_{i_1,\dots,i_k,j}(H \cap B_{2l}) \cup \text{all subspaces that are associated} \qquad (*)$$
$$\text{to points of the boundary of the above described space.}$$

By Proposition 3.7 the path-connectedness of such a neighbourhood is clear. It will be contained in $U(w, \varepsilon)$, provided that we choose the two variables l and δ as follows:
  (i) First choose l sufficiently big, such that $S_{i_1,\dots,i_k,j}(B_{2l})$ fits into $U(w, \varepsilon)$, including all its subspaces that are associated to points of this region. Since according to our construction in Step B2 of Section 2 every space that is associated to a ring $B_{2\nu} \setminus \operatorname{Int} B_{2\nu+1}$ with sufficiently high ν has to fit into a neighbourhood of diameter $(\frac{1}{2})^\nu$, this condition will be satisfied for sufficiently high l. In addition, by Remark 3.4, the ratio of dilatation of all $S_{i_1,\dots,i_k,j}$ is < 1.
  (ii) Let μ be the index so that v_j is an element of $\partial(B_{2\mu} \setminus \operatorname{Int} B_{2\mu+1})$.
  (iii) Now choose $\delta' < \varepsilon/2$ and smaller than the width of $S_{i_1,\dots,i_k}(B_{2\mu} \setminus \operatorname{Int} B_{2\mu+1})$ at w.
  (iv) Analyze how many subspaces that are associated to points of $U(w, \delta') \cap (S_{i_1,\dots,i_k}(H) \cup S_{i_1,\dots,i_k,j}((B_2 \setminus \operatorname{Int} B_3) \cup \cdots \cup (B_{2l-2} \setminus \operatorname{Int} B_{2l-1})))$ are still sticking out of $U(w, \varepsilon)$. Since each of them must have a diameter of at least $\varepsilon/2$, there can be only finitely many of them. Now choose $\delta \leqslant \delta'$ such that each of the attaching points of these sticking out subspaces lies more than δ apart from w.
Observe that according to our construction in $(*)$, it might happen that not the entire metric neighbourhood of radius δ of w belongs to the set constructed there. $U(w, \delta)$ might contain points from subspaces that are associated to points outside $U(w, \delta)$, but that are sticking into our metric neighbourhood. But if $U(w, \frac{\delta}{2})$ would still contain such points, then the corresponding subspaces would have to have a radius bigger than $\frac{\delta}{2}$, and hence these points could only come from finitely many of such subspaces. Hence, with repeating this argument from (iv) we see that at least some metric neighbourhood of w entirely belongs to the set that we constructed in $(*)$. A similar argument can then be used for all boundary-points that belong to $S_{i_1,\dots,i_k,j}((B_{2\nu} \setminus \operatorname{Int} B_{2\nu+1})$ ($\nu \geqslant l$), to $S_{i_1,\dots,i_k,j}(B_{2\nu} \setminus \operatorname{Int} B_{2\nu+1}) \cap U(w, \delta)$ ($\nu < l$) and to $S_{i_1,\dots,i_k}(H) \cap U(w, \delta)$. Since for the interior points of these sets and of the associated spaces it is clear that they possess full metric neighbourhoods in the set that was constructed in Formula $(*)$, we see that we have indeed constructed an open set.
(3) *Boundary points of a ring h_k which are not attaching points*: The case is analogous, but just simpler as the case before: The construction in Formula $(*)$ just corresponds to

$$\big(U(w, \delta) \cap S_{i_1,\dots,i_k}(H)\big) \cup \text{all subspaces that are associated}$$
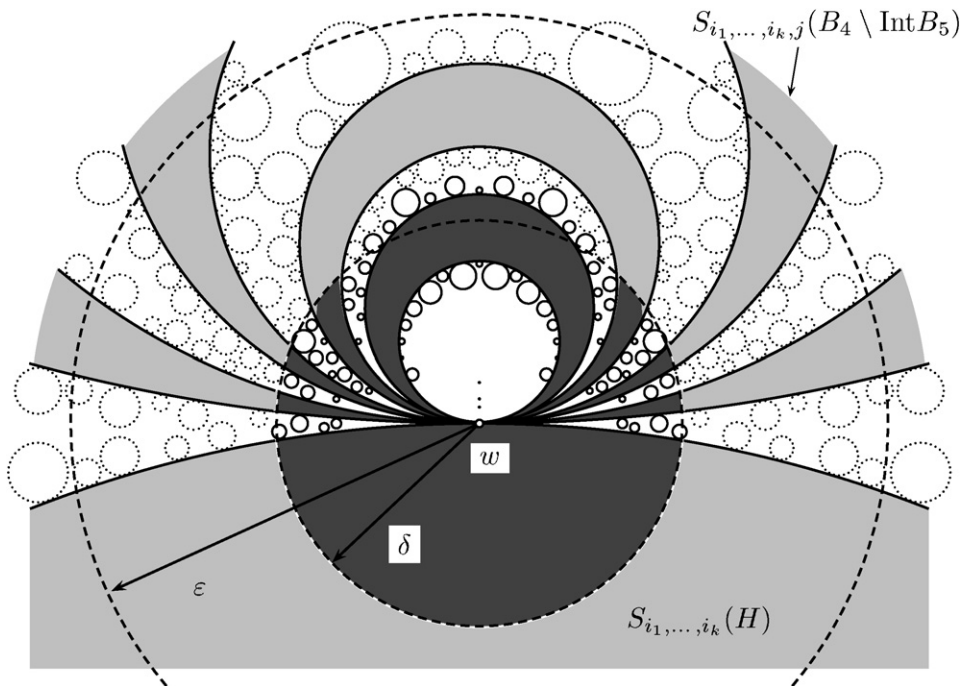$$\text{to points of the boundary of this space}$$

Fig. 5. This figure tries to give a geometric impression for how to construct a path-connected neighbourhood W inside an ε-neighbourhood: The areas in grey are solidly filled parts of $\tilde{H}_\infty$, but only those in dark grey belong to W. Smaller copies of $\tilde{H}_\infty$ are in this figure indicated by circles, and these circles are drawn solidly/dotted, according to whether these copies do belong entirely or do not belong at all to W, respectively.

and corresponding to the missing terms we have accordingly fewer steps in constructing this set.

(4) *Limit points*: Let P be a limit point and $(h_0, h_1, h_2, \ldots)$ be the associated sequence in the sense of Proposition 3.6. The basic idea is to construct a neighbourhood of P of the form $S_{i_1,\ldots,i_k}(C)$, where $h_k = S_{i_1,\ldots,i_k}(H_0)$. By choosing the index k big, such neighbourhoods which are path-connected by Proposition 3.7 can be made arbitrarily small. The only trouble comes from the fact that such a neighbourhood fails to be open at one point, namely at the attaching point of h_k to h_{k-1}. However, we can make it open by attaching to our h_k-neighbourhood an appropriate semidisk-neighbourhood in h_{k-1} of this attaching point w. Such a neighbourhood can be analogously as in the preceding steps (2) and (3) constructed by taking $U(w, \delta) \cap S_{i_1,\ldots,i_{k-1}}(H)$ for a suitable δ and by uniting to it all subspaces that are associated to points on the boundary of it.

Since we succeeded with the construction of such neighbourhoods for all types of points in $\tilde{H}_0$, the proof of Proposition 5.1 and hence of Property 1.2(a) is complete. □

Remark 5.2. Observe that for our Example A open neighbourhoods can analogously as in 5.1(1)–(4) be constructed, of course, not claiming any local pathwise connectivity in this case. However, we do claim, that also limit points of $\tilde{D}_\infty$ do have neighbourhoods that, as those from 5.1(4), consist of a homeomorphic copy of the entire space, in union with some metric neighbourhood of the attaching point, in union with all subspaces that were associated to this metric neighbourhood. Such a neighbourhood, in particular, will not contain any limit point on its boundary.

6. Proof that Examples A and B are simply connected

We are left with the task to show the last missing property of our Examples A & B, namely that the fundamental group is trivial (with $n = 2$ for Example B excepted). Before we can start this proof, we need one observation:

Proposition 6.1. *The set of limit points for our Examples A and B is pathwise totally disconnected.*
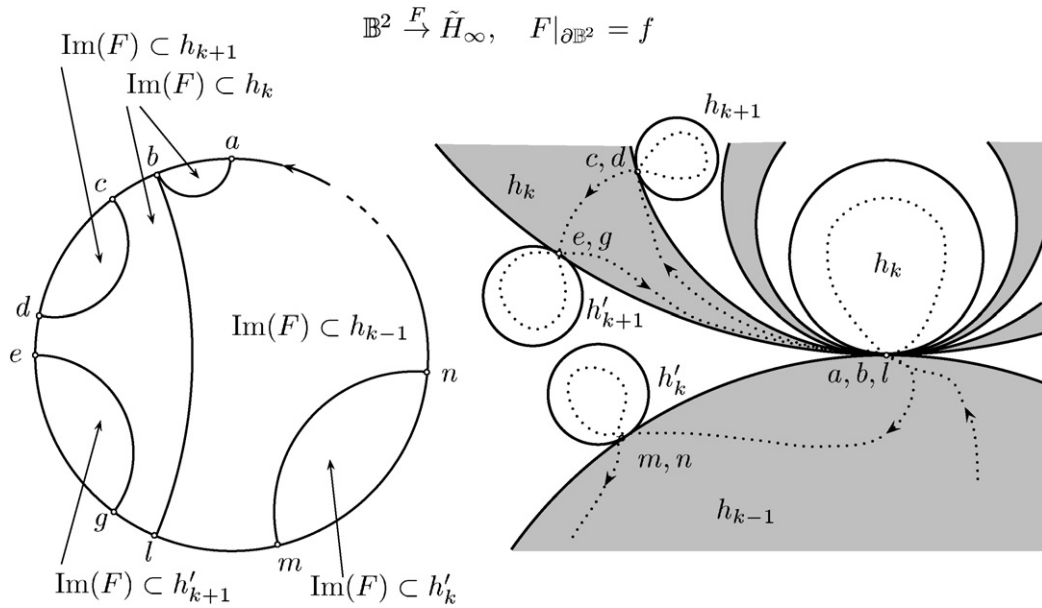
$$\mathbb{B}^2 \xrightarrow{F} \tilde{H}_\infty, \quad F|_{\partial \mathbb{B}^2} = f$$

Fig. 6. The figure shows the construction of F in the case of Example B. The subfigure on the right shows a part of $\tilde{H}_\infty$, drawn essentially as in Fig. 5. In dotting we also put a segment of a sample path u into this figure and indicated the parameter values of when this path passes through the attaching points. These values then are used in the left subfigure where the corresponding steps of constructing a Kenyon-diagram are indicated.

Proof. The proof of both cases is analogous, and is essentially only based on the analogous gluing structures of these spaces. We present it using Convention 2.1 to talk about both examples simultaneously. As in Proposition 3.6 for each limit point construct the unique sequence $(Y_0 = y_0, y_1, y_2, \ldots)$, such that each y_k is a building block of degree k and is glued to its predecessor. Since these sequences are unique, the corresponding sequences for two different limit points have to differ at least from some finite index on. If this happens, then the place from which on these sequences differ determines different similarity maps which map C to disjoint regions in $\mathbb{R}^n$. And these disjoint regions contain our limit points and can be used to construct disjoint open neighbourhoods for our points as well, which do not have any limit points on their boundaries (cf. 5.1(4)/5.2). Hence our points are not contained in the same connected component of the subset of limit points of $\tilde{Y}_\infty$. $\quad\square$

Remark 6.2. Observe that the building block for Example B in dimension $n = 2$ is the classical Hawaiian earring based on rings $\approx S^1$. Therefore we cannot expect Example B in this dimension to be simply connected. However, this obstruction vanishes for all higher n.

Proposition 6.3. *Example A for $n \geqslant 2$ and Example B for $n > 2$ are simply connected.*

Proof. Let $u : [0, 1] \to \tilde{Y}_\infty$ be an arbitrary closed path in our space, with $\tilde{Y}_\infty$ being defined according to Convention 2.1. Since it is just a homotopic process to shrink all constant domains of such a path to points, we can and will in the forthcoming proof assume that u is free of constant segments.

In order to show that $\pi_1(\tilde{Y}_\infty) = 1$, we need to construct a nullhomotopy for such a path u. According to a proposal of Kenyon [10], for that purpose we look at the closed unit-disk $\mathbb{B}^2$, and we will use in the forthcoming constructions the metric that can be put on such a disk to give the Poincaré Disk model for the hyperbolic (or "Lobachevskian") space $\mathbb{H}^2$ (cf. [12, §9.2]). Accordingly we will talk of "hyperbolic lines" (which are Euclidean semicircles that are perpendicular to the boundary) and similar things. We use $\partial \mathbb{B}^2$ as a parameter domain for our path u, and hence by using the parametrization for u we get a map $f : \partial \mathbb{B}^2 \to \tilde{Y}_\infty$. If we manage to extend this map f to a continuous map $F : \mathbb{B}^2 \to \tilde{Y}_\infty$, then we succeeded in proving that u was nullhomotopic.

F will be defined by an infinite iterative process where each iteration step enlarges the domain of definition of F; but in general only in the limit after infinitely many iterates we will have completely defined F. Although based on

the same ideas, we give in the following separate descriptions for the initial step of this process and for the step that will be infinitely many times iterated to enlarge the domain of definition of F.

Initial step: Let us, without loss of generality, assume that u passes through Y_0, i.e. through the only building block of degree zero (if it does not, then the building block of lowest degree through which u passes takes this role in the forthcoming proof). We treat it as evident that also the thickened double-comb space D or the thickened Hawaiian earring H for $n \geqslant 3$ has trivial fundamental group (a rigorous proof could be given by precisely the same method as this proof when treating the various n-dimensional cells and line-segments or rings of which D or H, respectively, are comprised as we treat the building blocks below). Our path might spend various different segments inside Y_0. According to the tree-like gluing structure of Y_∞, if u leaves Y_0 and goes into the building blocks of higher degrees, it has to come back to Y_0 through the same boundary point. Accordingly we can on $\partial\mathbb{B}^2$ mark those (closed) segments where our path is inside Y_0, and those (open) segments where it is outside Y_0, and we will find that f takes the same boundary point of Y_0 as value on the two endpoints of an interval where u is outside Y_0, for each such interval. We connect the start with the endpoint of each such interval by a hyperbolic line segment through $\mathbb{B}^2$, and, as a first step of extending f to F, we hereby agree that the restriction of F to such a line segment is the constant value that f has on both endpoints of such a hyperbolic line.

This construction gives a system of non-nested hyperbolic lines which seal off various "semi-disks" near the boundary of $\partial\mathbb{B}^2$ from a "central region". F has already been completely defined on the boundary of the central region, and the restriction of F to the boundary of the central region can be interpreted as a path which entirely remains in Y_0. Since $\pi_1(Y_0) = 1$, this path is nullhomotopic in Y_0, and we can use this nullhomotopy to assign to F now also a continuous definition inside the entire central region. Hence we reduced the problem of defining F now to the remaining semidisks near $\partial\mathbb{B}^2$. Observe that the boundary of each of those semidisks consists of two segments, one "outer boundary", which is lying on $\partial\mathbb{B}^2$, and an "inner boundary", which is a hyperbolic line-segment.

Enlargement step: For some $S = S_{i_1,\ldots,i_k}$ let $S(v)$ be the value that f takes on the endpoints of the outer boundary p of some semidisk, in which the definition of F is still missing. By assumption, $f(p)$ is contained in $S(C)$. The subset of $\partial\mathbb{B}^2$, where $f(p)$ is not contained in $S(Y)$, is open, and can in the worst case consist of countably many disjoint open subintervals. Here f will map to building blocks of degree $k + 1$ or higher. At the beginning and at the endpoint of each of those subintervals f will take on the same value. As in the Initial Step of this construction, connect each pair of endpoints of those subintervals by a hyperbolic line, and use the f-value from the endpoints of these lines as constant value for F on each line. These hyperbolic lines, together with the inner boundary of our semidisk and the appropriate segments of $\partial\mathbb{B}^2$ in between form a closed Jordan curve c, on which F is already defined and only takes on values in $S(Y)$. Since $\pi_1(Y) = 1$, a nullhomotopy for $F(c)$ exists inside $S(Y)$ and can be used to define F in the interior region of c. In addition, the geometric shape of Y_0 allows to choose this nullhomotopy in such a way, that its diameter does not exceed the diameter of the loop to be contracted. Hence we have extended the definition of F to some parts of our outer semidisk, and by iterating this construction we will continue to extend it to parts of the still missing smaller semidisks.

In the remainder of this section we will complete this proof by showing

  (i) that this definition fills the entire interior of $\mathbb{B}^2$,
 (ii) that the gluing structure of the various subdisks on which F is independently defined is locally finite in the interior and hence the resulting map is continuous, and
(iii) that this definition on $\mathrm{Int}(\mathbb{B}^2)$ continuously extends to the map that by the original path f is defined on $\partial\mathbb{B}^2$.

Proof of (i): Assume that our construction mechanism fails to associate to some point $P \in \mathrm{Int}(\mathbb{B}^2)$ a definition. This implies that P does not belong to the central region that we considered in the Initial step but has to lie inside one of the outer semidisks. And it has to lie inside one of the outer subsemidisks, when in a forthcoming step of this proof we extended our definition of F into this semidisk. The same is true for each of the infinitely many forthcoming steps of the construction. Since P had a fixed distance to $\partial\mathbb{B}^2$, the hyperbolic geometry does also not allow the endpoints of the inner boundaries of these semidisks to converge to each other in $\partial\mathbb{B}^2$. Denote these endpoints by $(\alpha_i)_{i\in\mathbb{N}}$ and $(\beta_i)_{i\in\mathbb{N}}$ and denote by $[\alpha_i, \beta_i]$ the appropriate segment on $\partial\mathbb{B}^2$, i.e. the appropriate segment that is the outer boundary of the ith semidisk. According to our construction we know that f left at the parameter value α_i a building block of degree $(i - 1)$, and did not return to this building block before the pa-

rameter value β_i. The sequences $(\alpha_i)_{i \in \mathbb{N}}$ and $(\beta_i)_{i \in \mathbb{N}}$ are monotone sequences in opposite directions on $\partial \mathbb{B}^2 \approx S^1$ and therefore converging. Hence $[\lim_i \alpha_i, \lim_i \beta_i]$ is an interval on which f can only take on limit points, and hence by Proposition 6.1 it can only be constant. However this has been ruled out at the beginning of our proof-construction.

Proof of (ii): If the segmentation by semidisks, according to the above construction should not be locally finite in the interior, the geometry of non-intersecting hyperbolic straight lines would force that we have to obtain essentially the same picture as in the preceding proof to have some accumulation in the interior of $\mathbb{B}^2$. Hence we get the same type of contradiction to this assumption.

Proof of (iii): By our construction in the Enlargement step, only something needs to be proven, if for some point P in $\partial \mathbb{B}^2$ we have infinitely many semidisks in its neighbourhood. In the forthcoming proof we need to distinguish cases depending on whether $f(P)$ is a limit point or an element of Y_∞.

(1) Let $f(P)$ be a limit point lying in some ε-neighbourhood $U(f(P), \varepsilon)$. The neighbourhoods that we constructed in 5.1(4)/5.2 for both of our examples are, after removing the semidisk-part, small homeomorphic copies of our space $\tilde{Y}_\infty$. Hence such a copy can be described as $S(\tilde{Y}_\infty)$, where the homeomorphism S has to be defined for each building block separately as a composition of the corresponding similarity mappings $S_{i_1,\ldots,i_k}$ and of their inverses for the appropriate choices of indices. By our gluing structure, the part $S(\tilde{Y}_\infty)$ is connected to the rest of our space $\tilde{Y}_\infty$ only through the point $S(v) = S_{i_1,\ldots,i_k}(v)$. By Remark 3.4 such subspaces have for sufficient high index k a diameter that is bounded from above by $\sum_{\nu=k}^{\infty} (\frac{1}{2})^\nu = (\frac{1}{2})^{k-1}$. Hence they will be contained in $U(f(P), \varepsilon)$. Since our continuous path f can have entered and left $S(\tilde{Y}_\infty)$ only through the point $S(v)$, there are corresponding parameter values on $\partial \mathbb{B}^2$ which define a δ-neighbourhood in which we know that f takes on only values inside $S(\tilde{Y}_\infty)$. By construction in the Enlargement step, these points on $\partial \mathbb{B}^2$ have been connected by a semidisk, and whatever has been defined as map F there, did also only take on values inside $S(\tilde{Y}_\infty)$. Hence this semidisk is precisely that type of neighbourhood that needed to be constructed in an elementary proof of continuity for F at the point P.

(2) Now let $P \in \partial \mathbb{B}^2$ be a point where $f(P)$ is some point in Y_∞. Hence $f(P)$ lies in a copy of Y with finite degree, and therefore only k semidisks of $\mathbb{B}^2$ can be nested at the point P. Hence, if P is an accumulation point of semidisks, this can only be due to side-by-side situated disks. Of course, we cannot rule out that inside some of these semidisks there are sitting finitely or infinitely many other semidisks, but there always do also exist semidisks of degree k or $k+1$ which are accumulating from aside to our point P. For our path this means that $f(P)$ will be on the boundary of a building block y_{k-1} which might or might not be the attaching point to some y_k. An accumulation of semidisks near P means that f in the neighbourhood of the point P either infinitely many times changes between y_k and y_{k-1}, or in this neighbourhood enters a lot of smaller building blocks that are in this region attached either to y_k or to y_{k-1}. Either way, we can always place an arbitrarily small semidisk around P so that such that all but finitely many filling regions for F from the Enlargement step are either completely inside or completely outside this semidisk. The continuity of F at P now follows from the continuity of f and from the fact that the filling nullhomotopies were chosen so as not to increase the diameter of the curves of f-values which they contract.

Since by the above arguments we have seen that our construction gave a continuous map that can be interpreted as a homotopy of our arbitrary path f, it has been proven to be nullhomotopic, and $\tilde{Y}_\infty$ to have trivial fundamental group. $\quad \square$

Acknowledgements

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