



A new Lindelöf topological group [☆]

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ABSTRACT

We show that the subsemigroup of the product of ω_1 -many circles generated by the L -space constructed by J. Moore is again an L -space. This leads to a new example of a Lindelöf topological group. The question whether all finite powers of this group are Lindelöf remains open.

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1. Introduction

This paper is devoted to one of the possible approaches to the problem posed by Arhangel'skiĭ [1] concerning existence of a Lindelöf topological group with non-Lindelöf square. This approach is based on the recent deep result of Moore [7] asserting that there exists an L -space in ZFC. We recall that an L -space is a regular hereditarily Lindelöf non-separable topological space. The connection between L -spaces and preservation of Lindelöfness by finite powers is given by the following result, which is a corollary of [10, Theorem 7.10] and its proof:

Theorem 1.1. *Suppose that X is a regular topological space with countable tightness and Y is a non-separable subspace of X . If all finite powers of X are Lindelöf, then there exist a c.c.c. poset \mathbb{P} and a family \mathcal{D} of dense subsets of \mathbb{P} of size $|\mathcal{D}| = \omega_1$ such that if there exists a filter $G \subset \mathbb{P}$ meeting each $D \in \mathcal{D}$, then Y has an uncountable discrete subspace.*

Consequently, if MA_{ω_1} holds and X is a regular topological space with countable tightness containing an L -subspace, then some of the finite powers of X are not Lindelöf.

The L -space constructed in [7] is a subspace of Σ_{ω_1} , the Σ -product of ω_1 many circles. It is well known [3] that all finite powers of this Σ -product have countable tightness. Theorem 1.1 suggests the following open question.

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Question 1.2. Let \mathcal{L} be the L -space constructed in [7]. Is the subgroup of Σ_{ω_1} generated by \mathcal{L} a Lindelöf group? More generally, can \mathcal{L} be embedded into a Lindelöf subgroup G of Σ_{ω_1} ?

The L -space constructed in [7] remains an L space in extensions by a wide class of forcing notions containing all c.c.c. ones. Therefore if the answer to Question 1.2 is positive, i.e. \mathcal{L} can be embedded into a Lindelöf subgroup G of Σ_{ω_1} , then Theorem 1.1 would imply that some of the finite powers of G are not Lindelöf in ZFC.

In this paper we make a step towards the solution of Question 1.2. Using the ideas of [7], we show in Section 2 that the subsemigroup of Σ_{ω_1} generated by \mathcal{L} is an L -space. Thus there exists an L -semigroup with cancellation, which seems to have not been noted elsewhere. On the other hand, the group generated by \mathcal{L} contains a copy of the one-point compactification of the discrete space of size ω_1 , and hence is not hereditarily Lindelöf.¹ In Section 3 we prove that the subgroup of the Tychonoff product of ω_1 -many circles generated by the union of \mathcal{L} and certain meager σ -compact subspace is Lindelöf, which speaks for the positive answer to Question 1.2. However, this group has uncountable tightness, and consequently it is not within the scope of applications of Theorem 1.1.

The authors were able to find only two consistent examples of a Lindelöf group G with non-Lindelöf square in the literature, see [5] and [11]. Malykhin’s example is constructed under $\text{cof}(\mathcal{M}) = \omega_1$ in terms of [2], while Todorčević uses the additional assumption that there exists a countably additive measure extending the Lebesgue measure and which is defined on all sets of reals. Both of these assertions contradict Martin’s axiom. The existence of such a group G is also consistent with MA: Soukup [8] constructed a model of ZFC + MA which contains an L -group of countable tightness (an L -group is a topological group whose underlying topological space is an L -space). Therefore Theorem 1.1 implies that some of the finite powers of G are not Lindelöf.

All spaces considered here are assumed to be Tychonoff.

2. L -semigroups with cancellation

We briefly discuss Theorem 1.1 before passing to L -semigroups.

Proof sketch of Theorem 1.1. The direct application of [10, Theorem 7.10] gives Theorem 1.1 only for spaces X such that all finite powers of X have countable tightness. However for a pair X, Y of spaces satisfying the premises of Theorem 1.1 one can easily construct a continuous map $f : X \rightarrow \Sigma_{\omega_1}$ such that $f(Y)$ is not separable, see, e.g., the proof of [6, Corollary 2.3]. Since all finite powers of $f(X)$ have countable tightness, we can apply to $f(X), f(Y)$ the same argument as in the proof of [10, Theorem 7.10] and then pull the conclusion back to X, Y . This way we get Theorem 1.1. \square

In the rest of this section we follow the notations from [7]. Developing the ideas of Todorčević [9], Moore considered the function $\text{osc} : \{(\alpha, \beta) \in \omega_1^2 : \alpha < \beta\} \rightarrow \omega$ having strong combinatorial properties. We shall give more detailed definition of this function in Example 3.1. For the purposes of this section the following fundamental result is sufficient.

Theorem 2.1. ([7, Theorem 4.3]) For every uncountable families of pairwise disjoint sets $\mathcal{A} \subset [\omega_1]^k$ and $\mathcal{B} \subset [\omega_1]^l$ and every $n \in \omega$, there exist $a \in \mathcal{A}$ and $b_m \in \mathcal{B}$, $m < n$, such that for all $i < k$, $j < l$, and $m < n$:

$$a < b_m \text{ and } \text{osc}(a(i), b_m(j)) = \text{osc}(a(i), b_0(j)) + m.$$

(Here $a < b$ means $\max a < \min b$.)

Let $(z_\alpha)_{\alpha < \omega_1}$ be a sequence of points on the circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ which are rationally independent. (We consider \mathbb{T} as a subgroup of $\mathbb{C} \setminus \{0\}$ with the multiplication.) Given any $\alpha < \beta < \omega_1$, set $o(\alpha, \beta) = z_\alpha^{\text{osc}(\alpha, \beta)+1}$. We define $w_\beta \in \mathbb{T}^{\omega_1}$ by letting

$$w_\beta(\alpha) = \begin{cases} o(\alpha, \beta), & \text{if } \alpha < \beta, \\ 1, & \text{otherwise.} \end{cases}$$

It was showed in [7, Theorem 7.11] that for every uncountable $X \subset \omega_1$ the space $\mathcal{L}_X = \{w_\beta | X : \beta \in X\}$ is an L -space. The methods developed in [7] allow one to slightly extend this result.

For a subset A of a group G , we denote by $\text{sgrp}(A)$ and $\text{grp}(A)$ the smallest subsemigroup and subgroup of G containing A , respectively. In particular, $\text{sgrp}(\mathcal{L}_X)$ stands for the subsemigroup of \mathbb{T}^X generated by \mathcal{L}_X . A semigroup with cancellation is a semigroup H such that both of the equalities $hh' = hh''$ and $h''h = h'h$ imply $h' = h''$, where $h, h', h'' \in H$.

Theorem 2.2. For every uncountable $X \subset \omega_1$ the subspace $\text{sgrp}(\mathcal{L}_X)$ of \mathbb{T}^X is an L -space. In particular, $\text{sgrp}(\mathcal{L}_X)$ is an L -subsemigroup of \mathbb{T}^X with cancellation.

¹ J. Moore informed us in a private communication that it is unlikely that his ideas can help to construct ZFC-examples of L -groups.

The following classical result independently proved by Kronecker and Tchebychef will be useful.

Theorem 2.3. Suppose that $z_i, i < k$, are elements of \mathbb{T} which are rationally independent. For every $\varepsilon > 0$ there exists a natural number n_ε such that if $u, v \in \mathbb{T}^k$, then there is an $m < n_\varepsilon$ such that $|u_i z_i^m - v_i| < \varepsilon$ for all $i < k$.

The next proposition resembles [7, Theorem 5.6].

Proposition 2.4. Let $\mathcal{A} \subset [\omega_1]^k$ and $\mathcal{B} \subset [\omega_1]^l$ be uncountable families of pairwise disjoint sets. Then for every sequence $(U_i)_{i < k}$ of open subsets of \mathbb{T} , every partitions $k = u_0 \sqcup u_1$ and $l = v_0 \sqcup v_1$, and arbitrary sequence $(n_j)_{j < l}$ of integers with the property $\sum_{j \in v_r} n_j \neq 0$ for all $r \in 2$, there exist $a \in \mathcal{A}$ and $b \in \mathcal{B}$ such that $a < b$ and

$$\prod_{j \in v_r} o(a(i), b(j))^{n_j} \in U_i$$

for all $i \in u_r$ and $r \in 2$.

Proof. There is no loss of generality in assuming that U_i is an ε -ball around a point t_i for some fixed $\varepsilon > 0$. Set $N_r = \sum_{j \in v_r} n_j, r \in 2$, and let $\delta = \varepsilon / \max\{|N_0|, |N_1|\}$. Passing to an uncountable subset of \mathcal{A} , if necessary, we may additionally assume that the numbers n_δ given by Theorem 2.3 for the sequence $z_{a(i)}, i \in k$, are the same for all $a \in \mathcal{A}$.

Let $a \in \mathcal{A}$ and $b_m \in \mathcal{B}, m < n_\delta$, be such as in Theorem 2.1, i.e. for all $i < k, j < l$, and $m < N$ we have $a < b_m$ and $\text{osc}(a(i), b_m(j)) = \text{osc}(a(i), b_0(j)) + m$. For each $r \in 2$ and $i \in u_r$ put $t'_i = \prod_{j \in v_r} o(a(i), b_0(j))^{n_j}$. Let t''_i be such that the N_r -th power of t''_i equals $t_i t_i^{-1}$, and let W_i be the δ -ball around t''_i , where $i \in k$. By the definition of n_δ , there exists $m < n_\delta$ such that

$$z_{a(i)}^m \in W_i$$

for all $i < k$. Set $b = b_m$. Then

$$\prod_{j \in v_r} o(a(i), b(j))^{n_j} \in \prod_{j \in v_r} o(a(i), b_0(j))^{N_r} z_{a(i)}^{m \cdot N_r} \subset t'_i W_i^{N_r}.$$

The W_i 's were chosen in such a way that $W_i^{N_r}$ is a subset of the ε -ball around $t_i t_i^{-1}$. This completes our proof. \square

The following proposition is reminiscent of [7, Theorem 7.10].

Proposition 2.5. If $X, Y \subset \omega_1$ are disjoint, then there is no continuous injection of any uncountable subspace of $\text{sgrp}(\mathcal{L}_X)$ into \mathcal{L}_Y .

Proof. Suppose to the contrary that such an injection g of an uncountable subset Q of $\text{sgrp}(\mathcal{L}_X)$ into \mathcal{L}_Y exists. Passing to an uncountable subset of Q , if necessary, we may assume that there exist $m \in \omega, a \Delta$ -system \mathcal{C} of subsets of X of size m with a root d , and a sequence $(n'_j)_{j < m}$ of positive integers, such that $s_c = \prod_{j \in m} w_{c(j)}^{n'_j} \in Q$ and $g : s_c \mapsto w_{f(c)}$, where $f : \mathcal{C} \rightarrow Y$ is an injection. It is also clear that there is no loss of generality in assuming that $d = \emptyset$.

For each $\xi < \omega_1$, let $c_\xi \in \mathcal{C}$ and $\zeta_\xi \in Y$ be such that $f(c_\xi) > \zeta_\xi$ and if $\xi < \xi'$, then $c_\xi < \zeta_{\xi'}$. Let $\Theta \subset \omega_1$ be uncountable such that for some open neighborhood $V \subset \mathbb{T}, w_{f(c_\xi)}(\zeta_\xi) \notin \bar{V}$ for all $\xi \in \Theta$.

Applying the continuity of g at s_c to $W_\xi = \{w \in \mathcal{L}_Y : w(\zeta_\xi) \notin \bar{V}\}$, we can find a basic open neighborhood U_ξ of s_{c_ξ} in Q such that $g(U_\xi) \subset W_\xi$. Applying the Δ -system lemma [4, Theorem 1.6] and the second countability of \mathbb{T} again, we see that there exist $k_0 \in \omega$, an uncountable $\Theta' \subset \Theta$, open neighborhoods $(U'_i)_{i \in k_0}$ in \mathbb{T} , and $a_\xi \in [X]^{k_0}$ such that for all $\xi \in \Theta'$:

- (i) $\{a_\xi : \xi \in \Theta'\}$ is a Δ -system with a root a ;
- (ii) the set $\{w \in Q : \forall i < k_0 (w(a_\xi(i)) \in U'_i)\}$ contains s_{c_ξ} and is a subset of U_ξ ;
- (iii) $|c_\xi \cap f(c_\xi)|$ does not depend on ξ ; and
- (iv) $|a_\xi \setminus a \cap \zeta_\xi|$ does not depend on ξ .

Let \mathcal{A} be the collection of all $a_\xi \cup \{\zeta_\xi\} \setminus a, \xi \in \Theta'$, and let k be the size of elements of \mathcal{A} . Let also \mathcal{B} be the collection of all $c_\xi \cup \{f(c_\xi)\}$, where $\xi \in \Theta'$, and $l = m + 1$.

Now, let $k = u_0 \sqcup u_1$ and $l = v_0 \sqcup v_1$ be the partitions of k and l defined as follows: $u_1 = \{|(a_\xi \setminus a) \cap \zeta_\xi|\}, v_1 = \{|c_\xi \cap f(c_\xi)|\}, u_0 = k \setminus u_1$, and $v_0 = l \setminus v_1$ (conditions (iv) and (iii) mean that the partitions do not depend on a particular $\xi \in \Theta'$). For every $j \in l$ we put

$$n_j = \begin{cases} n'_j, & \text{if } j < |c_\xi \cap f(c_\xi)|, \\ 1, & \text{if } j = |c_\xi \cap f(c_\xi)|, \\ n'_{j-1}, & \text{if } j > |c_\xi \cap f(c_\xi)|. \end{cases}$$

Finally, for every $i \in k$ we define U_i as follows:

$$U_i = \begin{cases} U'_{i+|a|}, & \text{if } i < |(a_\xi \setminus a) \cap \zeta_\xi|, \\ V, & \text{if } i = |(a_\xi \setminus a) \cap \zeta_\xi|, \\ U'_{i+|a|-1}, & \text{if } i > |(a_\xi \setminus a) \cap \zeta_\xi|. \end{cases}$$

Applying Proposition 2.4, it is possible to find $\xi < \xi' \in \Theta'$ such that

$$a = a_\xi \cup \{\zeta_\xi\} < c_{\xi'} \cup \{f(c_{\xi'})\} = b \quad \text{and} \\ \prod_{j \in v_r} o(a(i), b(j))^{n_j} \in U_i \quad \text{for all } i \in u_r \text{ and } r \in 2.$$

The U_i 's and n_j 's were defined in such a way that the second condition under $r = 1$ gives $w_{f(c_{\xi'})}(\zeta_\xi) \in V$, and for $r = 0$ this gives

$$s_{c_{\xi'}}(a_\xi(i)) = \prod_{j \in m} w_{c_{\xi'}(j)}(a_\xi(i))^{n_j} \in U'_i$$

for all $i \geq |a|$, while for $i < |a|$ the above trivially holds by (i) and (ii). But now $s_{c_{\xi'}} \in U_\xi$ even though $g(s_{c_{\xi'}})(\zeta_\xi) = w_{f(c_{\xi'})}(\zeta_\xi) \in V$, contradicting the choice of U_ξ . The proof is thus finished. \square

Proof of Theorem 2.2. The “+1” in the definition of the function o clearly ensures that the closure in $\text{sgrp}(\mathcal{L}_X)$ of any countable subset of $\text{sgrp}(\mathcal{L}_X)$ is countable. Indeed, suppose that H is a countable subset of \mathcal{L}_X and $\alpha \in \omega_1$ is such that $\alpha > \xi$ for all ξ with $w_\xi|X \in H$. Thus $t(\gamma) = 1$ for every $t \in \text{sgrp}(H)$ and $\gamma \geq \alpha$. Let us fix $s = \prod_{i \leq n} w_{\xi_i}^{m_i}|X \in \text{sgrp}(\mathcal{L}_X)$. Without loss of generality, $\xi_0 < \xi_1 < \dots < \xi_n$ and $m_n \neq 0$. If $\xi_n > \alpha$,

$$s(\max\{\alpha, \xi_{n-1}\}) = z_{\max\{\alpha, \xi_{n-1}\}}^{m_n(\text{osc}(\max\{\alpha, \xi_{n-1}\}, \xi_n)+1)} \neq 1,$$

and consequently s is not in the closure of $\text{sgrp}(H)$.

Therefore, if $\text{sgrp}(\mathcal{L}_X)$ were not hereditarily Lindelöf, it would contain an uncountable discrete subspace Q . The above means that for every $q \in Q$ there exists a basic open subset $U_q \ni q$ of \mathbb{T}^{ω_1} such that $U_q \cap Q = \{q\}$. Since each U_q depends on finitely many coordinates, we can find an uncountable $Y \subset X$ such that $|\omega_1 \setminus Y| = \omega_1$ and $Q|Y = \{q|Y : q \in Q\}$ is still discrete. Then any injection $g : Q|Y \rightarrow \mathcal{L}_{\omega_1 \setminus Y}$ is continuous, which contradicts Proposition 2.5. \square

The following technical statement will be crucial in the next section.

Corollary 2.6. Let $\mathcal{C} \subset [\omega_1]^I$ be an uncountable family of pairwise disjoint sets and $(n_j)_{j < I}$ be a sequence of integers with $\sum_{j < I} n_j \neq 0$. Then for every $X \subset \omega_1$ such that $\bigcup \mathcal{C} \subset X$, the subspace

$$\left\{ \prod_{j < I} w_{c(j)}^{n_j} | X : c \in \mathcal{C} \right\}$$

of Σ_X is hereditarily Lindelöf.

Proof. Almost literal repetition of the proof of Proposition 2.5 (just a couple of the first lines should be omitted) gives us that there is no continuous injection from any uncountable subspace of $\{\prod_{j < I} w_{c(j)}^{n_j} | X : c \in \mathcal{C}\}$ into \mathcal{L}_Y provided $Y \cap \bigcup \mathcal{C} = \emptyset$. Now it suffices to apply the same argument as in the proof of Theorem 2.2. \square

In the same way we can also prove the following proposition, which shows that it is essential in Theorem 1.1 to consider finite powers and not just finite products.

Proposition 2.7. For every finite family $\{X_0, \dots, X_n\}$ of uncountable pairwise disjoint subsets of ω_1 , the product $\text{sgrp}(\mathcal{L}_{X_0}) \times \dots \times \text{sgrp}(\mathcal{L}_{X_n})$ is an L -space.

On the other hand, it is easy to prove that $\text{grp}(\mathcal{L}_X)$ is not hereditarily Lindelöf. We shall use the following consequence of [7, Proposition 7.13].

Proposition 2.8. For every $\beta < \omega_1$ the set $\{w_\xi | \beta : \xi < \omega_1\}$ is countable.

For a cardinality τ we denote by $A(\tau)$ the one-point compactification of the discrete space of size τ . The following proposition corresponds to [7, Theorem 7.2].

Proposition 2.9. $\text{grp}(\mathcal{L}_X)$ contains a copy of $A(\omega_1)$.

Proof. Using Proposition 2.8 we can construct two increasing transfinite sequences $(\xi_\beta)_{\beta < \omega_1}$ and $(\zeta_\beta)_{\beta < \omega_1}$ of ordinals with the following properties:

- (i) $\zeta_\beta > \xi_\beta$ for all $\beta < \omega_1$;
- (ii) $\xi_{\beta'} > \zeta_\beta$ for all $\beta < \beta' < \omega_1$; and
- (iii) $w_{\xi_{\beta'}} \sup\{\zeta_\beta : \beta < \beta'\} = w_{\zeta_{\beta'}} \sup\{\xi_\beta : \beta < \beta'\}$ for all $\beta' < \omega_1$.

A direct verification shows that $\{w_{\zeta_\beta} \cdot w_{\xi_\beta}^{-1} : \beta < \omega_1\} \cup \{0\}$ is a copy of $A(\omega_1)$. \square

3. An example of a Lindelöf group

In this section we shall construct an example of a Lindelöf group G containing \mathcal{L}_X of the form $\text{grp}(\mathcal{L}_X \cup K)$ for a meager σ -compact subgroup K of \mathbb{T}^{ω_1} defined below. This group has uncountable tightness, and hence Theorem 1.1 cannot be used here to deduce that G^n is not Lindelöf for some $n \in \omega$. We do not know whether all finite powers of the group G constructed in Example 3.1 are Lindelöf.

Let

$$K = \left\{ (z_\alpha^{p_\alpha})_{\alpha < \omega_1} : \forall \alpha < \omega_1 (p_\alpha \in \mathbb{Z}) \wedge (\sup\{|p_\alpha| : \alpha < \omega_1\} < \infty) \right\}.$$

It is clear that K is a meager σ -compact subgroup of \mathbb{T}^{ω_1} . In addition, [7, Theorem 7.14] implies that $\mathcal{L}_X \cap \text{pr}_X K$ is at most countable for every $X \in [\omega_1]^{\omega_1}$.

Example 3.1. Let X be an uncountable subset of ω_1 and $G = \text{grp}(\mathcal{L}_X) \cdot \text{pr}_X K$. Then G is Lindelöf.

First we shall prove some auxiliary statements. At this point we need to go a bit deeper into the construction of the function o , see [7, Section 2]. Summarizing Facts 1 and 2 from [7] we conclude that there exists a function $L : \{(\alpha, \beta) \in \omega_1^2 : \alpha \leq \beta\} \rightarrow [\omega_1]^{<\omega}$ with the following properties:

- (i) $L(\alpha, \beta) \subset \alpha$ and $L(\alpha, \beta) = \emptyset$ if and only if $\alpha = 0$ or $\alpha = \beta$;
- (ii) if $\alpha \leq \beta \leq \gamma$ and $L(\beta, \gamma) \subset L(\alpha, \beta)$, then $L(\alpha, \gamma) = L(\beta, \gamma) \cup L(\alpha, \beta)$; and
- (iii) if β is limit, then $\lim_{\alpha \rightarrow \beta} \min L(\alpha, \beta) = \beta$.

The definition of o also involves such a standard object as a *coherent sequence* of functions, i.e. a sequence $(e_\alpha)_{\alpha \in \omega_1}$ such that $e_\alpha : \alpha \rightarrow \omega$, e_α is finite-to-one, and for arbitrary $\alpha < \beta$, the set $\{\xi < \alpha : e_\alpha(\xi) \neq e_\beta(\xi)\}$ is finite. Now, $\text{osc}(\alpha, \beta)$ is the cardinality of the set $\text{Osc}(e_\alpha, e_\beta, L(\alpha, \beta))$ defined as follows:

$$\left\{ \xi \in L(\alpha, \beta) \setminus \min L(\alpha, \beta) : e_\alpha(\xi^-) = e_\beta(\xi^-) \wedge e_\alpha(\xi) > e_\beta(\xi) \right\},$$

where ξ^- is the greatest element of $L(\alpha, \beta)$ smaller than ξ .

Lemma 3.2. Let $a \in [\omega_1]^k$ and $(n_i)_{i \in k}$ be a finite sequence of integers with the property $\sum_{i \in k} n_i = 0$. Then the set $\{\sum_{i \in k} \text{osc}(\alpha, a(i)) \cdot n_i : \alpha < a(0)\}$ is finite.

Proof. Assuming the converse, we can find an ordinal $\eta \leq a(0)$ and a sequence $(\xi_n)_{n \in \omega}$ of ordinals converging to η such that $\xi_n < \xi_{n+1}$ and $|\sum_{i \in k} \text{osc}(\xi_n, a(i)) \cdot n_i| \geq n$. Let $\gamma_0, \gamma_1 < \eta$ be such that $L(\eta, a(i)) \subset \gamma_0$ for all $i \in k$ and $L(\gamma, \eta) \supset \gamma_0$ for all $\gamma_1 \leq \gamma < \eta$, and $e_{a(i)}|_{(\gamma_0, \eta)} = e_{a(j)}|_{(\gamma_0, \eta)}$ for all $i, j \in k$ (this can be done by the facts above). Then for every $i \in k$ and $\gamma_1 \leq \gamma < \eta$, $L(\gamma, a(i)) = L(\gamma, \eta) \cup L(\eta, a(i))$, and hence $\text{Osc}(e_\gamma, e_{a(i)}, L(\gamma, a(i))) = \text{Osc}(e_\gamma, e_{a(i)}, L(\eta, a(i)) \cup L(\gamma, \eta))$. Let $q_\gamma = \text{Osc}(e_\gamma, e_{a(i)}, L(\gamma, \eta))$ (it does not depend on i by our choice of γ_1). Therefore

$$|\text{Osc}(e_\gamma, e_{a(i)}, L(\gamma, a(i)))| = |\text{Osc}(e_\gamma, e_{a(i)}, L(\eta, a(i)))| + q_\gamma + s_\gamma,$$

where $s_\gamma \in \{0, 1\}$ is the number indicating whether $\min L(\gamma, \eta)$ is included into

$$\text{Osc}(e_\gamma, e_{a(i)}, L(\gamma, a(i))) = \text{Osc}(e_\gamma, e_{a(i)}, L(\eta, a(i)) \cup L(\gamma, \eta))$$

or not. Set $M = \max_{i \in k} |L(\eta, a(i))|$. Then for every $\gamma \in (\gamma_1, \eta)$ we have

$$\begin{aligned} \left| \sum_{i \in k} \text{osc}(\gamma, a(i)) \cdot n_i \right| &= \left| \sum_{i \in k} |\text{Osc}(e_\gamma, e_{a(i)}, L(\gamma, a(i)))| \cdot n_i \right| \\ &= \left| \sum_{i \in k} (|\text{Osc}(e_\gamma, e_{a(i)}, L(\eta, a(i)))| + q_\gamma + s_\gamma) \cdot n_i \right| \\ &= \left| \sum_{i \in k} |\text{Osc}(e_\gamma, e_{a(i)}, L(\eta, a(i)))| \cdot n_i + \sum_{i \in k} q_\gamma \cdot n_i + \sum_{i \in k} s_\gamma \cdot n_i \right| \\ &= \left| \sum_{i \in k} |\text{Osc}(e_\gamma, e_{a(i)}, L(\eta, a(i)))| \cdot n_i + \sum_{i \in k} s_\gamma \cdot n_i \right| \leq (kM + 1) \sum_{i \in k} |n_i|, \end{aligned}$$

which is a contradiction. \square

Proof of Example 3.1. Assuming that G is not Lindelöf, we can find an increasing family $\{U_\alpha : \alpha < \omega_1\}$ of open subsets of \mathbb{T}^X covering G and an element $g_\alpha \in G \setminus U_\alpha$. Using the standard Δ -system argument, we can find an uncountable family $B \subset [X]^I$ of pairwise disjoint sets, a sequence $(n_j)_{j < I}$ of integers, $x \in \text{grp}(\mathcal{L}_X)$, and $\{y_b : b \in B\} \subset \text{pr}_X K$ such that

$$x \cdot \left\{ \prod_{j \in I} w_{b(j)}^{n_j} \cdot y_b : b \in B \right\} \subset \{g_\alpha : \alpha < \omega_1\},$$

and hence² the intersection

$$\left(x \cdot \left\{ \prod_{j \in I} w_{b(j)}^{n_j} : b \in B \right\} \cdot \text{pr}_X K \right) \cap \{g_\alpha : \alpha < \omega_1\}$$

is uncountable. Two cases are possible:

Case 1. $\sum_{j \in I} n_j \neq 0$. In this case Corollary 2.6 implies that $\{\prod_{j \in I} w_{b(j)}^{n_j} : b \in B\}$ is hereditarily Lindelöf, hence $x \cdot \{\prod_{j \in I} w_{b(j)}^{n_j} : b \in B\} \cdot \text{pr}_X K$ is Lindelöf being a continuous image of a product of a Lindelöf space with a σ -compact, and therefore this set is contained in some U_ξ , which contradicts the fact that it contains uncountably many g_α 's.

Case 2. $\sum_{j \in I} n_j = 0$. Passing to an uncountable subset of B , if necessary, we can additionally assume that $B = \{b_\xi : \xi < \omega_1\}$ and $b_\xi > b_\eta$ provided that $\eta < \xi$. Let $y'_b(\alpha) = \prod_{j \in I} w_{b(j)}^{n_j}(\alpha)$ if $\alpha < b(0)$, and $y'_b(\alpha) = 1$ otherwise, where $b \in B$. Applying Lemma 3.2, we conclude that $y'_b \in K$ for all $b \in B$. In addition, it is easy to see that $C = \{\prod_{j \in I} w_{b_\xi(j)}^{n_j} \cdot (y'_{b_\xi})^{-1} : \xi < \omega_1\} \cup \{1\}$ is a copy of $A(\omega_1)$. Therefore

$$x \cdot \left\{ \prod_{j \in I} w_{b(j)}^{n_j} : b \in B \right\} \cdot \text{pr}_X K = x \cdot \left\{ \prod_{j \in I} w_{b(j)}^{n_j} \cdot (y'_b)^{-1} \cdot y'_b : b \in B \right\} \cdot \text{pr}_X K \subset x \cdot C \cdot \text{pr}_X K \cdot \text{pr}_X K,$$

and the latter set is a σ -compact subset of G , and hence it is contained in some U_α , which is a contradiction. \square

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² Formally, we should have written $w_{-|X}$ instead of w_{-} throughout the proof.

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