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Continuous selections and σ-spaces ☆

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ABSTRACT

Assume that $X \subseteq \mathbb{R} \setminus \mathbb{Q}$, and each clopen-valued lower semicontinuous multivalued map $\Phi : X \Rightarrow \mathbb{Q}$ has a continuous selection $\phi : X \to \mathbb{Q}$. Our main result is that in this case, X is a σ-space. We also derive a partial converse implication, and present a reformulation of the Scheepers Conjecture in the language of continuous selections.

1. Introduction

All topological spaces considered in this note are assumed to have large inductive dimension 0, that is, disjoint closed sets can be separated by clopen sets.

By a *multivalued map* Φ from a set X into a set Y we understand a map from X into the power-set of Y, denoted by $P(Y)$, and we write $\Phi : X \Rightarrow Y$. Let X, Y be topological spaces. A multivalued map $\Phi : X \Rightarrow Y$ is *lower semi-continuous* (*lsc*) if for each open $V \subseteq Y$, the set

$$\Phi_{\cap}^{-1}(V) = \{x \in X:\ \Phi(x) \cap V \neq \emptyset\}$$

is open in X.

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A function $f : X \to Y$ is a *selection* of a multivalued map $\Phi : X \Rightarrow Y$ if $f(x) \in \Phi(x)$ for all $x \in X$. Let $\mathcal{C} \subseteq P(Y)$. A multivalued map $\Phi : X \Rightarrow Y$ is *$\mathcal{C}$-valued* if $\Phi(x) \in \mathcal{C}$ for all $x \in X$. Similarly, we define *clopen-valued*, *closed-valued*, and *open-valued*. A general reference for selections of multivalued mappings is [11].

Theorem 1 (Michael [9]). *Assume that X is a countable space, Y is a first-countable space, and $\Phi : X \Rightarrow Y$ is lsc. Then Φ has a continuous selection $\phi : X \to Y$.*

This result was extended in [17, Theorem 3.1], where it was proved that a space X is countable if and only if for each first-countable Y, each lsc multivalued map from X to Y has a continuous selection. In fact, their proof gives the following.

Theorem 2 (Yan and Jiang [17]). *A separable space X is countable if and only if for each first-countable space Y and each open-valued lsc map $\Phi : X \Rightarrow Y$, there is a continuous selection $\phi : X \to Y$.*

We extend Theorems 1 and 2 by considering a qualitative restriction on the space X (instead of the quantitative restriction "X is countable"). We also point out a connection to a conjecture of Scheepers.

2. σ-Spaces

Define a topology on $P(\mathbb{N})$ by identifying $P(\mathbb{N})$ with the Cantor space $\{0,1\}^{\mathbb{N}}$. The standard base of the topology of $P(\mathbb{N})$ consists of the sets of the form

$$[s;t] = \{A \subseteq \mathbb{N}:\ A \cap s = t\},$$

where s and t are finite subsets of $\mathbb{N}$. Let Fr denote the *Fréchet filter*, consisting of all cofinite subsets of $\mathbb{N}$, and let $[\mathbb{N}]^{\aleph_0}$ be the family of all infinite subsets of $\mathbb{N}$. Fr and $[\mathbb{N}]^{\aleph_0}$ are subspaces of $P(\mathbb{N})$ and are homeomorphic to $\mathbb{Q}$ and to $\mathbb{R} \setminus \mathbb{Q}$, respectively (see [7]). Let

$$\mathcal{B} = \big\{[s;\emptyset]:\ s \text{ is a finite subset of } \mathbb{N}\big\};$$

$$\mathcal{B}_{Fr} = \{B \cap Fr:\ B \in \mathcal{B}\}.$$

Note that $\mathcal{B}$ is the standard clopen base at the point $\emptyset \in P(\mathbb{N})$.

A topological space X is a *σ-space* if each F_σ subset of X is a G_δ subset of X [10].

The main result of this note is the following.

Theorem 3. *The following are equivalent*:

(1) *X is a σ-space*;
(2) *Each $\mathcal{B}_{Fr}$-valued lsc map $\Phi : X \Rightarrow Fr$ has a continuous selection.*

The proof of Theorem 3 and subsequent results use the following notions. A family $\mathcal{U} = \{U_n:\ n \in \mathbb{N}\}$ of subsets of a set X is a *γ-cover* of X if for each $x \in X$, $x \in U_n$ for all but finitely many n. A bijectively enumerated family $\mathcal{U} = \{U_n:\ n \in \mathbb{N}\}$ of subsets of a set X induces a *Marczewski map* $\mathcal{U} : X \to P(\mathbb{N})$ defined by

$$\mathcal{U}(x) = \{n \in \mathbb{N}:\ x \in U_n\}$$

for each $x \in X$ [8].

Remark 4. Marczewski maps can be naturally associated to any sequence of sets, not necessarily bijectively enumerated. Our restriction to bijective enumerations allows working with the classical notion of γ-cover. An alternative approach would be to use *indexed* γ-covers, that is, sequences of sets $(U_n:\ n \in \mathbb{N})$ such that each $x \in X$ belongs to U_n for all but finitely many n. All results of the present paper hold in this setting, too.

For a function $f : X \to Y$, $f[X]$ denotes $\{f(x):\ x \in X\}$, the image of f.

Lemma 5. *Let $\mathcal{U} = \{U_n:\ n \in \mathbb{N}\}$ be a bijectively enumerated family of subsets of a topological space X. Then*

(1) *$\mathcal{U}$ is a clopen γ-cover of X if and only if $\mathcal{U}[X] \subseteq Fr$ and $\mathcal{U} : X \to P(\mathbb{N})$ is continuous*;
(2) *$\mathcal{U}$ is an open γ-cover of X if and only if $\mathcal{U}[X] \subseteq Fr$ and the multivalued map $\Phi : X \Rightarrow Fr$ defined by $\Phi(x) = P(\mathcal{U}(x)) \cap Fr$ is lsc.*

Proof. The first assertion follows immediately from the corresponding definitions. To prove the second assertion, let us assume that $\mathcal{U} = \{U_n:\ n \in \mathbb{N}\}$ is an open γ-cover of X. Fix some finite subsets s, t of $\mathbb{N}$ and $x \in X$ such that $[s;t] \cap \Phi(x) \neq \emptyset$. There exists $A \in Fr$ such that $A \subseteq \mathcal{U}(x)$ and $A \cap s = t$.

Let $V = \bigcap_{n \in \mathcal{U}(x) \cap s} U_n$. The set V is open in X, being an intersection of finitely many open sets, and it contains x by definition of $\mathcal{U}$. Thus it suffices to show that $[s;t] \cap \Phi(y) \neq \emptyset$ for all $y \in V$. A direct verification indeed shows that $(A \cap s) \cup (\mathcal{U}(y) \setminus s)$ belongs to $[s;t]$ as well as to $\Phi(y)$.

To prove the converse implication, it suffices to note that $U_n = \Phi_{\cap}^{-1}[\{n\};\{n\}]$, and use the lower semi-continuity of Φ. $\square$

The following is a key result of Sakai. A cover $\{U_n \colon n \in \mathbb{N}\}$ of X is *γ-shrinkable* [12] if there is a clopen γ-cover $\{C_n \colon n \in \mathbb{N}\}$ of X such that $C_n \subseteq U_n$ for all n. Note that $\mathcal{U}$ is a γ-cover of X if and only if $\mathcal{U}[X] \subseteq Fr$.

Theorem 6 (Sakai [12]). *X is a σ-space if and only if each open γ-cover of X is γ-shrinkable.*

Proof of Theorem 3. $(2 \Rightarrow 1)$. Assume that each $\mathcal{B}_{Fr}$-valued lsc $\Phi : X \Rightarrow Fr$ has a continuous selection. We will show that X is a σ-set by using Sakai's characterization (Theorem 6).

Let $\mathcal{U}$ be an open γ-cover of X. Define $\Phi(x) = P(\mathcal{U}(x)) \cap Fr$. Φ is $\mathcal{B}_{Fr}$-valued, and by Lemma 5, Φ is lsc. By our assumption, Φ has a continuous selection. The following lemma implies that $\mathcal{U}$ is γ-shrinkable.

Lemma 7. *Let $\mathcal{U} = \{U_n \colon n \in \mathbb{N}\}$ be a bijectively enumerated open γ-cover of a space X. The following are equivalent*:

(1) *$\mathcal{U}$ is γ-shrinkable*;
(2) *The multivalued map $\Phi(x) = P(\mathcal{U}(x)) \cap Fr$ has a continuous selection.*

Proof. $(1 \Rightarrow 2)$. If $\mathcal{V} = \{V_n \colon n \in \mathbb{N}\}$ is a witness for (1), then the map $x \mapsto \mathcal{V}(x)$ is a continuous selection of Φ.

$(2 \Rightarrow 1)$. If $\phi : X \to Fr$ is a continuous selection of Φ, then $\{V_n := \{x \in X \colon \phi(x) \ni n\} \colon n \in \mathbb{N}\}$ is a clopen γ-cover of X with the property $V_n \subseteq U_n$, for all $n \in \mathbb{N}$. Indeed, if $x \in V_n$, then $n \in \phi(x) \in P(\mathcal{U}(x)) \cap Fr$, and hence $n \in \mathcal{U}(x)$, which is equivalent to $x \in U_n$. $\square$

$(1 \Rightarrow 2)$. Assume that X is a σ-space and $\Phi : X \Rightarrow Fr$ is lsc and $\mathcal{B}_{Fr}$-valued. The following is easy to verify.

Lemma 8. *For each $\mathcal{B}_{Fr}$-valued $\Phi : X \Rightarrow Fr$, there exists a map $\phi : X \to Fr$ such that $\Phi(x) = P(\phi(x)) \cap Fr$ for all $x \in X$.*

Conversely, for each map $\phi : X \to Fr$, the multivalued map $\Phi : X \Rightarrow Fr$ defined by $\Phi(x) = P(\phi(x)) \cap Fr$ is $\mathcal{B}_{Fr}$-valued.

Let ϕ be as in Lemma 8. For each n, let $U_n = \{x \in X \colon n \in \phi(x)\} = \{x \in X \colon \Phi(x) \cap [\{n\};\{n\}] \neq \emptyset\}$. Each U_n is open, and $\mathcal{U} = \{U_n \colon n \in \mathbb{N}\}$ is a γ-cover of X. By Sakai's Theorem 6, $\mathcal{U}$ is γ-shrinkable.

Note that the Marczewski map induced by the family $\mathcal{U}$ is exactly the map ϕ. Thus, by Lemma 7, $\Phi(x) = P(\phi(x)) \cap Fr = P(\mathcal{U}(x)) \cap Fr$ has a continuous selection. $\square$

Corollary 9. *If each clopen-valued lsc map $\Phi : X \Rightarrow Fr$ has a continuous selection, then X is a σ-space.*

Problem 10. Assume that $X \subseteq \mathbb{R}$ is a σ-space. Does each clopen-valued lsc map $\Phi : X \Rightarrow \mathbb{Q}$ have a continuous selection?

It is consistent (relative to ZFC) that each metrizable separable σ-space X is countable [10]. Thus, by Theorems 1 and 3, we have the following extension of Theorem 2.

Corollary 11. *It is consistent that the following statements are equivalent, for metrizable separable spaces X*:

(1) *Every clopen-valued lsc map $\Phi : X \Rightarrow \mathbb{Q}$ has a continuous selection*;
(2) *X is countable.*

Problem 12. Is Corollary 11 provable in ZFC?

$\mathfrak{b}$ is the minimal cardinality of a subset of $\mathbb{N}^{\mathbb{N}}$ which is unbounded with respect to $\leqslant^*$ ($f \leqslant^* g$ means: $f(n) \leqslant g(n)$ for all but finitely many n). $\mathfrak{b}$ is uncountable, and consistently, $\aleph_1 < \mathfrak{b}$ [2]. If $|X| < \mathfrak{b}$, then X is a σ-set [4,15]. By Theorem 3, we have the following quantitative result.

Corollary 13. *Assume that $|X| < \mathfrak{b}$. Then for each $\mathcal{B}_{Fr}$-valued lsc map $\Phi : X \Rightarrow Fr$, Φ has a continuous selection.*

3. $\mathfrak{b}$-Scales

Let $\mathbb{N}^{\uparrow\mathbb{N}}$ be the set of all (strictly) increasing elements of $\mathbb{N}^{\mathbb{N}}$. $B = \{b_\alpha \colon \alpha < \mathfrak{b}\} \subseteq \mathbb{N}^{\uparrow\mathbb{N}}$ is a $\mathfrak{b}$-*scale* if $b_\alpha \leqslant^* b_\beta$ for all $\alpha < \beta$, and B is unbounded with respect to $\leqslant^*$. $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ is a convergent sequence with the limit point ∞, which is assumed

$$\begin{tikzcd}
 & & \mathsf{U}_{\mathrm{fin}}(\mathcal{O},\varGamma) \arrow[r] & \mathsf{U}_{\mathrm{fin}}(\mathcal{O},\varOmega) \arrow[r] & \mathsf{S}_{\mathrm{fin}}(\mathcal{O},\mathcal{O}) \\
 & & \mathsf{S}_{\mathrm{fin}}(\varGamma,\varOmega) \arrow[ur] & & \\
\mathsf{S}_1(\varGamma,\varGamma) \arrow[r] \arrow[uurr] & \mathsf{S}_1(\varGamma,\varOmega) \arrow[rr] \arrow[ur] & & \mathsf{S}_1(\varGamma,\mathcal{O}) \arrow[uur] & \\
 & & \mathsf{S}_{\mathrm{fin}}(\varOmega,\varOmega) \arrow[uu] & & \\
\mathsf{S}_1(\varOmega,\varGamma) \arrow[r] \arrow[uu] & \mathsf{S}_1(\varOmega,\varOmega) \arrow[rr] \arrow[uu] \arrow[ur] & & \mathsf{S}_1(\mathcal{O},\mathcal{O}) \arrow[uu] &
\end{tikzcd}$$

Fig. 1. The Scheepers Diagram.

to be larger than all elements of $\mathbb{N}$. $\overline{\mathbb{N}}^{\uparrow\mathbb{N}}$ is the set of all nondecreasing elements of $\overline{\mathbb{N}}^{\mathbb{N}}$, and $Q = \{x \in \overline{\mathbb{N}}^{\uparrow\mathbb{N}}\colon\ (\exists m)\ (\forall n \geqslant m)\ x(n) = \infty\}$ is the set of all "eventually infinite" elements of $\overline{\mathbb{N}}^{\uparrow\mathbb{N}}$.

Sets of the form $B \cup Q$ where B is a $\mathfrak{b}$-scale were extensively studied in the literature (see [1,10,16] and references therein). $B \cup Q$ is concentrated on Q and is therefore not a σ-space. Consequently, it does not have the properties stated in Theorem 3. In fact, we have the following.

Theorem 14. *Let $X = B \cup Q$, where $B \subseteq \mathbb{N}^{\mathbb{N}}$ is a $\mathfrak{b}$-scale. Then there exists a clopen-valued lsc map $\Phi : X \Rightarrow \mathbb{Q}$ with the following properties*:

(i) *$\Phi(x) = \mathbb{Q}$, for all $x \in B$; and*
(ii) *For each $Y \subseteq X$ such that $Q \subseteq Y$, and each continuous $\phi : Y \to \mathbb{Q}$ such that $\phi(y) \in \Phi(y)$ for all $y \in Y$, $|Y| < |X|$.*

Proof. Write $Q = \{q_n\colon\ n \in \mathbb{N}\}$, and consider the γ-cover $\mathcal{U} = \{U_n\colon\ n \in \mathbb{N}\}$ of X, where $U_n = X \setminus \{q_n\}$, $n \in \mathbb{N}$.

Lemma 15. *For each $B' \subseteq B$ with $|B'| = \mathfrak{b}$, and each choice of clopen sets $V_n \subseteq U_n$, $n \in \mathbb{N}$, there is $x \in B'$ such that $\{n\colon\ x \notin V_n\}$ is infinite.*

Proof. Assuming the converse, we could find a clopen γ-cover $\{V'_n\colon\ n \in \mathbb{N}\}$ of $B' \cup Q$ such that $V'_n \subseteq U_n$. Let V_n be a closed subspace of $\overline{\mathbb{N}}^{\uparrow\mathbb{N}}$ such that $V_n \cap X = V'_n$. Then $W_n = \overline{\mathbb{N}}^{\uparrow\mathbb{N}} \setminus V_n$ is an open neighborhood of q_n in $\overline{\mathbb{N}}^{\uparrow\mathbb{N}}$. Set $G_n = \bigcup_{k \geqslant n} W_k$ and $G = \bigcap_{n \in \mathbb{N}} G_n$. For each $n \in \mathbb{N}$ the set $\mathbb{N}^{\uparrow\mathbb{N}} \cap (\overline{\mathbb{N}}^{\uparrow\mathbb{N}} \setminus G_n)$ is a cofinite subset of the compact space $\overline{\mathbb{N}}^{\uparrow\mathbb{N}} \setminus G_n$, and hence it is σ-compact.

Therefore $\mathbb{N}^{\uparrow\mathbb{N}} \cap (\overline{\mathbb{N}}^{\uparrow\mathbb{N}} \setminus G) = \bigcup_{n \in \mathbb{N}} \mathbb{N}^{\uparrow\mathbb{N}} \cap (\overline{\mathbb{N}}^{\uparrow\mathbb{N}} \setminus G_n)$ is a σ-compact subset of $\mathbb{N}^{\uparrow\mathbb{N}}$ as well. Since B' is unbounded, there exists $x \in B' \cap G$, and hence x belongs to W_n for infinitely many $n \in \mathbb{N}$, which implies that $\{n \in \mathbb{N}\colon\ x \notin V_n\} = \{n \in \mathbb{N}\colon\ x \notin V'_n\}$ is infinite, a contradiction. $\square$

Recall that Fr is homeomorphic to $\mathbb{Q}$. Thus, it suffices to construct an lsc $\Psi : X \Rightarrow Fr$ with the properties (i) and (ii). Set $\Psi(x) = P(\mathcal{U}(x)) \cap Fr$.

By Lemma 5, the multivalued map Ψ is lsc and there are no partial selections $f : Y \to Fr$ defined on subsets $Y \subseteq X$ such that $|Y| = |X| = \mathfrak{b}$ and $Y \supset Q$. Indeed, it suffices to use Lemmata 7 and 15, asserting that there is no clopen refinement $\{V_n\colon\ n \in \mathbb{N}\}$ of $\{U_n\colon\ n \in \mathbb{N}\}$ which is a γ-cover of such a subspace Y of X. $\square$

Theorem 14 can be compared with Theorem 1.7 and Example 9.4 of [9].

The undefined terminology in the following discussion is standard and can be found in, e.g., [13]. Lemma 15 motivates the introduction of the following covering property of a space X:

(θ) There exists an open γ-cover $\mathcal{U} = \{U_n\colon\ n \in \mathbb{N}\}$ of X and a countable $D \subseteq X$ such that for any family $\mathcal{V} = \{V_n\colon\ n \in \mathbb{N}\}$ of clopen subsets of X with $V_n \subseteq U_n$ for all n, if $\mathcal{V}$ is a γ-cover of some $Y \subseteq X$ such that $D \subseteq Y$, then $|Y| < |X|$.

Theorem 14 implies the following.

Corollary 16. *Assume that $X = B \cup Q$ where $B \subseteq \mathbb{N}^{\mathbb{N}}$ is a $\mathfrak{b}$-scale. Then X satisfies (θ).*

The property (θ) seems to stand apart from the classical selection principles considered in [13,6]. Fig. 1 (reproducing [6, Fig. 3, p. 245]) summarizes the relations among these properties.

Every countable space satisfies the strongest property in that figure, namely $\mathsf{S}_1(\varOmega,\varGamma)$ [5], and it is clear that countable spaces do not satisfy (θ). Moreover, by Sakai's Theorem 6, no σ-space satisfies (θ).

Assuming the Continuum Hypothesis there is a $\mathfrak{b}$-scale B such that $B \cup Q$ is not a σ-space, but satisfies $\mathsf{S}_1(\varOmega,\varGamma)$ [5] as well as (θ) (Corollary 16).

Consider the topological sum $X = \mathbb{R} \oplus (\mathbb{R} \setminus \mathbb{Q})$. The open sets $U_n = (-n, n) \oplus (\mathbb{R} \setminus \mathbb{Q})$, $n \in \mathbb{N}$, form a γ-cover of X and show that X satisfies (θ) for a trivial reason, and does not satisfy the weakest property in the Scheepers Diagram, namely $\mathsf{S}_{\mathrm{fin}}(\mathcal{O}, \mathcal{O})$, because it contains $(\mathbb{R} \setminus \mathbb{Q})$ as a closed subspace. A less trivial (zero-dimensional) example is given in the following consistency result.

Theorem 17. *Assume that $\mathfrak{b} = \mathfrak{d} = \mathrm{cf}(\mathfrak{c}) < \mathfrak{c}$. There is a set $X \subseteq \mathbb{R} \setminus \mathbb{Q}$ satisfying (θ) but not $\mathsf{S}_{\mathrm{fin}}(\mathcal{O}, \mathcal{O})$.*

Proof. Let $B = \{b_\alpha\colon \alpha < \mathfrak{b}\}$ be a $\mathfrak{b}$-scale and $\mathfrak{c} = \bigcup_{\alpha < \mathfrak{b}} \lambda_\alpha$ with $\lambda_\alpha < \mathfrak{c}$. Fix $D_\alpha \subseteq \mathbb{N}^{\uparrow\mathbb{N}}$ such that $|D_\alpha| = \lambda_\alpha$ and for each $f \in D_\alpha$, $|f(n) - b_\alpha(n)| < 2$ for all n.

Let $Y \subseteq \mathbb{N}^{\mathbb{N}}$ be a dominating family. The direct sum of $X = Q \cup \bigcup_{\alpha < \mathfrak{b}} D_\alpha$ and Y satisfies (θ) by the methods of Theorem 14. But Y is a closed subset of this space and does not satisfy $\mathsf{S}_{\mathrm{fin}}(\mathcal{O}, \mathcal{O})$ [13]. □

4. Connections with the Scheepers Conjecture

Let $\mathcal{A}$ and $\mathcal{B}$ be any two families. Motivated by works of Rothberger, Scheepers introduced the following prototype of properties [13]:

$\mathsf{S}_1(\mathcal{A}, \mathcal{B})$: For each sequence $\{\mathcal{U}_n\}_{n \in \mathbb{N}}$ of members of $\mathcal{A}$, there exist members $U_n \in \mathcal{U}_n$, $n \in \mathbb{N}$, such that $\{U_n\colon n \in \mathbb{N}\} \in \mathcal{B}$.

Let Γ and C_Γ be the collections of all open and clopen γ-covers of a set $X \subseteq \mathbb{R}$, respectively. Scheepers [14] has conjectured that the property $\mathsf{S}_1(\Gamma, \Gamma)$ is equivalent to a certain local property in the space of continuous real-valued functions on X. Sakai [12] and independently Bukovský and Haleš [3] proved that Scheepers' Conjecture holds if and only if $\mathsf{S}_1(\Gamma, \Gamma) = \mathsf{S}_1(C_\Gamma, C_\Gamma)$ for sets of reals.

Lemma 5 establishes a bijective correspondence between open γ-covers of a space X and maps $\phi : X \to Fr$ for which the multivalued map $\Phi(x) = P(\phi(x)) \cap Fr$ is lsc. This is used in the proof of the following characterizations, which give an alternative justification for the Scheepers Conjecture.

Theorem 18. *X satisfies $\mathsf{S}_1(C_\Gamma, C_\Gamma)$ if and only if for each continuous $\phi : X \to Fr^{\mathbb{N}}$ there is $f \in \mathbb{N}^{\mathbb{N}}$ such that $f(k) \in \phi(x)(k)$ for each $x \in X$ and all but finitely many k.*

Since the proof of Theorem 18 is easier than that of the following theorem, we omit it.

Theorem 19. *X satisfies $\mathsf{S}_1(\Gamma, \Gamma)$ if and only if for each $\phi : X \to Fr^{\mathbb{N}}$ such that the multivalued map $\Phi : x \mapsto \Pi_{k \in \mathbb{N}}(P(\phi(x)(k)) \cap Fr)$ is lsc, there is $f \in \mathbb{N}^{\mathbb{N}}$ such that $f(k) \in \phi(x)(k)$ for each $x \in X$ and all but finitely many k.*

Proof. Assume that X satisfies $\mathsf{S}_1(\Gamma, \Gamma)$. Fix a map $\phi : X \to Fr^{\mathbb{N}}$ as in the second assertion. The multivalued map $\Phi_k : X \Rightarrow Fr$ assigning to each point $x \in X$ the subset $\Phi_k(x) = P(\phi(x)(k)) \cap Fr$ of Fr, is lsc for all k.

The family $\{U_{k,n}\colon n \in \mathbb{N}\}$, where $U_{k,n} = \{x \in X\colon \Phi_k(x) \cap [\{n\}; \{n\}] \neq \emptyset\} = \{x \in X\colon n \in \phi(x)(k)\}$, is an open γ-cover of X. Since X satisfies $\mathsf{S}_1(\Gamma, \Gamma)$, there exists $f \in \mathbb{N}^{\mathbb{N}}$ such that $\{U_{k,f(k)}\colon k \in \mathbb{N}\}$ is a γ-cover of X. This implies that $f(k) \in \phi(x)(k)$ for all $x \in \mathbb{N}$ and all but finitely many k.

The proof of the converse implication is similar, using Lemma 5. □

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References

[1] T. Bartoszynski, B. Tsaban, Hereditary topological diagonalizations and the Menger–Hurewicz conjectures, Proc. Amer. Math. Soc. 134 (2006) 605–615.
[2] A.R. Blass, Combinatorial cardinal characteristics of the continuum, in: M. Foreman, A. Kanamori, M. Magidor (Eds.), Handbook of Set Theory, Kluwer Academic Publishers, Dordrecht, in press, http://www.math.lsa.umich.edu/~ablass/hbk.pdf.
[3] L. Bukovský, J. Haleš, QN-space, wQN-space and covering properties, Topology Appl. 154 (2007) 848–858.
[4] D.H. Fremlin, A.W. Miller, On some properties of Hurewicz, Menger and Rothberger, Fund. Math. 129 (1988) 17–33.
[5] F. Galvin, A. Miller, γ-Sets and other singular sets of real numbers, Topology Appl. 17 (1984) 145–155.
[6] W. Just, A.W. Miller, M. Scheepers, P.J. Szeptycki, The combinatorics of open covers II, Topology Appl. 73 (1996) 241–266.
[7] A. Kechris, Classical Descriptive Set Theory, Grad. Texts in Math., vol. 156, Springer-Verlag, Berlin, 1995.
[8] E. Marczewski (Szpilrajn), The characteristic function of a sequence of sets and some of its applications, Fund. Math. 31 (1938) 207–233.
[9] E. Michael, Continuous selections and countable sets, Fund. Math. 111 (1981) 1–10.
[10] A. Miller, Special subsets of the real line, in: K. Kunen, J.E. Vaughan (Eds.), Handbook of Set-Theoretic Topology, North-Holland, Amsterdam, 1984, pp. 201–235.
[11] D. Repovš, P.V. Semenov, Continuous Selections of Multivalued Mappings, vol. 455, Kluwer Academic Publishers, Dordrecht, 1998.
[12] M. Sakai, The sequence selection properties of $C_p(X)$, Topology Appl. 154 (2007) 552–560.

[13] M. Scheepers, Combinatorics of open covers I: Ramsey theory, Topology Appl. 69 (1996) 31–62.
[14] M. Scheepers, Sequential convergence in $C_p(X)$ and a covering property, East-West J. Math. 1 (1999) 207–214.
[15] M. Scheepers, B. Tsaban, The combinatorics of Borel covers, Topology Appl. 121 (2002) 357–382.
[16] B. Tsaban, L. Zdomskyy, Scales, fields, and a problem of Hurewicz, J. Eur. Math. Soc., in press.
[17] P.-F. Yan, S.-L. Jiang, Countable sets, BCO spaces and selections, Topology Appl. 148 (2005) 1–5.