Topology and its Applications 157 (2010) 466–481

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

o-Boundedness of free topological groups ☆

Taras Banakh [a,b], Dušan Repovš [c], Lyubomyr Zdomskyy [d,*]

[a] Department of Mathematics, Ivan Franko National University of Lviv, Ukraine
[b] Instytut Matematyki, Uniwersytet Humanistyczno-Przyrodniczy Jana Kochanowskiego w Kielcach, Poland
[c] Faculty of Mathematics and Physics, and Faculty of Education, University of Ljubljana, Jadranska 19, Ljubljana 1000, Slovenia
[d] Kurt Gödel Research Center for Mathematical Logic, University of Vienna, Währinger Straße 25, A-1090 Wien, Austria

ARTICLE INFO

Article history:
Received 31 March 2009
Received in revised form 21 September 2009
Accepted 10 October 2009

MSC:
primary 54H11, 54D20
secondary 20N02, 57S05

Keywords:
Free (abelian) topological group
Q-point
(o,ω)-bounded group
Scheepers property
Semifilter
F-Menger property

ABSTRACT

Assuming the absence of Q-points (which is consistent with ZFC) we prove that the free topological group $F(X)$ over a Tychonov space X is o-bounded if and only if every continuous metrizable image T of X satisfies the selection principle $\bigcup_{fin}(\mathcal{O},\Omega)$ (the latter means that for every sequence $\langle u_n\rangle_{n\in\omega}$ of open covers of T there exists a sequence $\langle v_n\rangle_{n\in\omega}$ such that $v_n\in[u_n]^{<\omega}$ and for every $F\in[X]^{<\omega}$ there exists $n\in\omega$ with $F\subset\bigcup v_n$). This characterization gives a consistent answer to a problem posed by C. Hernándes, D. Robbie, and M. Tkachenko in 2000.

1. Introduction

In this paper we present a (consistent) characterization of topological spaces X whose free topological group $F(X)$ is o-bounded, thus resolving the corresponding problem posed by C. Hernándes, D. Robbie, and M. Tkachenko in [7].

We recall that a topological group G is said to be *o-bounded* if for every sequence $\langle U_n\rangle_{n\in\omega}$ of neighborhoods of the neutral element e of G there is a sequence $\langle F_n\rangle_{n\in\omega}$ of finite subsets of G such that $G=\bigcup_{n\in\omega}U_n\cdot F_n$. This notion was introduced by O. Okunev and M. Tkachenko as a covering counterpart of a σ-bounded group, see e.g. [15] for the discussion of this subject. It is clear that each σ-bounded group (that is, a subgroup of a σ-compact topological group) is o-bounded.

Our aim is to detect topological spaces X with o-bounded free (abelian) topological group. By the free abelian topological group over a Tychonov space X we understand an abelian topological group $A(X)$ that contains X as a subspace and such that each continuous map $\phi: X\to G$ into an abelian topological group G extends to a unique continuous group homomorphism $\Phi: A(X)\to G$. Deleting the adjective "abelian", we obtain the definition of the free topological group $F(X)$ over X.

☆ The authors were supported in part by the Slovenian Research Agency grants P1-0292-0101-04, J1-9643-0101, and BI-UA/07-08-001. The third author would also like to acknowledge the support of the FWF grant P19898-N18.
* Corresponding author.
E-mail addresses: tbanakh@yahoo.com (T. Banakh), dusan.repovs@guest.arnes.si (D. Repovš), lzdomsky@gmail.com (L. Zdomskyy).

0166-8641/$ – see front matter © 2009 Published by Elsevier B.V.
doi:10.1016/j.topol.2009.10.006

Tychonov spaces X with o-bounded free abelian topological group $A(X)$ were characterized by the third author [18] as spaces all of whose continuous metrizable images are Scheepers spaces. We recall that a topological space X satisfies the selection principle $\bigcup_{fin}(\mathcal{O},\mathcal{O})$ (resp. $\bigcup_{fin}(\mathcal{O},\Omega)$) or else is said to be *Menger* (resp. *Scheepers*) if for any sequence $\langle\mathcal{U}_n\rangle_{n\in\omega}$ of open covers of X each cover $\mathcal{U}_n$ contains a finite subfamily $\mathcal{V}_n\subset\mathcal{U}_n$ such that $\langle\bigcup\mathcal{V}_n\rangle_{n\in\omega}$ is a cover (an ω-cover) of X. A cover $\mathcal{U}$ of a set X is called an *ω-cover* if each finite subset $F\subset X$ lies in some element $U\in\mathcal{U}$ of the cover $\mathcal{U}$.

It is clear that each Scheepers space is a Menger space. The converse follows from the inequality $\mathfrak{u}<\mathfrak{g}$ (which holds in some models of ZFC), see [17]. On the other hand, under CH there is a Menger metrizable space that fails to be Scheepers, see [8, Theorem 2.8].

The following theorem proved in [18] characterizes Tychonov spaces X with o-bounded free abelian topological group $A(X)$.

Theorem 1.1. *The free abelian topological group $A(X)$ over a Tychonov space X is o-bounded if and only if each continuous metrizable image of X is Scheepers.*

In this paper, assuming the absence of Q-points, we shall prove a similar characterization of o-bounded free topological groups $F(X)$. By a *Q-point* we understand a free ultrafilter $\mathcal{U}$ on ω such that each finite-to-one function $f:\omega\to\omega$ is injective on a suitable set $U\in\mathcal{U}$.

The following characterization is the principal result of this paper.[1]

Theorem 1.2. *If there is no Q-point, then for every Tychonov space X the following conditions are equivalent*:

(1) *All finite powers of the free topological group $F(X)$ are o-bounded*;
(2) *The free topological group $F(X)$ is o-bounded*;
(3) *The free abelian topological group $A(X)$ is o-bounded*;
(4) *Every continuous metrizable image of X is Scheepers.*

The existence of Q-points is independent from ZFC. One can easily construct Q-points under Continuum Hypothesis. On the other hand, since under NCF for every ultrafilter $\mathcal{F}$ there exists a monotone surjection $\phi:\omega\to\omega$ such that $\phi(\mathcal{F})$ is generated by $\mathfrak{u}$ sets and $\mathfrak{u}<\mathfrak{d}$ by [1, Theorem 1], and obviously no Q-point can be generated by less than $\mathfrak{d}$ sets, NCF implies that there are no Q-points. In particular, there are no Q-points in the models of $\mathfrak{u}<\mathfrak{g}$.[2] Here NCF (abbreviated from the *Near Coherence of Filters*) stands for the following statement: For arbitrary ultrafilters $\mathcal{U},\mathcal{F}$ on ω there exists a monotone surjection $\phi:\omega\to\omega$ such that $\phi(\mathcal{U})=\phi(\mathcal{F})$. This statement follows from $\mathfrak{u}<\mathfrak{g}$ but contradicts the Martin's Axiom, see [5].

At this point we would like to mention that one of the most important problems regarding the o-boundedness is whether this property is preserved by squares of topological groups. This problem was first solved in [10] under additional set-theoretic assumptions, which were further weakened to $\mathfrak{r}\geqslant\mathfrak{d}$ by Mildenberger [11]. The apparently most intriguing part of this problem left open after [10] and [11] is whether the square of an o-bounded group is o-bounded provided NCF (or even $\mathfrak{u}<\mathfrak{g}$) holds. From the above it follows that free topological groups can not be used to solve this problem.

The proof of Theorem 1.2 is nontrivial and relies on the theory of multicovered spaces developed in [4]. The necessary information related to multicovered spaces is collected in Section 2. In that section we also define the operation of the semi-direct product of multicovered spaces. In Section 3 we recall some information related to the F-Menger property in multicovered spaces and prove an important Theorem 3.6 on preservation of the $[\mathcal{F}]$-Menger property by semi-direct products of multicovered spaces. In Section 4 we characterize the F-Menger property in (semi)multicovered groupoids. Those are multicovered spaces endowed with a binary and unary operations that are compactible with the multicover in a suitable sense. In Sections 6–8 we apply the obtained results about semi-multicovered groupoids to detecting the F-Menger property in topological groups and topological monoids. These results provide us with tools for the proof of Theorem 1.2 which is given in Section 9.

The properties of topological spaces and groups considered in this paper are often called *selection principles* (or *covering properties*) *in topology*. More information about this rapidly growing area may be found in surveys [9,14,16].

2. Multicovered spaces

By a *multicovered space* we understand a pair (X,μ) consisting of a set X and a family μ of covers of X. Such a family μ is called the *multicover* of X. Multicovered spaces naturally appear in many situations. In particular,

- each topological space X has the canonical multicover $\mu_{\mathcal{O}}$ consisting of all open covers of X;

[1] It was announced in [4] that Theorem 1.2 is true without any additional set-theoretic assumptions, but the proof does not work under the existence of a Q-point, see our Theorem 6.4.

[2] The assertion "$\mathfrak{b}=\mathfrak{d}$ and there are no Q-points" is consistent as well, see [12].

- each uniform space $(X,\mathcal{U})$ possesses the canonical multicover $\mu_{\mathcal{U}} = \{\{U(x):\ x \in X\}:\ U \in \mathcal{U}\}$ consisting of uniform covers of X;
- each metric space (X,d) has the canonical multicover $\mu_d = \{\{B(x,\varepsilon):\ x \in X\}:\ \varepsilon > 0\}$ consisting of covers by balls of fixed radius;
- each topological group G with topology τ carries four natural multicovers:
  - the left multicover $\mu_L = \{\{xU:\ x \in G\}:\ 1 \in U \in \tau\}$;
  - the right multicover $\mu_R = \{\{Ux:\ x \in G\}:\ 1 \in U \in \tau\}$;
  - the two-sided multicover $\mu_{L\wedge R} = \{\{Ux \cap xU:\ x \in G\}:\ 1 \in U \in \tau\}$;
  - the Rölke multicover $\mu_{L\vee R} = \{\{UxU:\ x \in G\}:\ 1 \in U \in \tau\}$.

A multicovered space (X,μ) is *centered* if for any covers $u, v \in \mu$ there is a cover $w \in \mu$ such that each w-bounded subset $B \subset X$ is both u-bounded and v-bounded. We define a subset $B \subset X$ to be *u-bounded* with respect to a cover u of X if $B \subset \bigcup u'$ for some finite subfamily $u' \subset u$.

Observe that all the examples of multicovered spaces presented above are centered. Each multicover μ on X can be transformed into a centered multicover $\mathsf{cen}(\mu)$ consisting of the covers

$$u_1 \wedge \cdots \wedge u_n = \{U_1 \cap \cdots \cap U_n:\ \forall i \leqslant n,\ U_i \in u_i\} \quad \text{where } u_1, \ldots, u_n \in \mu.$$

From now on *all multicovered spaces will be assumed to be centered.*

A multicovered space X is called *ω-bounded* if each cover $u \in \mu$ contains a countable subcover $u' \subset u$. If, in addition, there is a cover $v \in \mu$ such that each v-bounded subset of X is u'-bounded, then the multicovered space (X,μ) is *properly ω-bounded*.

By the *cofinality* $\mathrm{cf}(X,\mu)$ of a multicovered space (X,μ) we understand the smallest cardinality $|\mu'|$ of a subfamily $\mu' \subset \mu$ such that for every cover $u \in \mu$ there is a cover $u' \in \mu'$ such that each u'-bounded subset $B \subset X$ is u-bounded. For example, the multicover of any metric space has countable cofinality.

Multicovered spaces are objects of a category whose morphisms are uniformly bounded maps. We define a map $f : X \to Y$ between two multicovered spaces (X,μ_X) and (Y,μ_Y) to be *uniformly bounded* if for every cover $v \in \mu_Y$ there is a cover $u \in \mu_X$ such that for every u-bounded subset $B \subset X$ the image $f(B)$ is v-bounded in Y. Observe that every uniformly continuous map $f : X \to Y$ between uniform spaces is uniformly bounded with respect to the multicovers induced by the uniformities on X and Y, respectively.

Two multicovers μ_0, μ_1 on a set X are *equivalent* if the identity maps $\mathrm{id} : (X,\mu_0) \to (X,\mu_1)$ and $\mathrm{id} : (X,\mu_1) \to (X,\mu_0)$ are uniformly bounded (and hence are isomorphisms in the category of multicovered spaces).

Proposition 2.1. *A multicovered space (X,μ) is properly ω-bounded if and only if μ is equivalent to a multicover ν on X consisting of countable disjoint covers.*

Proof. Suppose that μ is an equivalent to a multicover ν of X consisting of countable covers. Given any $u \in \mu$, we can find $v \in \nu$ such that each v-bounded subset of X is u-bounded. This means that for every $V \in v$ there exists $u_V \in [u]^{<\omega}$ such that $V \subset \bigcup u_V$. Set $u' = \bigcup_{V \in v} u_V$ and observe that u' is a countable subcover of u such that each v-bounded subset is u'-bounded.

Let $u_1 \in \mu$ be such that each u_1-bounded subset of X is v-bounded. Then each u_1-bounded subset of X is u'-bounded, and hence μ is properly ω-bounded.

Now, suppose that μ is properly ω-bounded. For every $u \in \mu$ fix $u' \in [u]^{\omega}$ and $u'' \in \mu$ such that $\bigcup u' = X$ and each u''-bounded subset of X is u'-bounded. A direct verification shows that the multicover $\mu' = \{u':\ u \in \mu\}$ of X is equivalent to μ. Let us write u' in the form $\{U_n:\ n \in \omega\}$ and let $V_n = U_n \setminus \bigcup_{i<n} V_n$, $n \in \omega$, and $v_u = \{V_n:\ n \in \omega\}$. Then the multicover $\nu = \{v_u:\ u \in \mu\}$ of X is equivalent to μ' (and hence to μ) and consists of countable disjoint covers. $\square$

Sometimes we shall need the following obvious observation:

Observation 2.2. If (centered) multicovers μ_0 and μ_1 of a set X are equivalent, then for every $\nu_0 \subset \mu_0$ there exist (centered) submulticovers $\nu_0' \subset \mu_0$ and $\nu_1 \subset \mu_1$ equivalent to ν_0' such that $\nu_0 \subset \nu_0'$ and $|\nu_1| \leqslant |\nu_0'| \leqslant \max\{|\nu_0|, \omega\}$.

A multicovered space (X,μ) is *uniformizable* if μ is equivalent to the multicover

$$\mu_{\mathcal{U}} = \big\{\big\{U(x):\ x \in X\big\}:\ U \in \mathcal{U}\big\}$$

generated by a suitable uniformity $\mathcal{U}$ on X (here for an entourage $U \in \mathcal{U}$ by $U(x) = \{y \in X:\ (x,y) \in U\}$ we denote the U-ball centered at x). There is a simple criterion of the uniformizability of ω-bounded multicovered spaces.

Proposition 2.3. *An ω-bounded multicovered space is uniformizable if and only if it is centered and properly ω-bounded.*

Proof. To prove the "only if" part, assume that the multicover μ is equivalent to the multicover $\mu_{\mathcal{U}}$ generated by some uniformity $\mathcal{U}$ on X. Since the multicover $\mu_{\mathcal{U}}$ is centered, so is the multicover μ. Next, we check that μ is properly ω-bounded.

Given any $u \in \mu$, we can find an entourage $U \in \mathcal{U}$ such that each U-ball $U(x) = \{x' \in X\colon (x, x') \in U\}$ is u-bounded. Let $V \in \mathcal{U}$ be an entourage such that $V = V^{-1}$ and $V \circ V \subset U$.

The ω-boundedness of the multicover μ implies that of $\mu_{\mathcal{U}}$. Consequently, there is a countable subset $C \subset X$ such that $\bigcup_{x\in C} V(x) = X$. By the choice of the entourage U, there is a countable subfamily $u' \subset u$ such that each U-ball $U(c)$, $c \in C$, is u'-bounded. Observe that for every $x \in X$ we can find $c \in C$ with $x \in V(c)$. Consequently, $V(x) \subset V \circ V(c) \subset U(c) \subset \bigcup u'_f$ for some finite subfamily $u'_f \subset u'$. This means that each V-ball $V(x)$, $x \in X$, is u'-bounded.

Since the multicovers μ and $\mu_{\mathcal{U}}$ are equivalent, we can find a cover $v \in \mu$ such that each v-bounded subset of X is $\{V(x)\}_{x\in X}$-bounded and hence u'-bounded, witnessing that the multicovered space (X, μ) is properly ω-bounded.

To prove the "if" part, assume that a multicovered space (X, μ) is centered and properly ω-bounded. By Proposition 2.1, we can assume that the multicover μ consists of countable disjoint covers. For every finite subfamily $\mu' \subset \mu$ consider the entourage of the diagonal

$$W_{\mu'} = \bigl\{(x, y) \in X \times X\colon \forall u \in \mu'\ \exists U \in u,\ x, y \in U\bigr\}.$$

Those entourages generate a uniformity $\mathcal{W}$ on X such that the multicover $\mu_{\mathcal{W}}$ consisting of $\mathcal{W}$-uniform covers of X is equivalent to the multicover μ. $\square$

Now we describe some operations in the category of multicovered spaces.

Each subset A of a multicovered space (X, μ) carries the *induced multicover* $\mu{\restriction}A$ consisting of the covers $u{\restriction}A = \{U \cap A\colon U \in u\}$, $u \in \mu$. The multicovered space $(A, \mu{\restriction}A)$ is called a *subspace* of (X, μ).

Each map $f : X \to Y$ from a set X to a multicovered space (Y, μ_Y) induces the multicover $f^{-1}(\mu)$ consisting of the covers $f^{-1}(u) = \{f^{-1}(U)\colon U \in u\}$, $u \in \mu_Y$.

For two multicovers μ_0, μ_1 on a set X their *meet* is the multicover

$$\mu_0 \wedge \mu_1 = \{u \wedge v\colon u \in \mu_0,\ v \in \mu_1\}$$

where

$$u \wedge v = \{U \cap V\colon U \in u,\ V \in v\}.$$

The product $X \times Y$ of two multicovered spaces (X, μ_X) and (Y, μ_Y) possesses the multicover

$$\mu_X \boxtimes \mu_Y = \{u \boxtimes v\colon u \in \mu_X,\ v \in \mu_Y\} \quad \text{where } u \boxtimes v = \{U \times V\colon U \in u,\ V \in v\}.$$

The obtained multicovered space $(X \times Y, \mu_X \boxtimes \mu_Y)$ is called the *direct product* of the multicovered spaces (X, μ_X) and (Y, μ_Y).

Besides the multicover $\mu_X \boxtimes \mu_Y$ on the Cartesian product $X \times Y$ of multicovered spaces (X, μ_X) and (Y, μ_Y) there are three less evident multicovers:

- $\mu_X \ltimes \mu_Y = \{\{U \times V\colon U \in u,\ V \in v_U\}\colon u \in \mu_X,\ \{v_U\}_{U\in u} \subset \mu_Y\}$, the multicover of the left semi-direct product,
- $\mu_X \rtimes \mu_Y = \{\{U \times V\colon V \in v,\ U \in u_V\}\colon v \in \mu_Y,\ \{u_V\}_{V\in v} \subset \mu_X\}$, the multicover of the right semi-direct product, and
- $\mu_X \bowtie \mu_Y = (\mu_X \ltimes \mu_Y) \wedge (\mu_X \rtimes \mu_Y)$, the multicover of the semi-direct product.

The multicovered spaces

$$X \ltimes Y = (X \times Y, \mu_X \ltimes \mu_Y), \qquad X \rtimes Y = (X \times Y, \mu_X \rtimes \mu_Y), \qquad X \bowtie Y = (X \times Y, \mu_X \bowtie \mu_Y)$$

are called respectively: the *left semi-direct product*, the *right semi-direct product*, and the *semi-direct product* of the multicovered spaces (X, μ_X) and (Y, μ_Y).

The following proposition shows that the (semi)direct products respect the equivalence relation.

Proposition 2.4. *Let $\langle \mu_X, \nu_X\rangle$ and $\langle \mu_Y, \nu_Y\rangle$ be pairs of equivalent centered multicovers of sets X and Y, respectively. Then $\mu_X \boxtimes \mu_Y$, $\mu_X \ltimes \mu_Y$, $\mu_X \rtimes \mu_Y$, and $\mu_X \bowtie \mu_Y$ are equivalent to $\nu_X \boxtimes \nu_Y$, $\nu_X \ltimes \nu_Y$, $\nu_X \rtimes \nu_Y$, and $\nu_X \bowtie \nu_Y$, respectively.*

Proof. We shall prove the equivalence of the multicovers $\mu_X \ltimes \mu_Y$ and $\nu_X \ltimes \nu_Y$. For the other pairs of multicovers the proof is analogous.

In order to prove the uniform boundedness of the identity map

$$(X \times Y, \mu_X \ltimes \mu_Y) \to (X \times Y, \nu_X \ltimes \nu_Y),$$

take any cover $w \in \nu_X \ltimes \nu_Y$ and find a cover $u \in \nu_X$ and a family of covers $\{v_U\}_{U\in u} \subset \nu_Y$ such that $w = \{U \times V\colon U \in u,\ V \in v_U\}$.

Since the cover μ_X is equivalent to ν_X, there is a cover $u' \in \mu_X$ such that each u'-bounded subset of X is u-bounded. Consequently, for each $U' \in u'$ there is a finite subfamily $u_{U'} \subset u$ with $U' \subset \bigcup u_{U'}$. Since the multicover μ_Y is equivalent to the centered multicover ν_Y, there is a cover $v'_{U'} \in \mu_Y$ such that each $v'_{U'}$-bounded subset of Y is v_U-bounded for every set $U \in u'_{U'}$. Now consider the cover

$$w' = \{U' \times V' \colon U' \in u',\ \mathcal{V}' \in v'_{U'}\} \in \mu_X \ltimes \mu_Y$$

and observe that each w'-bounded subset of $X \times Y$ is w-bounded.

Analogously we prove the uniform boundedness of the identity map

$$(X \times Y, \nu_X \ltimes \nu_Y) \to (X \times Y, \mu_X \ltimes \mu_Y). \qquad \square$$

The (semi-)direct product operations also respect the (proper) ω-boundedness. This follows from Propositions 2.1, 2.4 and Observation 2.2 above.

Proposition 2.5. *Let (X, μ_X) and (Y, μ_Y) be two multicovered spaces. If the multicovers μ_X and μ_Y are (properly) ω-bounded, then so are the multicovers $\mu_X \boxtimes \mu_Y$, $\mu_X \ltimes \mu_Y$, $\mu_X \rtimes \mu_Y$, and $\mu_X \bowtie \mu_Y$ on $X \times Y$.*

Finally we prove the following useful reduction lemma.

Lemma 2.6. *Let (X, μ_X) and (Y, μ_Y) be two properly ω-bounded multicovered spaces. For every countable subfamily $\mu' \subset \mu_X \bowtie \mu_Y$ there are countable subfamilies $\mu'_X \subset \mu_X$ and $\mu'_Y \subset \mu_Y$ such that the identity map $\mathrm{id} : (X \times Y, \mu'_X \bowtie \mu'_Y) \to (X \times Y, \mu')$ is uniformly bounded.*

Proof. Observation 2.2 and Propositions 2.1, 2.4 reduce the proof of the lemma to the case when μ_X and μ_Y consist of countable disjoint covers. Also, there is no loss of generality in assuming that $\mu' = \{w\}$, where $w = w_0 \wedge w_1$ for some $w_0 \in \mu_X \rtimes \mu_Y$ and $w_1 \in \mu_X \ltimes \mu_Y$. In its turn, we can write w_0 and w_1 in the form $w_0 = \{U \times V \colon V \in v,\ U \in u_V\}$, where $v \in \mu_Y$ and $\langle u_V \rangle_{V \in v} \in \mu_X^v$; and $w_1 = \{U \times V \colon U \in u,\ V \in v_U\}$, where $u \in \mu_X$ and $\langle v_U \rangle_{U \in u} \in \mu_Y^u$. Now, any countable centered $\mu'_X \subset \mu_X$ and $\mu'_Y \subset \mu_Y$ containing $\{u\} \cup \{u_V \colon V \in v\}$ and $\{v\} \cup \{v_U \colon U \in u\}$, respectively, are easily seen to be as asserted. $\square$

3. F-Menger multicovered spaces

The notions of a Menger or Scheepers topological space can be easily generalized to multicovered spaces. Namely, we define a multicovered space (X, μ) to be *Menger* (resp. *Scheepers*) if for any sequence of covers $\langle u_n \rangle_{n \in \omega} \in \mu^\omega$ there is a cover (resp. ω-cover) $\langle B_n \rangle_{n \in \omega}$ of X by u_n-bounded sets $B_n \subset X$, $n \in \omega$. These two properties are particular cases of the F-Menger property of a multicovered space, where F is a suitable family of semifilters.

By a *semifilter* we understand a family $\mathcal{F}$ of infinite subsets of ω such that for each set $A \in \mathcal{F}$ the free filter $\mathcal{F}_A = \{B \subset \omega \colon |A \setminus B| < \aleph_0\}$ generated by A lies in $\mathcal{F}$. Intuitively, sets belonging to a given semifilter can be thought of as large in a suitable sense. An evident example of a semifilter is any free (ultra)filter on ω. By SF we denote the family of all semifilters and by UF the subfamily of SF consisting of all free ultrafilters (so, $\mathsf{UF} = \beta\omega \setminus \omega$). More information on semifilters can be found in [2] and [3].

There is an important equivalence relation on the set of semifilters SF, called the *coherence relation*, which is defined with the help of finite-to-finite multifunctions.

By a multifunction on ω we understand a subset $\Phi \subset \omega \times \omega$ thought of as a set-valued function $\Phi : \omega \Rightarrow \omega$ assigning to each number $n \in \omega$ the subset $\Phi(n) = \{m \in \omega \colon (n, m) \in \Phi\}$ of ω. Each multifunction Φ has the inverse $\Phi^{-1} = \{(n, m) \colon (m, n) \in \Phi\}$. A multifunction $\Phi : \omega \Rightarrow \omega$ is called *finite-to-finite* if for every $a \in \omega$ the sets $\Phi(a)$ and $\Phi^{-1}(a)$ are finite and nonempty.

We say that a semifilter $\mathcal{F}$ is *subcoherent* to a semifilter $\mathcal{U}$ and denote this by $\mathcal{F} \Subset \mathcal{U}$ if there is a finite-to-finite multifunction $\Phi : \omega \Rightarrow \omega$ such that $\Phi(\mathcal{F}) = \{\Phi(F) \colon F \in \mathcal{F}\} \subset \mathcal{U}$. Semifilters $\mathcal{F}, \mathcal{U}$ on ω are *coherent* (denoted by $\mathcal{F} \asymp \mathcal{U}$) if $\mathcal{F} \Subset \mathcal{U}$ and $\mathcal{U} \Subset \mathcal{F}$. The coherence of semifilters is an equivalence relation dividing the set SF of semifilters into the coherence classes $[\mathcal{F}] = \{\mathcal{U} \in \mathsf{SF} \colon \mathcal{U} \asymp \mathcal{F}\}$ of semifilters $\mathcal{F}$. By Proposition 5.5.2 and Theorem 5.5.3 of [2], a semifilter $\mathcal{F}$ is coherent to a filter $\mathcal{U}$ if and only if $\varphi(\mathcal{F}) = \varphi(\mathcal{U})$ for some monotone surjection $\varphi : \omega \to \omega$. Thus NCF is equivalent to the statement that all ultrafilters are coherent in the above sense.

Let $\mathsf{F} \subset \mathsf{SF}$ be a family of semifilters. Following [4], we define an indexed cover $\langle U_n \rangle_{n \in \omega}$ of a set X to be an F-*cover* if there is a semifilter $\mathcal{F} \in \mathsf{F}$ such that for every $x \in X$ the set $\{n \in \omega \colon x \in U_n\}$ belongs to $\mathcal{F}$. Observe that $\langle U_n \rangle_{n \in \omega}$ is an ω-cover if and only if it is an UF-cover.

A multicovered space (X, μ) is said to be F-*Menger* if for each sequence $\langle U_n \rangle_{n \in \omega} \subset \mu$ of covers of X there is an F-cover $\langle B_n \rangle_{n \in \omega}$ of X by u_n-bounded subsets $B_n \subset X$, $n \in \omega$. More information on F-Menger multicovered spaces can be found in [4].

Proposition 3.1. *A multicovered space X is Menger (resp. Scheepers) if and only if X is* SF*-Menger (resp.* UF*-Menger).*

Proof. The "if" part trivially follows from the definitions.

To prove the "only if" part, assume that a multicovered space (X,μ) is Menger. To prove that it is SF-Menger, fix a sequence of covers $\langle u_n\rangle_{n\in\omega}\in\mu^\omega$. Since X is Menger, for every $k\in\omega$ there is a cover $\langle B_{n,k}\rangle_{n\geqslant k}$ of X by u_n-bounded subsets $B_{n,k}\subset X$. It follows that for every $n\in\omega$ the finite union $B_n=\bigcup_{k\leqslant n}B_{n,k}$ is u_n-bounded in X. We claim that $\langle B_n\rangle_{n\in\omega}$ is an SF-cover of X. Observe that for every point $x\in X$ and every $k\in\omega$ the set $\{n\geqslant k\colon\ x\in B_{n,k}\}$ is not empty. Consequently, the set $\{n\in\omega\colon\ x\in B_n\}\supset\bigcup_{k\in\omega}\{n\geqslant k\colon\ x\in B_{n,k}\}$ is infinite and hence the family $\{\{n\in\omega\colon\ x\in B_n\}\colon\ x\in X\}$ can be enlarged to a semifilter $\mathcal{F}\in\mathsf{SF}$, witnessing that $\langle B_n\rangle_{n\in\omega}$ is an SF-cover of X.

If the multicovered space X is Scheepers, then the covers $\langle B_{n,k}\rangle_{n\in\omega}$, $k\in\omega$, can be chosen to be ω-covers. This means that for any finite subset $F\subset X$ and every $k\in\omega$ the set $\{n\geqslant k\colon F\subset B_{n,k}\}$ is not empty and consequently, the set $\{n\in\omega\colon\ F\subset B_n\}\supset\bigcup_{k\in\omega}\{n\geqslant k\colon\ F\subset B_{n,k}\}$ is infinite. It follows that the family $\{\{n\in\omega\colon\ x\in B_n\}\colon\ x\in X\}$ can be enlarged to a free ultrafilter $\mathcal{F}\in\mathsf{UF}$ witnessing that $\langle B_n\rangle_{n\in\omega}$ is an UF-cover of X. $\square$

The F-Menger property behaves nicely only for relatively large families $\mathsf{F}\subset\mathsf{SF}$, containing together with each semifilter $\mathcal{F}$ all its finite-to-one images. We recall that a function $\varphi:\omega\to\omega$ is called *finite-to-one* if for each $y\in\omega$ the preimage $\varphi^{-1}(y)$ is finite and nonempty. It is clear that each monotone surjection $\varphi:\omega\to\omega$ is finite-to-one.

Given a family of semifilters $\mathsf{F}\subset\mathsf{SF}$ consider two its extensions:

$$\mathsf{F}_{\asymp}=\bigcup_{\mathcal{F}\in\mathsf{F}}[\mathcal{F}]\quad\text{and}\quad\mathsf{F}_{\downarrow}=\bigcup_{\mathcal{F}\in\mathsf{F}}\mathcal{F}_{\downarrow}$$

where $\mathcal{F}_\downarrow=\{\varphi(\mathcal{F})\mid\varphi:\omega\to\omega\text{ is a monotone surjection}\}$. It is clear that

$$\mathsf{F}\subset\mathsf{F}_\downarrow\subset\mathsf{F}_\asymp.$$

The following important lemma allows us to reduce the F-Menger property of centered multicovered spaces to the $\mathcal{F}_\downarrow$-Menger property for a suitable semifilter $\mathcal{F}\in\mathsf{F}$.

Lemma 3.2. *If a multicovered space (X,μ) with countable cofinality is $\mathsf{F}_\asymp$-Menger for some family of semifilters $\mathsf{F}\subset\mathsf{SF}$, then (X,μ) is $\mathcal{F}_\downarrow$-Menger for some semifilter $\mathcal{F}\in\mathsf{F}$.*

Proof. Let $\langle u_n\rangle_{n\in\omega}\in\mu^\omega$ be a witness for $\mathrm{cf}(X,\mu)=\omega$ such that each u_{n+1}-bounded subset of X is u_n-bounded. Since (X,μ) is $\mathsf{F}_\asymp$-Menger, there exists $\mathcal{F}\in\mathsf{F}$, a finite-to-finite multifunction $\Phi:\omega\Rightarrow\omega$, and a cover $\langle A_n\rangle_{n\in\omega}$ of X such that each A_n is u_n-bounded and $\Phi(\mathcal{N}_x)\in\mathcal{F}$, where $\mathcal{N}_x=\{n\in\omega\colon\ x\in A_n\}$, $x\in X$.

Let $\langle n_m\rangle_{m\in\omega}$ be an increasing sequence of natural numbers such that $n_0=0$ and $n_{m+1}>1+\max(\Phi\cup\Phi^{-1})(\{0,\dots,n_m\})$ for $m\in\omega$. It follows that $n_{m+1}>m+1$ and $\Phi([n_{m+1},n_{m+2}))\subset[n_m,n_{m+3})$ for every $m\in\omega$.

Set $C_m=\bigcup_{n\in[n_{m-1},n_{m+3})}A_n$ and note that it is u_m-bounded. Let $\phi:\omega\to\omega$ be the monotone surjection such that $\phi^{-1}(m)=[n_m,n_{m+1})$. We claim that $\{m\in\omega\colon\ x\in C_m\}\in\phi(\mathcal{F})$ for every $x\in X$. For this sake we shall show that $\{m\in\omega\colon x\in C_m\}^*\supset\phi(\Phi(\mathcal{N}_x))$. Indeed, if $n\in\mathcal{N}_x\cap[n_{m+1},n_{m+2})$ for some $m\in\omega$, then $\phi(\Phi(n))\subset\{m,m+1,m+2\}$. By the definition of C_j's, $A_n\subset\bigcup_{n\in[n_{m+1},n_{m+2})}A_n\subset C_j$ for every $j\in\{m,m+1,m+2\}$, which means that $\{m,m+1,m+2\}\subset\{m\in\omega\colon\ x\in C_m\}$, and hence $\phi(\Phi(\mathcal{N}_x))\subset^*\{m\in\omega\colon\ x\in C_m\}$. Thus $\langle C_m\rangle_{m\in\omega}$ is an $\{\phi(\mathcal{F})\}$-cover of X by u_m-bounded subsets.

For every $x\in X$ we denote by $\mathcal{N}'(x)$ the set $\{m\in\omega\colon\ x\in C_m\}$. Given $\langle v_k\rangle_{k\in\omega}\in\mu^\omega$, we can find an increasing sequence $\langle m_k\rangle_{k\in\omega}$ of natural numbers such that each u_{m_k}-bounded subset of X is v_k-bounded. Let $B_k=\bigcup_{m\in[m_k,m_{k+1})}C_m$ and $\psi:\omega\to\omega$ be such that $\psi^{-1}(k)=[m_k,m_{k+1})$. By the definition of m_k's, B_k is v_k-bounded. It follows from the above that

$$(\psi\circ\phi)(\mathcal{F})\ni\psi\big(\mathcal{N}'_x\big)\subset^*\{k\in\omega\colon\ x\in B_k\}$$

for every $x\in X$, and hence $\psi\circ\phi$ is a witness for $\langle B_k\colon\ k\in\omega\rangle$, being an $\mathcal{F}_\downarrow$-cover of X by v_k-bounded subsets, which completes our proof. $\square$

Corollary 3.3. *Let $\mathsf{F}\subset\mathsf{SF}$ be a family of semifilters. A centered multicovered space X is $\mathsf{F}_\asymp$-Menger if and only if X is $\mathsf{F}_\downarrow$-Menger.*

Proof. Observe that a multicovered space (X,μ) is $\mathsf{F}_\asymp$-Menger (resp. $\mathsf{F}_\downarrow$-Menger) if and only if such is also (X,ν) for every countable centered $\nu\subset\mu$. We can now apply Lemma 3.2. $\square$

Therefore for any semifilter $\mathcal{F}$ the $[\mathcal{F}]$-Menger and $\mathcal{F}_\downarrow$-Menger properties of centered multicovered spaces are equivalent. Of these two properties, the $\mathcal{F}_\downarrow$-Menger property is better for applications while the $[\mathcal{F}]$-Menger property is more convenient for proofs.

For any free filter $\mathcal{F}$ on ω the class of $\mathcal{F}_\downarrow$-Menger spaces has the following nice inheritance properties:

Theorem 3.4. *Let $\mathcal{F}$ be a free filter on ω. Then*:

(1) *Each subspace of an $\mathcal{F}_\downarrow$-Menger multicovered space is $\mathcal{F}_\downarrow$-Menger.*
(2) *A multicovered space X is $\mathcal{F}_\downarrow$-Menger if it can be written as the countable union $X=\bigcup_{n\in\omega}A_n$ of $\mathcal{F}_\downarrow$-Menger subspaces $A_n\subset X$, $n\in\omega$.*
(3) *A multicovered space Y is $\mathcal{F}_\downarrow$-Menger if it is the image of an $\mathcal{F}_\downarrow$-Menger multicovered space X under a uniformly bounded surjective map $f:X\to Y$.*
(4) *The direct product $X\times Y$ of two $\mathcal{F}_\downarrow$-Menger multicovered spaces is $\mathcal{F}_\downarrow$-Menger.*
(5) *The meet $\mu_0\wedge\mu_1$ of two $\mathcal{F}_\downarrow$-Menger multicovers μ_0 and μ_1 on a set X is $\mathcal{F}_\downarrow$-Menger.*

Proof. (1) The first assertion is obvious.

(2) To prove the second property, denote by μ the underlying multicover of X and fix any sequence of covers $\langle u_m\rangle_{m\in\omega}\in\mu^\omega$.

For every $n\in\omega$ find a cover $\langle B_{n,m}\rangle_{m\in\omega}$ of the $\mathcal{F}_\downarrow$-Menger subspace $X_n\subset X$ by u_m-bounded subsets $B_{n,m}\subset X$ such that

$$\bigl\{\{m\in\omega:\ x\in B_{n,m}\}:\ x\in X_n\bigr\}\in\mathcal{F}_n$$

for some semifilter $\mathcal{F}_n\in\mathcal{F}_\downarrow$.

For every $m\in\omega$ consider the u_m-bounded subset $B_m=\bigcup_{n\leqslant m}B_{n,m}$ of X. We claim that

$$\bigl\{\{m\in\omega:\ x\in B_m\}:\ x\in X\bigr\}\in\mathcal{F}_\infty$$

where $\mathcal{F}_\infty=\bigcup_{n\in\omega}\mathcal{F}_n$. Indeed, given any $x\in X$ we can find $n\in\omega$ such that $x\in X_n$ and conclude that

$$\mathcal{F}_\infty\supset\mathcal{F}_n\ni\{m\geqslant n:\ x\in B_{n,m}\}\subset\{m\in\omega:\ x\in B_m\}$$

and thus $\{m\in\omega:\ x\in B_m\}\in\mathcal{F}_\infty$.

We claim that $\mathcal{F}_\infty\in[\mathcal{F}]$. It follows that for every $n\in\omega$ there is a finite-to-finite multifunction $\Phi_n:\omega\Rightarrow\omega$ with $\Phi_n(\mathcal{F}_n)\subset\mathcal{F}$. It is easy to construct a finite-to-finite multifunction $\Phi:\omega\Rightarrow\omega$ such that for every $n\in\omega$ the inclusion $\Phi_n(k)\subset\Phi(k)$ holds for all but finitely many numbers k. Then for every $n\in\omega$ we get

$$\Phi(\mathcal{F}_n)\subset\Phi_n(\mathcal{F}_n)\subset\mathcal{F}$$

and thus $\Phi(\mathcal{F}_\infty)\subset\mathcal{F}$, witnessing $\mathcal{F}_\infty\Subset\mathcal{F}$. The inverse relation $\mathcal{F}_\infty\Supset\mathcal{F}$ follows from $\mathcal{F}\asymp\mathcal{F}_0\subset\mathcal{F}_\infty$.

Therefore the space X is $[\mathcal{F}]$-Menger and hence $\mathcal{F}_\downarrow$-Menger by Corollary 3.3.

(3) Assume that X is an $\mathcal{F}_\downarrow$-Menger multicovered space and let $f:X\to Y$ be a uniformly bounded map onto a multicovered space Y. To show that Y is $\mathcal{F}_\downarrow$-Menger, fix a sequence of covers $\langle u_n\rangle_{n\in\omega}\in\mu_Y^\omega$. By the uniform boundedness of f there is a sequence of covers $\langle v_n\rangle_{n\in\omega}\in\mu_X^\omega$ such that for every $n\in\omega$ the image $f(B)$ of any v_n-bounded subset $B\subset X$ is u_n-bounded. Using the $\mathcal{F}_\downarrow$-Menger property of X, find an $\mathcal{F}_\downarrow$-cover $\langle B_n\rangle_{n\in\omega}$ of X by v_n-bounded subsets $B_n\subset X$. Then each set $f(B_n)$ is u_n-bounded in Y. We claim that $\langle f(B_n)\rangle_{n\in\omega}$ is an $\mathcal{F}_\downarrow$-cover of Y. Since $\langle B_n\rangle_{n\in\omega}$ is an $\mathcal{F}_\downarrow$-cover of X, there is a semifilter $\mathcal{F}'\in\mathcal{F}_\downarrow$ such that $\{\{n\in\omega:\ x\in B_n\}:\ x\in X\}\in\mathcal{F}'$.

Fix a point $y\in Y$ and find any $x\in f^{-1}(y)$ (which exists by the surjectivity of f). Observe that $x\in B_n$ implies $y\in f(B_n)$ and thus

$$\mathcal{F}'\ni\{n\in\omega:\ x\in B_n\}\subset\bigl\{n\in\omega:\ y\in f(B_n)\bigr\}$$

and finally $\{n\in\omega:\ y\in f(B_n)\}\in\mathcal{F}'\in\mathcal{F}_\downarrow$, witnessing that $\langle f(B_n)\rangle_{n\in\omega}$ is an $\mathcal{F}_\downarrow$-cover of Y.

(4) Let (X,μ_X) and (Y,μ_Y) be two $\mathcal{F}_\downarrow$-Menger multicovered spaces. Since the $\mathcal{F}_\downarrow$-Menger and $[\mathcal{F}]$-Menger properties are equivalent, it suffices to check that the direct product $(X\times Y,\mu_X\boxtimes\mu_Y)$ is $[\mathcal{F}]$-Menger. Fix a sequence of covers $\langle w_k\rangle_{k\in\omega}\in(\mu_X\boxtimes\mu_Y)^\omega$. For every $k\in\omega$ we can write w_k in the form $w_k=u_k\boxtimes v_k$, where $u_k\in\mu_X$ and $v_k\in\mu_Y$. Without loss of generality, each u_{k+1}- (resp. v_{k+1}-) bounded subset is u_k- (resp. v_k-) bounded. Using the $[\mathcal{F}]$-Menger property of X and Y, find $[\mathcal{F}]$-covers $\langle A_k\rangle_{k\in\omega}$ and $\langle B_k\rangle_{k\in\omega}$ of X and Y respectively such that each A_k (resp. B_k) is u_k-bounded (resp. v_k-bounded). Find a finite-to-finite multifunction Φ such that

$$\Phi\Bigl(\bigl\{\{n\in\omega:\ x\in A_n\}:\ x\in X\bigr\}\cup\bigl\{\{n\in\omega:\ y\in B_n\}:\ y\in Y\bigr\}\Bigr)\subset\mathcal{F}.$$

Let $\langle n_k\rangle_{k\in\omega}$ be an increasing sequence of natural numbers such that $n_0=0$ and $n_{k+1}>\max(\Phi\cup\Phi^{-1}(\{0,\dots,n_k\})$ for $k\in\omega$. It follows that $\Phi([n_{k+1},n_{k+2}))\subset[n_k,n_{k+3})$ for every $k\in\omega$. Set $A'_k=\bigcup_{n\in[n_k,n_{k+3})}A_n$, $B'_k=\bigcup_{n\in[n_k,n_{k+3})}B_n$, and $\Psi(k)=[n_k,n_{k+3})$. We claim that the multifunction $\Psi:\omega\Rightarrow\omega$ is a witness for $\langle A'_k\times B'_k\rangle_{k\in\omega}$ being an $[\mathcal{F}]$-cover of $X\times Y$. Indeed, let us fix $(x,y)\in X\times Y$ and set $F_x=\{n\in\omega:\ x\in A_n\}$, $F_y=\{n\in\omega:\ y\in B_n\}$, and $F'_{x,y}=\{k\in\omega:\ (x,y)\in A'_k\times B'_k\}$. It suffices to show that $\Phi(F_x)\cap\Phi(F_y)\subset\Psi(F_{x,y})$. Given any $m\in\Phi(F_x)\cap\Phi(F_y)$, find $n_x\in F_x$ and $n_y\in F_y$ such that $m\in\Phi(n_x)\cap\Phi(n_y)$. Let k be such that $m\in[n_{k+1},n_{k+2})$. By our choice of the sequence $\langle n_k\rangle_{k\in\omega}$, $\{n_x,n_y\}\subset[n_k,n_{k+3})$, and hence $x \in A_{n_x} \subset A'_k$ and $y \in B_{n_y} \subset B'_k$, which means that $k \in F_{x,y}$. By the definition of Ψ, $m \in [n_{k+1}, n_{k+2}) \subset \Psi(k)$, which completes our proof.

(5) The fifth property is a direct consequence of the previous ones. Indeed, consider the diagonal $\Delta_X = \{(x,x)\colon x \in X\}$ of the product $X \times X$ endowed with the induced multicover $\mu = \mu_1 \boxtimes \mu_2 \restriction \Delta_X$. It is easy to see that the map

$$f : (\Delta_X, \mu) \to (X, \mu_0 \wedge \mu_1), \qquad f : (x,x) \mapsto x$$

is uniformly bounded. By the fourth property, the direct product $(X \times X, \mu_0 \boxtimes \mu_1)$ is $\mathcal{F}_\downarrow$-Menger, and hence so is its subspace (Δ_X, μ). Finally, the third property implies that $(X, \mu_0 \wedge \mu_1)$ is $\mathcal{F}_\downarrow$-Menger as well. $\square$

Corollary 3.5. *Let F be a family of free filters and (X, μ) be an $\mathsf{F}_\downarrow$-Menger multicovered space. Then all finite powers of X are $\mathsf{F}_\downarrow$-Menger. In particular, the Scheepers property is preserved by finite powers.*

Proof. It suffices to prove that $(X^2, \mu \boxtimes \mu)$ is $\mathsf{F}_\downarrow$-Menger (by induction this will imply that the powers X^{2^n}, $n \in \omega$, all are $\mathsf{F}_\downarrow$-Menger). Given a sequence $\langle w_n \rangle_{n\in\omega} \in (\mu \boxtimes \mu)^\omega$, write each cover w_n in the form $u_n \boxtimes v_n$, and set $\nu = \mathrm{cen}(\{u_n, v_n\colon n \in \omega\})$. Applying Lemma 3.2 we conclude that (X, ν) is $[\mathcal{F}]$-Menger for some $\mathcal{F} \in \mathsf{F}$, and hence so is the square $(X^2, \nu \boxtimes \nu)$ by Theorem 3.4(4). It follows that there exists an $[\mathcal{F}]$-cover $\langle C_n \rangle_{n\in\omega}$ of X^2 such that each C_n is a w_n-bounded subset of X^2, which finishes our proof.

The last assertion follows from the equivalence of the Scheepers and UF-Menger properties, proved in Proposition 3.1. $\square$

For ultrafilters $\mathcal{F}$ coherent to no Q-point, the fourth assertion of Theorem 3.4 can be generalized to semi-direct products of $\mathcal{F}_\downarrow$-Menger multicovered spaces. For this we shall need a characterization of such ultrafilters $\mathcal{F}$ in terms of the left subcoherence.

Following [2, 10.2.1], we define a semifilter $\mathcal{F}$ to be *left subcoherent* to a semifilter $\mathcal{U}$ (and denote this by $\mathcal{F} \underset{\leftarrow}{\subset} \mathcal{U}$) if for every monotone unbounded $f : \omega \to \omega$ there is a finite-to-finite multifunction $\Phi : \omega \Rightarrow \omega$ such that $\Phi(\mathcal{F}) \subset \mathcal{U}$ and $\Phi(n) \subset [0, f(n)]$ for all $n \in \omega$. By Proposition 10.2.2 of [2], for two semifilters $\mathcal{F}, \mathcal{U}$, the relations $\mathcal{F} \underset{\leftarrow}{\subset} \mathcal{F} \Subset \mathcal{U}$ imply $\mathcal{F} \underset{\leftarrow}{\subset} \mathcal{U}$. By Proposition 10.2.3 of [2], an ultrafilter is not coherent to a Q-point if and only if $\mathcal{U} \underset{\leftarrow}{\subset} \mathcal{U}$.

Theorem 3.6. *If a free ultrafilter $\mathcal{F}$ on ω is not coherent to a Q-point, then for any centered $\mathcal{F}_\downarrow$-Menger multicovered spaces X, Y their semi-direct product $X \bowtie Y$ is $\mathcal{F}_\downarrow$-Menger.*

Proof. By definition, the multicover $\mu_X \bowtie \mu_Y$ of the semi-direct product $X \bowtie Y$ is the meet

$$\mu_X \bowtie \mu_Y = (\mu_X \ltimes \mu_Y) \wedge (\mu_X \rtimes \mu_Y)$$

of the multicovers of the left and right semi-direct products of the multicovered spaces (X, μ_X) and (Y, μ_Y). According to Theorem 3.4(5) and Corollary 3.3, the $\mathcal{F}_\downarrow$-Menger property of the multicovered space $(X \times Y, \mu_X \bowtie \mu_Y)$ will follow as soon as we prove the $[\mathcal{F}]$-Menger property for the left and right semi-direct products $X \ltimes Y$ and $X \rtimes Y$.

To prove the $[\mathcal{F}]$-Menger property of $X \ltimes Y$, fix a sequence $\langle w_n \rangle_{n\in\omega} \in (\mu_X \ltimes \mu_Y)^\omega$ of covers of $X \times Y$. It follows from the definition of the multicover $\mu_X \ltimes \mu_Y$ that for every cover w_n there is a cover $u_n \in \mu_X$ such that for every u_n-bounded subset $B \subset X$ there is a cover $v \in \mu_Y$ such that for every v-bounded subset $D \subset Y$ the product $B \times D$ is w_n-bounded.

The $[\mathcal{F}]$-Menger property of the multicovered space (X, μ_X) yields an $[\mathcal{F}]$-cover $\langle B_n \rangle_{n\in\omega}$ of X by u_n-bounded subsets $B_n \subset Y$. For every $n \in \omega$ find a cover $v_n \in \mu_Y$ such that for each v_n-bounded subset $D \subset Y$ the product $B_n \times D$ is w_n-bounded. Since (Y, μ_Y) is centered we can additionally assume that each v_{n+1}-bounded subset of Y is v_n-bounded for $n \in \omega$.

The $[\mathcal{F}]$-Menger property of the muticovered space (Y, μ_Y) yields an $[\mathcal{F}]$-cover $\langle C_n \rangle_{n\in\omega}$ of Y by v_n-bounded subsets $C_n \subset Y$. Since $\langle B_n \rangle_{n\in\omega}$ and $\langle C_n \rangle_{n\in\omega}$ are $[\mathcal{F}]$-covers, there are semifilters $\mathcal{F}_X, \mathcal{F}_Y \in [\mathcal{F}]$ such that

$$\big\{\{n \in \omega\colon x \in B_n\}\colon x \in X\big\} \subset \mathcal{F}_X \text{ and } \big\{\{n \in \omega\colon y \in C_n\}\colon y \in Y\big\} \subset \mathcal{F}_Y.$$

From $\mathcal{F}_X \Subset \mathcal{F}$ we conclude that there is a finite-to-finite multifunction $\Phi : \omega \Rightarrow \omega$ such that $\Phi(\mathcal{F}_X) \subset \mathcal{F}$. Since the coherence class $[\mathcal{F}] = [\mathcal{F}_Y]$ contains no Q-point, we can apply Propositions 10.2.2 and 10.2.3 of [2] to conclude that $\mathcal{F}_Y \underset{\leftarrow}{\subset} \mathcal{F}$. Consequently, there is a finite-to-finite multifunction $\Psi : \omega \Rightarrow \omega$ such that

$$\Psi(\mathcal{F}_Y) \subset \mathcal{F} \quad \text{and} \quad \max \Psi(n) < \min \Phi\big([n, +\infty)\big) \quad \text{for all } n \in \omega.$$

It follows that $\Psi^{-1} \circ \Phi(n) \subset [n, \infty)$ for all $n \in \omega$.

Let $D_n = \bigcup_{m \in \Psi^{-1} \circ \Phi(n)} C_m$ for every $n \in \omega$. It follows from $\Psi^{-1} \circ \Phi(n) \subset [n, \infty)$ that each set D_n is v_n-bounded. Consequently, the product $B_n \times D_n$ is w_n-bounded. We claim that $\langle B_n \times D_n \rangle_{n\in\omega}$ is an $[\mathcal{F}]$-cover of $X \times Y$. Given any pair $(x, y) \in X \times Y$ it suffices to check that $\Phi(F_{(x,y)}) \in \mathcal{F}$ where $F_{(x,y)} = \{n \in \omega\colon (x, y) \in B_n \times D_n\}$. Consider the sets $F_x = \{n \in \omega\colon x \in B_n\} \in \mathcal{F}_X$ and $F_y = \{n \in \omega\colon n \in C_n\} \in \mathcal{F}_Y$.

The inclusion $\Phi(F_{(x,y)}) \in \mathcal{F}$ will follow as soon as we check that $\Phi(F_x) \cap \Psi(F_y) \subset \Phi(F_{(x,y)})$. Take any number $m \in \Phi(F_x) \cap \Psi(F_y)$ and find points $n \in \Phi^{-1}(m) \cap F_x$ and $k \in \Psi^{-1}(m) \cap F_y$. It follows that $k \in \Psi^{-1} \circ \Phi(n)$ and thus $y \in C_k \subset D_n$. Consequently, $(x, y) \in B_n \times D_n$ and hence $n \in F_{(x,y)}$. Now we see that $m \in \Phi(n) \subset \Phi(F_{(x,y)})$ and thus $\Phi(F_x) \cap \Psi(F_y) \subset \Phi(F_{(x,y)})$. This completes the proof of the $[\mathcal{F}]$-Menger property of the left semi-direct product $X \ltimes Y$. By Corollary 3.3, the space $X \ltimes Y$ has the $\mathcal{F}_{\downarrow}$-Menger property.

The same property of the right semi-direct product $X \rtimes Y$ can be proved by analogy. □

4. (Semi-)multicovered groupoids

In this section we shall introduce a new mathematical object – a semi-multicovered groupoid – in which algebraic and multicover structures are connected.

By a *groupoid* we shall understand a set X endowed with one binary operation $\cdot : X \times X \to X$ and one unary operation $(\cdot)^{-1} : X \to X$.

Note that a groupoid X is a group if the binary operation is associative, X has a two-sided unit 1 and for every $x \in X$ we get $xx^{-1} = 1 = x^{-1}x$. However, the notion of a groupoid is much more general than one can expect. For example, each set X endowed with a binary operation $\cdot : X \times X \to X$ can be thought of as a groupoid whose unary operation $(\cdot)^{-1} : X \to X$ is the identity.

A subset A of a groupoid X is called a *sub-groupoid* of X if $x \cdot y \in A$ and $x^{-1} \in A$ for all $x, y \in A$. We say that a groupoid X is *algebraically generated* by a subset $A \subset X$ if X coincides with the smallest sub-groupoid of X that contains the subset A.

By a *multicovered groupoid* (resp. *semi-multicovered groupoid*) we understand a groupoid X endowed with a centered multicover μ such that the unary operation $(\cdot)^{-1} : X \to X$ of X is uniformly bounded and the binary operation $\cdot : X \times X \to X$ of X is uniformly bounded as a function from $(X \times X, \mu \boxtimes \mu)$ (resp. $(X \times X, \mu \bowtie \mu)$) to (X, μ). It is clear that each multicovered groupoid is a semi-multicovered groupoid.

Topological groups endowed with the two-sided multicover are typical examples of semi-multicovered groupoids.

Proposition 4.1. *A topological group G endowed with the two-sided multicover $\mu_{L\wedge R}$ is a semi-multicovered groupoid. If G is commutative, then $(G, \mu_{L\wedge R})$ is a multicovered groupoid.*

Proof. The second ("commutative") part is fairly easy. Therefore we give the proof only of the first part.

The uniform boundedness of the inversion operation $(\cdot)^{-1} : G \to G$ trivially follows from the equality

$$(xU \cap Ux)^{-1} = U^{-1}x^{-1} \cap x^{-1}U^{-1}$$

holding for any point $x \in X$ and any neighborhood $U \subset G$ of the unit $1 \in G$.

To prove the uniform boundedness of the binary operation $\cdot : G \bowtie G \to G$, we first prove that this operation is uniformly bounded as a map from $(G \times G, \mu_L \rtimes \mu_L)$ to (G, μ_L).

Given any cover $u = \{xU\colon\ x \in G\} \in \mu_L$, find an open neighborhood U_1 of 1 such that $U_1 \cdot U_1 \subset U$. Given any $x \in G$, find an open neighborhood V_x of 1 such that $V_x \cdot x \subset U_1$. Set $u_1 = \{xU_1\colon\ x \in G\} \in \mu_L$ and for every $x \in G$ set $v_x = \{y \cdot V_x\colon\ y \in G\} \in \mu_L$. It follows from the above that if $A \subset G \times G$ is $\{yV_x \times xU_1\colon\ x, y \in G\}$-bounded, then $\cdot(A) = \{a \cdot b\colon\ (a, b) \in A\}$ is u-bounded. Since u was chosen arbitrary, $\cdot : (G, \mu_L) \rtimes (G, \mu_L) \to (G, \mu_L)$ is uniformly bounded. In the same way we can prove that $\cdot : (G, \mu_R) \ltimes (G, \mu_R) \to (G, \mu_R)$ is uniformly bounded, and hence $\cdot : G \times G \to G$ is uniformly bounded with respect to the multicovers $(\mu_L \rtimes \mu_L) \wedge (\mu_R \ltimes \mu_R)$ and $\mu_L \wedge \mu_R$. Since the identity maps

$$\mathrm{id} : (G \times G, \mu_{L\wedge R} \bowtie \mu_{L\wedge R}) \to \big(G \times G, (\mu_R \rtimes \mu_R) \wedge (\mu_L \ltimes \mu_L)\big)$$

and $\mathrm{id} : \mu_L \wedge \mu_R \to \mu_{L\wedge R}$ are uniformly bounded and the composition of two uniformly bounded maps is uniformly bounded, the map

$$\cdot : (G \times G, \mu_{L\wedge R} \bowtie \mu_{L\wedge R}) \to (G, \mu_{L\wedge R})$$

is uniformly bounded as well, which finishes our proof. □

5. The F-Menger property in (semi-)multicovered groupoids

In this section we characterize the F-Menger property in (semi-)multicovered groupoids. We start with the ω-boundedness of groupoids. For topological groups the following proposition was proved by I. Guran [6].

Proposition 5.1. *A semi-multicovered groupoid X is ω-bounded if and only if it is algebraically generated by an ω-bounded subspace $A \subset X$.*

Proof. The "only if" part is trivial. To prove the "if" part, assume that X is algebraically generated by an ω-bounded subspace $A \subset X$. Let $A_0 = A$ and $A_{n+1} = A_n \cup A_n^{-1} \cup (A_n \cdot A_n)$ for $n \in \omega$. By induction we shall show that each subspace $A_n \subset X$ is ω-bounded.

This is clear for $n = 0$. Suppose that for some $n \in \omega$ the subspace A_n is ω-bounded. The set $A_n^{-1} = \{a^{-1}\colon a \in A_n\}$ is ω-bounded, being the image of the ω-bounded multicovered space A under the uniformly bounded map $(\cdot)^{-1} : X \to X$. By Proposition 2.5, the semi-direct product $A_n \bowtie A_n$ is ω-bounded and so is its image $A_n \cdot A_n \subset X$ under the uniformly bounded binary operation $\cdot : X \bowtie X \to X$. Consequently, the subspace $A_{n+1} = A_n \cup A_n^{-1} \cup (A_n \cdot A_n)$ is ω-bounded, being the finite union of ω-bounded subspaces.

Now we see that the space X is ω-bounded, being a union of ω-bounded subspaces A_n, $n \in \omega$. $\square$

The following two theorems characterize the F-Menger property in (semi-)multicovered groupoids.

Theorem 5.2. *Let $\mathsf{F} = \mathsf{F}_\downarrow$ be a family of free filters on ω. A multicovered groupoid X is F-Menger if and only if it is algebraically generated by an F-Menger subspace $A \subset X$.*

Proof. The "only if" part is obvious. To prove the "if" part, suppose that some F-Menger subspace A of X algebraically generates X. First assume that the multicovered space X has countable cofinality. In this case we can apply Lemma 3.2 to conclude that the multicovered space A is $\mathcal{F}_\downarrow$-Menger for some filter $\mathcal{F} \in \mathsf{F}$. Let $A_0 = A$ and $A_{n+1} = A_n \cup A_n^{-1} \cup (A_n \cdot A_n)$ for $n \in \omega$. By induction we shall show that for every $n \in \omega$ the multicovered subspace A_n of X is $\mathcal{F}_\downarrow$-Menger.

This is clear for $n = 0$. Assuming that for some n the space A_n is $\mathcal{F}_\downarrow$-Menger, we can apply Theorem 3.4(3) and the uniform boundedness of the inversion $(\cdot)^{-1} : X \to X$ to conclude that the subset A_n^{-1} of X is $\mathcal{F}_\downarrow$-Menger. Since the multicovered space A_n is $\mathcal{F}_\downarrow$-Menger, we can apply Theorem 3.4(4) and conclude that the direct product $A_n \times A_n$ is $\mathcal{F}_\downarrow$-Menger. Now the uniform boundedness of the multiplication $\cdot : X \times X \to X$ implies that the image $A_n \cdot A_n$ of $A_n \times A_n$ is $\mathcal{F}_\downarrow$-Menger and so is the union $A_{n+1} = A_n \cup A_n^{-1} \cup (A_n \cdot A_n)$ according to Theorem 3.4(2).

By Theorem 3.4(2), the union $\bigcup_{n\in\omega} A_n$ is $\mathcal{F}_\downarrow$-Menger. The latter union coincides with X because X is algebraically generated by A. Therefore the multicovered space X is $\mathcal{F}_\downarrow$-Menger. Since $\mathcal{F}_\downarrow \subset \mathsf{F}_\downarrow = \mathsf{F}$, the space X is F-Menger.

Now we consider the general case (of an arbitrary cofinality of X). Let μ be the multicover of X. To prove that (X, μ) is F-Menger, take any sequence of covers $\langle u_n\rangle_{n\in\omega} \in \mu^\omega$. Using the uniform boundedness of the multiplication and inversion it is easy to find a countable centered subfamily $\mu' \subset \mu$ such that $\{u_n\}_{n\in\omega} \subset \mu'$ and the operations $\cdot : (X \times X, \mu' \times \mu') \to (X, \mu')$, $(\cdot)^{-1} : (X, \mu') \to (X, \mu')$ are uniformly bounded.

Since the identity map $\mathrm{id} : (X, \mu) \to (X, \mu')$ is uniformly bounded, the F-Menger property of $(A, \mu{\restriction}A)$ implies the F-property of $(A, \mu'{\restriction}A)$. Since the multicovered space (X, μ') has countable cofinality, the preceding case guarantees that (X, μ') is F-Menger because the multicovered groupoid (X, μ') is algebraically generated by the F-Menger subspace $A \subset (X, \mu')$. Consequently, for the sequence $\langle u_n\rangle_{n\in\omega} \in (\mu')^\omega$ there is an F-cover $\langle B_n\rangle_{n\in\omega}$ by u_n-bounded subsets $B_n \subset X$. This proves that the multicovered space (X, μ) is F-Menger. $\square$

A similar result also holds for semi-multicovered groupoids.

Theorem 5.3. *Assume that a family $\mathsf{F} = \mathsf{F}_\downarrow$ of free ultrafilters on ω contains no Q-point. A semi-multicovered groupoid X is F-Menger if X is algebraically generated by an F-Menger subspace $A \subset X$ and one of the following conditions holds*:

(1) $\mathsf{F} \subset [\mathcal{F}]$ *for some* $\mathcal{F} \in \mathsf{F}$;
(2) *The multicovered space X has countable cofinality*;
(3) *The multicovered space X is properly ω-bounded*;
(4) *The multicovered space X is uniformizable.*

Proof. (1) Assume that $\mathsf{F} \subset [\mathcal{F}]$ for some $\mathcal{F} \in \mathsf{F}$. Replacing the direct products in the proof of Theorem 5.2 by semi-direct products and applying Theorem 3.6 instead of Theorem 3.4(4) we can show that X is F-Menger.

(2) If the multicovered space X has countable cofinality, then by Lemma 3.2 the F-Menger property of A implies the $\mathcal{F}_\downarrow$-Menger property of A for some ultrafilter $\mathcal{F} \in \mathsf{F}$. By the first case, the multicovered space X is $\mathcal{F}_\downarrow$-Menger and consequently F-Menger.

(3) If the multicovered space X is properly ω-bounded, then we can repeat the argument in the proof of Theorem 5.2, replacing direct products with semi-direct products everywhere and applying Theorem 3.6 instead of Theorem 3.4(4). The existence of μ', which was straightforward for direct products, now follows from Lemma 2.6.

(4) Assume that the multicovered space X is uniformizable. The subspace $A \subset X$, being F-Menger, is ω-bounded. By Propositions 5.1 and 2.3, the uniformizable space X is (properly) ω-bounded. Now the previous case completes the proof. $\square$

Theorems 5.2, 5.3 and Proposition 3.1 imply the following characterization of the Scheepers property in (semi-)multicovered groupoids.

Corollary 5.4. *A multicovered groupoid X is Scheepers if and only if X is algebraically generated by a Scheepers subspace $A \subset X$.*

Corollary 5.5. *If no Q-point exists, then a semi-multicovered groupoid X is Scheepers provided X is algebraically generated by a Scheepers subspace $A \subset X$ and one of the following conditions holds*:

(1) *Any two ultrafilters are coherent*;
(2) *The multicovered space X has countable cofinality*;
(3) *The multicovered space X is properly ω-bounded*;
(4) *The multicovered space X is uniformizable.*

Note that the first condition of Corollary 5.5 is nothing else but the NCF principle. Recall that no Q-points exist under NCF.

6. The F-Menger property in topological groups

In this section we apply the general results proved in the preceeding section to studying the F-Menger property in topological groups. According to Proposition 4.1, each topological group G endowed with the two-sided multicover $\mu_{L\wedge R}$ is a semi-multicovered groupoid. Now we can apply Corollaries 5.4 and 5.5 in order to prove:

Corollary 6.1. *Let $\mathsf{F} = \mathsf{F}_{\downarrow}$ be a family of free filters on ω. An abelian topological group G endowed with the two-sided multicover $\mu_{L\wedge R}$ is F-Menger if and only if the group G is algebraically generated by an F-Menger subspace $A \subset G$.*

Corollary 6.2. *Assume that a family $\mathsf{F} = \mathsf{F}_{\downarrow}$ of free ultrafilters on ω contains no Q-points. A topological group G endowed with the multicover $\mu_{L\wedge R}$ is F-Menger if and only if the group G is algebraically generated by an F-Menger subspace $A \subset G$.*

Corollary 6.3. *If no Q-point exists, then a topological group G endowed with the multicover $\mu_{L\wedge R}$ is Scheepers if and only if the group G is algebraically generated by a Scheepers subspace $A \subset G$.*

The Q-point assumption cannot be removed from Theorems 3.6, 5.3 and Corollaries 5.5, 6.2, and 6.3. To construct a suitable counterexample, consider the homeomorphism group $H(\mathbb{R}_+)$ of the half-line $\mathbb{R}_+ = [0,\infty)$, endowed with the compact-open topology. This group will be considered as a semi-multicovered groupoid endowed with the two-sided multicover $\mu_{L\wedge R}$.

Theorem 6.4.

(1) *The semi-multicovered groupoid $H(\mathbb{R}_+)$ is not Menger and hence not Scheepers.*
(2) *For every Q-point $\mathcal{F}$ there is an $\mathcal{L}_{\downarrow}$-Menger subspace $A \subset (H(\mathbb{R}_+), \mu_{L\wedge R})$ generating the group $H(\mathbb{R}_+)$ in the sense that $H(\mathbb{R}_+) = A \circ A \circ A \circ A$.*

Proof. We recall that the compact-open topology on $H = H(\mathbb{R}_+)$ is generated by the base consisting of sets of the form

$$[f,n] = \bigl\{g \in H\colon \forall x \leqslant n\colon \bigl(\bigl|f(x) - g(x)\bigr| < 2^{-n}\bigr)\bigr\},$$

where $f \in H$ and $n \in \omega$. Let $U_n = [\mathrm{id}_{\mathbb{R}_+}, n]$. The family $\{U_n\colon n \in \omega\}$ is a local base at the neutral element $\mathrm{id}_{\mathbb{R}_+}$ of H and $U_{n+1}^2 \subset U_n$.

(1) The failure of the Menger property of the multicovered spaces $(H, \mu_{L\wedge R})$ and (H, μ_L) will follow as soon as we show that $H \neq \bigcup_{n\in\omega} F_n \circ U_n$ for every sequence $\langle F_n\rangle_{n\in\omega}$ of finite subsets of H. Given such a sequence, put $a_n = \max\{f(n+1)\colon f \in F_n\} + 1$. For every $n \in \omega$ and $(f,g) \in F_n \times U_n$ we have $f(g(n)) \leqslant f(n+1) < a_n$. Let $h \in H$ be the piecewise linear map such that $h(n) = a_n$. It follows that $h \notin \bigcup_{n\in\omega} F_n \circ U_n$.

(2) In order to prove the second assertion it suffices to find a sequence $\langle K_n\rangle_{n\in\omega}$ of finite subsets of H such that the set

$$\Bigl(\bigcup_{L\in\mathcal{L}}\bigcap_{n\in L} U_n K_n\Bigr) \bigcap \Bigl(\bigcup_{L\in\mathcal{L}}\bigcap_{n\in L} U_n K_n\Bigr)^{-1}$$

generates H. Let us consider a sequence $\langle K_n\colon n \in \omega\rangle$ of finite subsets of H such that $K_n = K_n^{-1}$ and $W_n := \{f \in H\colon \forall x \leqslant n\ (\frac{1}{3}x < f(x) < 3x)\} \subset K_n U_n^{-1} \cap U_n K_n$ for all $n \in \omega$. We claim that this sequence is as required. Indeed, let us note that

$$\Bigl(\bigcup_{L\in\mathcal{L}}\bigcap_{n\in L} U_n K_n\Bigr) \bigcap \Bigl(\bigcup_{L\in\mathcal{L}}\bigcap_{n\in L} U_n K_n\Bigr)^{-1} = \bigcup_{L\in\mathcal{L}}\bigcap_{n\in L} U_n K_n \cap K_n U_n^{-1} \supset \bigcup_{L\in\mathcal{L}}\bigcap_{n\in L} W_n.$$

Let us fix $h \in H$ such that $h(x) > x$ for all $x > 0$. Since $\mathcal{L}$ is a Q-point, there exists $L = \{l_i\colon i \in \omega\} \in \mathcal{L}$ such that $l_0 > 3$ and $l_{i+1} > 2h(l_i + 1)$ for all $i \in \omega$. Now let ϕ be the piecewise linear function such that $\phi(l_i) = 3l_i - 1/2$ and $\phi(l_i + 1) = 3l_{i+1} - 1$ for all i. It is clear that $\phi \in \bigcap_{n\in L} W_n$. Then $h(x) < (\phi \circ \phi)(x)$ for every $x > 0$. Indeed, fix $x > 0$. The following cases are possible.

(a) $x \in [l_i, l_i + 1)$ for some i. Then $\phi(x) \geqslant 3l_i - 1/2 > l_i + 1$, hence $\phi(\phi(x)) > \phi(l_i + 1) \geqslant 3l_{i+1} - 1 > l_{i+1}$, and consequently $\phi(\phi(x)) > h(l_i + 1) > h(x)$.

(b) $x \in [l_i + 1, l_{i+1})$. Then $\phi(x) \geqslant 3l_{i+1} - 1/2 > l_{i+1} + 1$, and hence $\phi(\phi(x)) > \phi(l_{i+1} + 1) = 3l_{i+2} - 1 > l_{i+2} > h(l_{i+1}) > h(x)$.

Thus $h(x) < \phi(\phi(x))$ for every x. Consequently $\phi^{-1}(x) < (\phi^{-1} \circ h)(x) < \phi(x)$, which means that $\phi^{-1} \circ h \in \bigcap_{n\in L} W_n$, and hence $h = \phi \circ \phi^{-1} \circ h \in (\bigcap_{n\in L} W_n) \circ (\bigcap_{n\in L} W_n)$. It suffices to note that the set of all $h \in H$ such that $h(x) > x$ for all $x > 0$ generates H. $\square$

7. Topological monoids and (semi-)multicovered binoids

Looking at the results of the preceding section the reader can notice that among four canonical multicovers μ_L, μ_R, $\mu_{L\wedge R}$, $\mu_{L\vee R}$ on topological groups, we distinguished one: the two-sided multicover $\mu_{L\wedge R}$. The reason is that the inversion operation is uniformly bounded with respect to this multicover. This is not true anymore for the multicovers μ_L and μ_R. To treat topological groups endowed with those multicovers we will simply forget about the inversion operation and think of groups as sets with a binary operation. Such algebraic structures are called *binoids*.

More precisely, a *binoid* is a set endowed with a binary operation $\cdot : X \times X \to X$. If this operation is associative, then X is called a *semigroup*. A semigroup with a two-sided unit 1 is called a *monoid*. It is clear that each group is a monoid. Each binoid X, endowed with the identity unary operation

$$(\cdot)^{-1} : X \to X, \qquad (\cdot)^{-1} : x \mapsto x,$$

becomes a groupoid. So all the results about groupoids concern also binoids.

By a *multicovered binoid* (resp. *semi-multicovered binoid*) we understand a binoid X endowed with a multicover μ making the binary operation $\cdot : X \times X \to (X, \mu)$ of X uniformly bounded with respect to the multicover $\mu \boxtimes \mu$ (resp. $\mu \bowtie \mu$) on $X \times X$. Each (semi-)multicovered binoid can be thought of as a (semi-)multicovered groupoid with the identity unary operation.

Let us now return to topological groups and observe that they are examples of topological monoids. By a *topological monoid* we understand a monoid X endowed with a topology τ making the binary operation $\cdot : X \times X \to X$ of X continuous. The four multicovers considered earlier on topological groups can be equally defined on topological monoids.

Namely, for each topological monoid (X, τ) we can consider:

- the left multicover $\mu_L = \{\{xU\colon x \in X\}\colon 1 \in U \in \tau\}$;
- the right multicover $\mu_R = \{\{Ux\colon x \in X\}\colon 1 \in U \in \tau\}$;
- the two-sided multicover $\mu_{L\wedge R} = \{\{Ux \cap xU\colon x \in X\}\colon 1 \in U \in \tau\}$;
- the Rölke multicover $\mu_{L\vee R} = \{\{UxU\colon x \in X\}\colon 1 \in U \in \tau\}$;
- the multicover $\mu_L \wedge \mu_R$.

Let us observe that each left shift $l_a : X \to X$, $l_a : x \mapsto ax$, is uniformly bounded with respect to the left multicover μ_L. Similarly, all the right shifts are uniformly bounded with respect to the right multicover μ_R.

For each commutative topological monoid the multicovers μ_L, μ_R and $\mu_{L\wedge R}$ coincide while $\mu_{L\vee R}$ is equivalent to $\mu_{L\wedge R}$ and to $\mu_L \wedge \mu_R$. For topological monoids the multicover $\mu_L \wedge \mu_R$ is more important than $\mu_{L\wedge R}$ (which is equivalent to $\mu_L \wedge \mu_R$ in topological groups).

If (X, τ) is a topological group then those multicovers are generated by suitable uniformities on X. The same is true for ω-bounded topological monoids.

Proposition 7.1. *Let X be a topological monoid endowed with a multicover $\mu \in \{\mu_L, \mu_R, \mu_L \wedge \mu_R, \mu_{L\wedge R}, \mu_{L\vee R}\}$. If the multicovered space (X, μ) is ω-bounded, then it is properly ω-bounded and hence is uniformizable.*

Proof. Assume that the multicovered space (X, μ_L) is ω-bounded. To prove that it is properly ω-bounded, fix any cover $u \in \mu_L$ and find a neighborhood $U \subset X$ of the unit $1 \in X$ such that $u = \{xU\colon x \in X\}$. By the continuity of the semigroup operation at 1, there is a neighborhood $V \subset U$ of 1 such that $VV \subset U$. Consider the cover $v = \{xV\colon x \in X\} \in \mu_L$. By the ω-boundedness of the multicover μ_L there is a countable subset $C \subset X$ such that $X = CV$. It follows that $u' = \{cU\colon c \in C\}$ is a countable subcover of u. We claim that each v-bounded subset of X is u'-bounded. It suffices to check that for every $x \in X$ the set xV is u'-bounded. Since $CV = X$, there is a point $c \in C$ such that $x \in cV$. Then $xV \subset cVV \subset cU \in u'$. Thus the multicover μ_L is properly ω-bounded and, being centered, is uniformizable according to Proposition 2.3.

The proof of the fact that the ω-boundedness of μ_R implies its proper ω-boundedness is completely analogous.

If the multicover $\mu_L \wedge \mu_R$ is ω-bounded, then the multicovers μ_L and μ_R are (properly) ω-bounded and so is their meet $\mu_L \wedge \mu_R$.

Concerning the multicovers $\mu_{L\wedge R}$ and $\mu_{L\vee R}$, one just has to make a minor changes in the proof above. We demonstrate this on $\mu_{L\wedge R}$. Suppose that $\mu_{L\wedge R}$ is ω-bounded. To prove that it is properly ω-bounded, fix any cover $u\in\mu_{L\wedge R}$ and find a neighborhood $U\subset X$ of the unit $1\in X$ such that $u=\{xU\cap Ux\colon x\in X\}$. By the continuity of the semigroup operation at 1, there is a neighborhood $V\subset U$ of 1 such that $VV\subset U$. Consider the cover $v=\{xV\cap Vx\colon x\in X\}\in\mu_{L\wedge R}$. By the ω-boundedness of the multicover $\mu_{L\wedge R}$ there is a countable subset $C\subset X$ such that $X=\bigcup_{c\in C}cV\cap Vc$. It follows that $u'=\{cU\cap Uc\colon c\in C\}$ is a countable subcover of u. We claim that each v-bounded subset of X is u'-bounded. It suffices to check that for every $x\in X$ the set $xV\cap Vx$ is u'-bounded. Since $X=\bigcup_{c\in C}cV\cap Vc$, there is a point $c\in C$ such that $x\in cV\cap Vc$. Then $xV\subset cVV\subset cU\in u'$ and $Vx\subset VVc\subset Uc\in u'$, and consequently $xV\cap Vx\subset cU\cap Uc$, and hence is u'-bounded. □

Now we shall detect multicovers on topological monoids turning them into (semi-)multicovered binoids.

Proposition 7.2. *Each commutative topological monoid X endowed with any of the equivalent multicovers μ_L, μ_R, $\mu_L\wedge\mu_R$, $\mu_{L\wedge R}$ or $\mu_{L\vee R}$ is a multicovered binoid.*

Proof. The commutativity of X easily implies that all of these multicovers are equivalent. Therefore it is enough to check that (X,μ_L) is a multicovered binoid. Given $u\in\mu_L$, write it in the form $\{xU\colon x\in X\}$ for some open neighborhood U of 1, and find $V\ni 1$ such that $V\cdot V\subset U$. Set $v=\{xV\colon x\in X\}\in\mu_L$. In order to prove that the operation $\cdot:(X\times X,\mu_L\boxtimes\mu_L)\to(X,\mu_L)$ is uniformly bounded we need to show that $(xV)\cdot(yV)$ is u-bounded for arbitrary $x,y\in X$. Since X is commutative, $xVyV=xyVV\subset xyU\in u$, which finishes our proof. □

For non-commutative topological monoids the situation is a bit more complicated. Let us define a point $x\in X$ of a topological monoid X to be *left balanced* (resp. *right balanced*) if for every neighborhood $U\subset X$ of the unit 1 of X there is a neighborhood $V\subset X$ of 1 such that $Vx\subset xU$ (resp. $xV\subset Ux$). Observe that x is left balanced if the left shift $l_x:X\to X$, $l_x:y\mapsto xy$, is open at 1.

Let B_L and B_R denote respectively the sets of all left and right balanced points of the monoid X.

A topological monoid X is defined to be *left balanced* (resp. *right balanced*) if $X=B_L\cdot U$ (resp. $X=U\cdot B_R$) for every neighborhood $U\subset X$ of the unit 1 in X. If a topological monoid X is left and right balanced, then we say that X is *balanced*.

Observe that the class of balanced topological monoids includes all commutative topological monoids.

Proposition 7.3. *If a topological monoid X is balanced* (*resp. left balanced, right balanced*)*, then X endowed with the multicover $\mu_L\wedge\mu_R$ (resp. μ_L, μ_R) is a semi-multicovered binoid.*

Proof. Assume that a topological monoid X is left balanced and let B_L be the set of left balanced points of X. We shall show that the semigroup operation $\cdot:(X\times X,\mu_L\rtimes\mu_L)\to(X,\mu_L)$ is uniformly bounded.

Given any cover $u\in\mu_L$, find a neighborhood $U_0\subset X$ of the unit 1 of X such that $u=\{xU_0\colon x\in X\}$. By the continuity of the operation at 1, there is a neighborhood $W\subset X$ of 1 such that $WWW\subset U_0$. The monoid X, being left balanced, is equals to $B_L\cdot W$. So, for every $y\in X$ we can find a left balanced point $b_y\in B_L$ such that $y\in b_yW$. The left balanced property of b_y allows us to find a neighborhood $W_y\subset X$ of 1 such that $W_yb_y\subset b_yW$.

Now consider the cover $v=\{yW\colon y\in X\}$ and for every set $V\in v$ find a point $y\in X$ with $V=yW$ and consider the cover $u_V=\{xW_y\colon x\in X\}\in\mu_L$. Set $w=\{U\times V\colon V\in v,\ U\in u_V\}\in\mu_L\rtimes\mu_L$. The uniform boundedness of the operation $\cdot:(X\times X,\mu_L\rtimes\mu_L)\to(X,\mu_L)$ will follow as soon as we show that for every set $U\times V\in w$ the set $U\cdot V$ is u-bounded. Find $y\in X$ with $V=yW$ and $x\in X$ with $U=xW_y$. Now we see that

$$U\cdot V=xW_yyW\subset xW_yb_yWW\subset xb_yWWW\subset xb_yU_0\in u.$$

By analogy we can prove that if the topological monoid X is right balanced, then X endowed with the multicover μ_R is a semi-multicovered binoid.

Now assume that the topological monoid X is balanced. Since X is both left and right balanced, the maps $\cdot:(X\times X,\mu_L\bowtie\mu_L)\to(X,\mu_L)$ and $\cdot:(X\times X,\mu_R\bowtie\mu_R)\to(X,\mu_R)$ are uniformly bounded, and hence so is the map

$$\cdot:\big(X\times X,(\mu_L\bowtie\mu_L)\wedge(\mu_R\bowtie\mu_R)\big)\to(X,\mu_L\wedge\mu_R).$$

Taking into account that the identity map

$$\mathrm{id}:\big(X\times X,(\mu_L\wedge\mu_R)\bowtie(\mu_L\wedge\mu_R)\big)\to\big(X\times X,(\mu_L\bowtie\mu_L)\wedge(\mu_R\bowtie\mu_R)\big)$$

is uniformly bounded, we see that so is the map

$$\cdot:\big(X\times X,(\mu_L\wedge\mu_R)\bowtie(\mu_L\wedge\mu_R)\big)\to(X,\mu_L\wedge\mu_R). \qquad \square$$

A typical example of a topological monoid is the semigroup ω^ω of all maps from ω to ω endowed with the Tychonov product topology. The semigroup operation on ω^ω is given by the composition of functions.

Besides the five multicovers generated by the algebraic structure, the monoid ω^ω carries the product multicover μ_p on ω^ω. It consists of the uniform covers $u_n = \{[s]\colon s \in \omega^n\}$, $n \in \omega$, where $[s] = \{y \in \omega^\omega\colon y{\restriction}n = s\}$ (we identify n with $\{0, \dots, n-1\}$ and write $f[n]$ for $\{f(0), \dots, f(n-1)\}$). Observe that the multicover μ_p coincides with the multicover μ_ρ generated by the (standard) metric $\rho(x, y) = \inf\{2^{-n}\colon x(n) = y(n)\}$ on ω^ω.

The following proposition characterizes left and right balanced points in the topological monoid ω^ω. We define a function $f : \omega \to \omega$ to be *eventually injective* if the restriction $f{\restriction}(\omega \setminus n)$ is injective for some $n \in \omega$.

Proposition 7.4. *An element $f \in \omega^\omega$ of the topological monoid ω^ω is*

(1) *left balanced if and only if the function f is bounded or surjective*;
(2) *right balanced if and only if f is constant or eventually injective.*

Proof. (1) To prove the "if" part of the first assertion, consider two cases.

(1a) Suppose that f is bounded and fix $m \in \omega$. Set $M = \max \operatorname{rng}(f) + 1$. We claim that $[\mathrm{id}{\restriction}M] \circ f \subset f \circ [\mathrm{id}{\restriction}m]$. Indeed, a direct verification shows that $[\mathrm{id}{\restriction}M] \circ f = \{f\} \subset f \circ [\mathrm{id}{\restriction}m]$.

(1b) f is surjective. Let us fix $m \in \omega$. We claim that $[\mathrm{id}{\restriction}f(m)] \circ f \subset f \circ [\mathrm{id}{\restriction}m]$. Indeed, fix $g \in [\mathrm{id}{\restriction}f(m)]$, for every $k \geqslant m$ find $h(k)$ such that $f(h(k)) = g(f(k))$ (it exists by the surjectivity of f), and set $h{\restriction}m = \mathrm{id}{\restriction}m$. Then $g \circ f = f \circ h$ and $h \in [\mathrm{id}{\restriction}m]$, and consequently $g \circ f \in f \circ [\mathrm{id}{\restriction}m]$, which means that f is left balanced.

To prove the "only if" part suppose that f is unbounded and there exists $p \in \omega \setminus \operatorname{rng}(f)$. We claim that $[\mathrm{id}{\restriction}M] \circ f \not\subset f \circ \omega^\omega$ for all $M \in \omega$. Indeed, given M find $n \in \omega$ such that $f(n) \geqslant M$ and set $g(f(n)) = p$ and $g{\restriction}(\omega \setminus \{f(n)\}) = \mathrm{id}$. Then $p \in \operatorname{rng}(g \circ f)$ but $p \notin \operatorname{rng}(f \circ h)$ for all h, and consequently $g \circ f \notin f \circ \omega^\omega$, which finishes our proof.

(2) To prove the "if" part of the second assertion consider two cases:

(2a) $f \in \omega^\omega$ is constant, i.e. there exists $n_0 \in \omega$ such that $f(n) = n_0$ for all n. Let us fix $m \in \omega$ and set $M = m$. We claim that $f \circ [\mathrm{id}{\restriction}M] \subset [\mathrm{id}{\restriction}m] \circ f$. Indeed, for every $g \in [\mathrm{id}{\restriction}M]$ and $n \in \omega$ we have $(f \circ g)(n) = n_0 = f(n)$, and hence $f \circ g = f = \mathrm{id} \circ f \in [\mathrm{id}{\restriction}m] \circ f$, and consequently f is left balanced.

(2b) f is eventually injective. Let us fix $m \in \omega$ and find $M \geqslant m$ such that

$$\big(m \cup f(M)\big) \cap f\big([M, +\infty)\big) = \emptyset.$$

We claim that $f \circ [\mathrm{id}{\restriction}M] \subset [\mathrm{id}{\restriction}m] \circ f$. Fix $g \in [\mathrm{id}{\restriction}M]$ and for every $k \geqslant M$ set $h(f(k)) = f(g(k))$. Since $f{\restriction}[M, +\infty)$ is injective, this well defines a function $h : f([M, +\infty)) \to \omega$. Extend h to the function on ω by letting $h(l) = l$ for all $l \notin f([M, +\infty))$. The choice of M guarantees that $h \in [\mathrm{id}{\restriction}m]$. A direct verification shows that $f \circ g = h \circ f$, which finishes the proof that f is right balanced.

To prove the "only if" part suppose that f is not constant and for every $n \in \omega$ there are distinct $m, l \geqslant n$ such that $f(l) = f(m)$. Let us fix $p, q \in \omega$ such that $f(p) \neq f(q)$. We claim that $f \circ [\mathrm{id}{\restriction}M] \not\subset \omega^\omega \circ f$ for all $M \in \omega$. Given $M \in \omega$, find distinct $r, l \geqslant M$ such that $f(l) = f(r)$ and set $g(r) = p$, $g(l) = q$, and $g(n) = n$ otherwise. Then $(f \circ g)(r) \neq (f \circ g)(l)$, while $(h \circ f)(r) = (h \circ f)(l)$ for all $h \in \omega^\omega$, and consequently $f \circ g \notin \omega^\omega \circ f$, which finishes the proof. □

We are now able to prove that the topological monoid ω^ω is balanced. We recall that a multicovered space (X, μ) is called *totally bounded* if X is u-bounded for every cover $u \in \mu$. In this case we also say that the multicover μ is totally bounded. It is clear that any two totally bounded multicovers on a set X are equivalent.

Proposition 7.5.

(1) *The topological monoid ω^ω is balanced.*
(2) *The multicover μ_R is totally bounded.*
(3) *The multicovers μ_L and $\mu_L \wedge \mu_R$ are equivalent to the product multicover μ_p.*

Proof. (1) Fix $m \in \omega$ and $x \in \omega^\omega$.

In order to prove that ω^ω is left balanced we need to find a left balanced $y \in \omega^\omega$ and $g \in [\mathrm{id}{\restriction}m]$ such that $x = y \circ g$. Let g be any injection with $|\omega \setminus \operatorname{rng}(g)| = \omega$ and $g{\restriction}m = \mathrm{id}{\restriction}m$. Define $y{\restriction}\operatorname{rng} g$ by $y(g(k)) = x(k)$ (the correctness follows from the injectivity of g) and extend y onto ω in such a way that $\operatorname{rng}(y{\restriction}\omega \setminus \operatorname{rng}(g)) = \omega$. Being surjective, y is a left balanced according to Proposition 7.4, $g \in [\mathrm{id}{\restriction}m]$, and $x = y \circ g$.

Next, we prove that ω^ω is right balanced. We need to find a right balanced $y \in \omega^\omega$ and $g \in [\mathrm{id}{\restriction}m]$ such that $x = g \circ y$. Set $y{\restriction}m = x{\restriction}m$, $F = m \cup x[m]$, and $g{\restriction}F = \mathrm{id}$. Let $y{\restriction}[m, +\infty)$ be an injection into $\omega \setminus F$ and $g{\restriction}\omega \setminus F$ be such that $g(y(k)) = x(k)$ for all $k \geqslant m$. It follows that y is eventually injective, $g \in [\mathrm{id}{\restriction}m]$, and $x = g \circ y$. By Proposition 7.4, y is right balanced.

(2) In fact, for every n there exists $f \in \omega^\omega$ such that $[\mathrm{id}{\restriction}n] \circ f = \omega^\omega$. Indeed, any injection from ω into $[n, +\infty)$ is obviously as required.

(3) Since $x \circ [\mathrm{id}{\restriction}m] \subset [x{\restriction}m] \in u_m$, the identity map $\mathrm{id} : (\omega^\omega, \mu_L) \to (\omega^\omega, \mu_p)$ is uniformly bounded. To show that $\mathrm{id} : (\omega^\omega, \mu_p) \to (\omega^\omega, \mu_L)$ is uniformly bounded we need to prove that for every $n \in \omega$ there exists $m \in \omega$ such that $[s]$ is $\{x \circ [\mathrm{id}{\restriction}n]:\ x \in \omega^\omega\}$-bounded for all $s \in \omega^m$. We claim that $m = n$ is as required. Indeed, let us fix $s \in \omega^n$ and define $x \in \omega^\omega$ letting $x{\restriction}n = s{\restriction}n$ and extending it onto ω in such a way that $\mathrm{rng}(y{\restriction}[n, +\infty)) = \omega$. Given $y \in [s]$, for every $k \geqslant m$ find $h(k)$ such that $y(k) = x(h(k))$ (this is possible by the definition of x) and set $h{\restriction}m = \mathrm{id}{\restriction}m$. It follows that $y = x \circ h \in x \circ [\mathrm{id}{\restriction}n]$. Since y was chosen arbitrary, we conclude that $[s] \subset x \circ [\mathrm{id}{\restriction}n]$, which finishes our proof of the equivalence of μ_p and μ_L.

Since μ_R is totally bounded, $\mu_L \wedge \mu_R$ is equivalent to μ_L and hence to μ_p as well. □

Other natural examples of balanced topological monoids arise from paratopological groups. We recall that a *paratopological group* is a group G endowed with a topology τ making the group operation continuous. The following statement is a direct consequence of Propositions 7.2 and 7.3.

Corollary 7.6.

1. *Each abelian paratopological group G endowed with the multicover $\mu_L \wedge \mu_R$ is a multicovered binoid.*
2. *Each paratopological group G is a balanced topological monoid and endowed with one of the multicovers μ_L, μ_R or $\mu_L \wedge \mu_R$ is a semi-multicovered binoid.*

8. The F-Menger property in topological monoids

In this section we shall characterize the F-Menger property in topological monoids. Combining Proposition 7.2 with Theorem 5.2 we get the following corollaries:

Corollary 8.1. *Let $\mathsf{F} = \mathsf{F}_\downarrow$ be a family of free filters on ω. A commutative topological monoid X endowed with the multicover $\mu_L \wedge \mu_R$ is* F*-Menger if and only if X is algebraically generated by an* F*-Menger subspace $A \subset X$.*

Corollary 8.2. *A commutative topological monoid X endowed with the multicover $\mu_L \wedge \mu_R$ is Scheepers if and only if X is algebraically generated by a Scheepers subspace $A \subset X$.*

The analogous results for non-commutative topological monoids follow from Propositions 5.1, 7.1, and 7.3 and Theorem 5.3(3).

Corollary 8.3. *Assume that a family $\mathsf{F} = \mathsf{F}_\downarrow$ of free ultrafilters on ω contains no Q-point. A balanced* (*resp. left balanced, right balanced*) *topological monoid X endowed with the multicover $\mu_L \wedge \mu_R$* (*resp. μ_L, μ_R*) *is* F*-Menger if and only if the monoid X is algebraically generated by an* F*-Menger subspace $A \subset X$.*

Corollary 8.4. *If no Q-point exists, then a balanced* (*resp. left balanced, right balanced*) *topological monoid X endowed with the multicover $\mu_L \wedge \mu_R$* (*resp. μ_L, μ_R*) *is Scheepers if and only if X is algebraically generated by a Scheepers subspace $A \subset X$.*

9. Characterizing the F-Menger property in free topological groups

We are now in a position to present the proof of Theorem 1.2, which is a special case of Corollary 9.2.

A topological space X is defined to be F-*Menger* if such is the multicovered space $(X, \mu_{\mathcal{O}})$ where $\mu_{\mathcal{O}}$ is the multicover consisting of all open covers of X. If the space X is Lindelöf, then the multicover $\mu_{\mathcal{O}}$ is equivalent to the multicover $\mu_{\mathcal{U}}$ consisting of the uniform covers with respect to the universal uniformity of X, see [18, Corollary 15]. The multicover $\mu_{\mathcal{U}}$ is equivalent to the multicover consisting of the covers $\{B_d(x):\ x \in X\}$ by 1-balls with respect to all continuous pseudometrics d on X.

Theorem 9.1. *Assume that a family $\mathsf{F} = \mathsf{F}_\downarrow$ of free ultrafilters on ω contains no Q-point. For a Tychonov space X the following conditions are equivalent*:

(1) *All continuous metrizable images of X are* F*-Menger*;
(2) *All continuous Lindelöf regular images of X are* F*-Menger*;
(3) *The multicovered space $(X, \mu_{\mathcal{U}})$ is* F*-Menger*;
(4) $(F(X), \mu_{L\wedge R})$ *is* F*-Menger*;
(5) $(F(X), \mu_L)$ *is* F*-Menger*;
(6) $(F(X), \mu_R)$ *is* F*-Menger*;
(7) $(F(X), \mu_{L\vee R})$ *is* F*-Menger*;
(8) $(A(X), \mu_{L\wedge R})$ *is* F*-Menger.*

Proof. The implications (4) $\Rightarrow$ (5), (6) $\Rightarrow$ (7) $\Rightarrow$ (8) are straightforward.

The implication (8) $\Rightarrow$ (3) follows from the fact that the restriction to X of the natural uniformity of $A(X)$ coincides with $\mathcal{U}_X$ [13] and the $\mathsf{F}_\downarrow$-Mengerness is preserved by taking subspaces, see Theorem 3.4(1).

(3) $\Rightarrow$ (2). Let $f : X \to T$ be a continuous map onto a Lindelöf space T. Then f is uniformly continuous with respect to the universal uniformities $\mathcal{U}_X$ and $\mathcal{U}_T$, and hence uniformly bounded with respect to the multicovers $\mu_{\mathcal{U}_X}$ and $\mu_{\mathcal{U}_T}$, and consequently the multicovered space $(T, \mu_{\mathcal{U}_T})$ is F-Menger by Theorem 3.4(3). Applying [18, Corollary 15], we conclude that the multicovers $\mu_{\mathcal{O}_T}$ and $\mu_{\mathcal{U}_T}$ of T are equivalent, and consequently T is F-Menger as a topological space.

The implication (2) $\Rightarrow$ (1) is trivial.

(1) $\Rightarrow$ (3). Suppose that $(X, \mu_{\mathcal{U}})$ is not F-Menger and fix a sequence $\langle U_n\rangle_{n\in\omega} \in \mathcal{U}^\omega$ such that $\langle U_n(K_n)\rangle_{n\in\omega}$ is an F-cover of X for no sequence $\langle K_n\rangle_{n\in\omega}$ of finite subsets of X. Let d be a continuous pseudometric on X such that $\{(x, y)\colon d(x, y) < 1/n\} \subset U_n$ and Y be the quotient space of X with respect to the equivalence relation $x \equiv y \leftrightarrow d(x, y) = 0$. A direct verification shows that Y is a metrizable image of X which fails to be F-Menger.

(3) $\Rightarrow$ (4). Assume that the multicovered space $(X, \mu_{\mathcal{U}})$ is F-Menger. Consider the canonical embedding $i : X \to F(X)$ and observe that it is uniformly bounded as a map from $(X, \mu_{\mathcal{U}})$ into $(F(X), \mu_{L\wedge R})$. By Theorem 3.4(3), the image $i(X) \subset F(X)$ is F-Menger and so is the semi-multicovered group $(F(X), \mu_{L\wedge R})$ according to Corollary 6.2. $\square$

Corollary 9.2. *Assume that there is no Q-point. For a Tychonov space X the following conditions are equivalent*:

(1) *All continuous metrizable images of X are Scheepers*;
(2) *All continuous Lindelöf regular images of X are Scheepers*;
(3) *The multicovered space $(X, \mu_{\mathcal{U}})$ is Scheepers*;
(4) *$(F(X), \mu_{L\wedge R})$ is Scheepers*;
(5) *$(F(X), \mu_{L\vee R})$ is Scheepers*;
(6) *$(A(X), \mu_{L\wedge R})$ is Scheepers*;
(7) *All finite powers of $F(X)$ are o-bounded*;
(8) *$F(X)$ is o-bounded*;
(9) *$A(X)$ is o-bounded.*

Proof. The equivalence of the conditions (1)–(6) is a special case of Theorem 9.1 for $\mathsf{F} = \mathsf{UF}$. The implications (7) $\Rightarrow$ (8) and (8) $\Rightarrow$ (9) are obvious. Concerning (9) $\Rightarrow$ (1), it follows from Theorem 1.1. Finally, the implication (4) $\Rightarrow$ (7) follows from Proposition 3.5. $\square$

Remark 9.3. In spite of the example from Theorem 6.4 we do not know if Theorem 9.1 is true for all families $\mathsf{F} = \mathsf{F}_\downarrow$ of ultrafilters (in particular, those containing Q-points).

References

[1] T. Banakh, A. Blass, The number of near-coherence classes of ultrafilters is either finite or $2^{\mathfrak{c}}$, in: Joan Bagaria, Stevo Todorcevic (Eds.), Set Theory, in: Trends Math., Birkhäuser, 2006, pp. 257–273.
[2] T. Banakh, L. Zdomskyy, Coherence of Semifilters, book in progress, http://www.franko.lviv.ua/faculty/mechmat/Departments/Topology/booksite.html.
[3] T. Banakh, L. Zdomskyy, The coherence of semifilters: A survey, in: L. Kočinac (Ed.), Selection Principles and Covering Properties in Topology, in: Quad. Mat., vol. 18, Dept. Math., Seconda Univ., Napoli, Caserta, 2006, pp. 53–99.
[4] T. Banakh, L. Zdomskyy, Selection principles and infinite games on multicovered spaces, in: Selection Principles and Covering Properties in Topology, L. Kočinac (Ed.), in: Quad. Mat., vol. 18, Dept. Math., Seconda Univ. Napoli, Caserta, 2006, pp. 1–51.
[5] A. Blass, C. Laflamme, Consistency results about filters and the number of inequivalent growth types, J. Symbolic Logic 54 (1989) 50–56.
[6] I. Guran, Topological groups: Subspaces and their properties, PhD thesis, Moscow State Univ., 1981 (in Russian).
[7] C. Hernándes, D. Robbie, M. Tkachenko, Some properties of o-bounded and strictly o-bounded topological groups, Appl. Gen. Topol. 1 (2000) 29–43.
[8] W. Just, A. Miller, M. Scheepers, P. Szeptycki, The combinatorics of open covers. II, Topology Appl. 73 (1996) 241–266.
[9] L.D.R. Kočinac, Some covering properties in topological and uniform spaces, Tr. Mat. Inst. Steklova 252 (2006) 134–149.
[10] M. Machura, S. Shelah, B. Tsaban, Squares of Menger-bounded groups, Trans. Amer. Math. Soc., in press.
[11] H. Mildenberger, Cardinal characteristics for Menger-bounded subgroups, Topology Appl. 156 (2008) 130–137.
[12] A.W. Miller, There are no Q-points in Laver's model for the Borel conjecture, Proc. Amer. Math. Soc. 78 (1980) 103–106.
[13] V. Pestov, Some properties of free topological groups, Vestnik Moskov. Univ. Ser. I Mat. Mekh. 1 (1982) 35–37 (in Russian).
[14] M. Scheepers, Selection principles and covering properties in topology, Note Mat. 22 (2003) 3–41.
[15] M. Tkachenko, Introduction to topological groups, Topology Appl. 86 (1998) 179–231.
[16] B. Tsaban, Selection principles and special sets of reals, in: Elliott Pearl (Ed.), Open Problems in Topology II, Elsevier Sci. Publ., 2007, pp. 91–108.
[17] L. Zdomskyy, A semifilter approach to selection principles, Comment. Math. Univ. Carolin. 46 (2005) 525–539.
[18] L. Zdomskyy, o-Boundedness of free objects over a Tychonoff space, Mat. Stud. 25 (2006) 10–28.