Spectral isometries onto algebras having a separating family of finite-dimensional irreducible representations

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A R T I C L E   I N F O

A B S T R A C T

We prove that if $\mathcal{A}$ is a complex, unital semisimple Banach algebra and $\mathcal{B}$ is a complex, unital Banach algebra having a separating family of finite-dimensional irreducible representations, then any unital linear operator from $\mathcal{A}$ onto $\mathcal{B}$ which preserves the spectral radius is a Jordan morphism.

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1. Introduction and statement of results

An old problem of Kaplansky asks whether every unital linear surjective mapping T between (complex, unital) semisimple Banach algebras $\mathcal{A}$ and $\mathcal{B}$ which preserves invertible elements must be a Jordan morphism, that is, $T(a)^2 = T(a^2)$ for all a in $\mathcal{A}$ [5]. This question was partly motivated by the fact that it was known to have a positive answer in the case when $\mathcal{B}$ is commutative (the Gleason–Kahane–Żelazko theorem), or $\mathcal{A} = \mathcal{B} = \mathcal{M}_n$ the space of all $n \times n$ complex matrices (the Marcus–Purves theorem); see, e.g., [9]. Aupetit proved the conjecture is also true in the case when $\mathcal{B}$ has a separating family of finite-dimensional irreducible representations.

Theorem 1. *(See [1, Theorem 2].) Let $\mathcal{A}$ be a complex, unital Banach algebra and $\mathcal{B}$ a complex, unital Banach algebra having a separating family of finite-dimensional irreducible representations. If $T : \mathcal{A} \to \mathcal{B}$ is linear, surjective and such that $T1 = 1$, and a invertible in $\mathcal{A}$ implies that $T(a)$ is invertible in $\mathcal{B}$, then T is a Jordan morphism.*

Denoting by $\sigma(a)$ the spectrum of a Banach algebra element a, then $T : \mathcal{A} \to \mathcal{B}$ linear, unital and invertibility-preserving implies $\sigma(T(a)) \subseteq \sigma(a)$ for each a in A. This leads us to the study of spectrum-preserving mappings, that is, T satisfying $\sigma(T(a)) = \sigma(a)$ for each a. Aupetit proved in [3] that if T is a surjective spectrum-preserving linear mapping between two von Neumann algebras, then T is a Jordan morphism. It is not known if the same is true for general semisimple Banach algebras. In this case, one may consider the more general problem of characterizing the unital surjective spectral isometries in terms of Jordan morphisms; if $\mathcal{A}$ and $\mathcal{B}$ are semisimple unital Banach algebras and $T : \mathcal{A} \to \mathcal{B}$ is linear, unital, surjective and satisfies

$$\rho\big(T(a)\big) = \rho(a) \quad (a \in \mathcal{A}), \tag{1}$$

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must then T be a Jordan morphism? (For an element a in a Banach algebra $\mathcal{A}$ we have denoted by $\rho(a)$ its spectral radius.) For example, we know the answer to be positive where $\mathcal{A}$ and $\mathcal{B}$ are commutative (the Nagasawa theorem [2, p. 78]) or $\mathcal{A} = \mathcal{B} = \mathcal{L}(X)$, the set of all bounded linear operators on a Banach space X [4]. No answer is known in the case when $\mathcal{A}$ and $\mathcal{B}$ are both supposed to be general von Neumann algebras.

We refer the reader to [8] for some basic facts about spectral isometries. See also [6] and references therein for some more background information and some of the history of the problem. It is asked in [6, p. 302] whether an analogue of Theorem 1 holds in the case of mappings preserving the spectral radius. In this paper we give a positive answer to this question.

Theorem 2. *Let $\mathcal{A}$ be a complex, unital semisimple Banach algebra and $\mathcal{B}$ a complex, unital Banach algebra having a separating family of finite-dimensional irreducible representations. If $T : \mathcal{A} \to \mathcal{B}$ is linear, unital, surjective and satisfies* (1)*, then T is a Jordan morphism.*

2. Proofs

Let us recall first that if $\mathcal{A}$ and $\mathcal{B}$ are complex, unital, semisimple Banach algebras and $T : \mathcal{A} \to \mathcal{B}$ is a linear surjective spectral isometry, then T is automatically continuous and invertible [8]. Then $T^{-1} : \mathcal{B} \to \mathcal{A}$ is also a spectral isometry. For example, this holds when we are under the hypothesis of Theorem 2, since the conditions satisfied by $\mathcal{B}$ imply that it is also semisimple.

Given $S \subseteq \mathcal{A}$, we shall denote by $S^c = \{a \in \mathcal{A}: as = sa, \ \forall s \in S\}$. The key ingredient in the proof of Theorem 2 is the following result.

Lemma 3. *Suppose we are under the hypothesis of Theorem 2. Let $a \in \mathcal{A}$ and denote $\mathcal{A}_1 = \{a\}^{cc}$. Then $\mathcal{B}_1 := T(\mathcal{A}_1)$ is a closed subalgebra of $\mathcal{B}$.*

Proof. Let π be a finite-dimensional irreducible representation of $\mathcal{B}$. Then the Jacobson density theorem implies that $\pi(\mathcal{B}) = \mathcal{M}_n$ for some $n \geqslant 1$ [1]. So $\pi : \mathcal{B} \to \mathcal{M}_n$, and π is surjective. Let us also observe that by [2, Theorem 5.5.2] we have that π is also continuous. For $x \in \mathcal{A}$, $y \in \mathcal{B}$ and $k = 1, 2, \ldots, n$, consider the entire function

$$\lambda \mapsto S_k\big[\pi\big(T\big(e^{-\lambda x}T^{-1}(y)e^{\lambda x}\big)\big)\big], \tag{2}$$

where S_k, for $k = 1, 2, \ldots, n$, is the kth symmetric function on the eigenvalues of matrices of $\mathcal{M}_n$. By (1), we have

$$\begin{aligned}
\rho_{\mathcal{M}_n}\big(\pi\big(T\big(e^{-\lambda x}T^{-1}(y)e^{\lambda x}\big)\big)\big) &\leqslant \rho_{\mathcal{B}}\big(T\big(e^{-\lambda x}T^{-1}(y)e^{\lambda x}\big)\big) \\
&= \rho_{\mathcal{A}}\big(e^{-\lambda x}T^{-1}(y)e^{\lambda x}\big) \\
&= \rho_{\mathcal{A}}\big(T^{-1}(y)\big) = \rho_{\mathcal{B}}(y)
\end{aligned}$$

for all $\lambda \in \mathbb{C}$. This implies that the entire function defined by (2) is bounded on the complex plane. By Liouville's theorem, it is constant on $\mathbb{C}$. Taking $\lambda = 0$, we get

$$S_k\big[\pi(y)\big] = S_k\big[\pi\big(T\big(e^{-\lambda x}T^{-1}(y)e^{\lambda x}\big)\big)\big] \quad (k = 1, \ldots, n; \ \lambda \in \mathbb{C}).$$

Thus, for all λ we have that $\pi(T(e^{-\lambda x}T^{-1}(y)e^{\lambda x}))$ and $\pi(y)$ have the same characteristic polynomial, which in turn implies that

$$\sigma_{\mathcal{M}_n}\big(\pi(y)\big) = \sigma_{\mathcal{M}_n}\big(\pi\big(T\big(e^{-\lambda x}T^{-1}(y)e^{\lambda x}\big)\big)\big) \quad (x \in \mathcal{A}; \ y \in \mathcal{B}; \ \lambda \in \mathbb{C}). \tag{3}$$

Fix now $x \in \mathcal{A}$ and for $\lambda \in \mathbb{C}$ define $R_\lambda : \mathcal{B} \to \mathcal{M}_n$ by putting

$$R_\lambda(y) = \pi\big(T\big(e^{-\lambda x}T^{-1}(y)e^{\lambda x}\big)\big) \quad (y \in \mathcal{B}).$$

Then R_λ is linear and surjective, $R_\lambda(1) = 1$ and by (3) we have $\sigma_{\mathcal{M}_n}(R_\lambda(y)) \subseteq \sigma_{\mathcal{B}}(y)$. By [1, Theorem 1] we have that R_λ is either an algebra morphism or an algebra antimorphism. Let us also remark that $R_0 : \mathcal{B} \to \mathcal{M}_n$ is an algebra morphism. Also, if $n = 1$ then R_λ is an algebra morphism for all $\lambda \in \mathbb{C}$. Suppose now that $n \geqslant 2$ and define

$$\Lambda = \big\{\lambda \in \mathbb{C}: R_\lambda(b_1 b_2) = R_\lambda(b_1)R_\lambda(b_2), \ \forall b_1, b_2 \in \mathcal{B}\big\}.$$

Then $0 \in \Lambda$, and using the continuity one can easily see that $\Lambda \subseteq \mathbb{C}$ is a closed subset. In order to prove that Λ is the whole complex plane, we shall prove that $\Lambda \subseteq \mathbb{C}$ is also open. So suppose, to the contrary, that there exists a sequence $(\lambda_k)_{k \geqslant 1} \subseteq \mathbb{C} \setminus \Lambda$ such that $\lambda_k \to \lambda_0 \in \Lambda$. By what we have proved above, R_{λ_k} is an antimorphism for each $k = 1, 2, \ldots$. Therefore, for all $b_1, b_2 \in \mathcal{B}$ we have that $R_{\lambda_k}(b_1 b_2) = R_{\lambda_k}(b_2)R_{\lambda_k}(b_1)$ for $k = 1, 2, \ldots$. Passing with k to infinity we obtain that $R_{\lambda_0}(b_1 b_2) = R_{\lambda_0}(b_2)R_{\lambda_0}(b_1)$. Since $\lambda_0 \in \Lambda$ then $R_{\lambda_0}(b_1 b_2) = R_{\lambda_0}(b_1)R_{\lambda_0}(b_2)$. Thus $R_{\lambda_0}(b_2)R_{\lambda_0}(b_1) = R_{\lambda_0}(b_1)R_{\lambda_0}(b_2)$ for all $b_1, b_2 \in \mathcal{B}$; since R_{λ_0} is surjective, we obtain that $\mathcal{M}_n$ is commutative, thus arriving at a contradiction. We have therefore

proved that R_λ is a morphism of algebras for each $\lambda \in \mathbb{C}$. That is,

$$\pi\left(T\left(e^{-\lambda x}T^{-1}(b_1 b_2)e^{\lambda x}\right)\right) = \pi\left(T\left(e^{-\lambda x}T^{-1}(b_1)e^{\lambda x}\right)\right)\pi\left(T\left(e^{-\lambda x}T^{-1}(b_2)e^{\lambda x}\right)\right), \tag{4}$$

equality which holds for all $x \in \mathcal{A}$, $b_1, b_2 \in \mathcal{B}$ and $\lambda \in \mathbb{C}$.

Let now $a \in \mathcal{A}$ and define $\mathcal{A}_1$ and $\mathcal{B}_1$ as in the statement. Since $\mathcal{A}_1$ is a closed subspace of $\mathcal{A}$ and $T : \mathcal{A} \to \mathcal{B}$ is a linear topological isomorphism, then $\mathcal{B}_1 \subseteq \mathcal{B}$ is a closed subspace. In order to prove that it is a subalgebra, consider $b_1, b_2 \in \mathcal{B}_1$. Then $T^{-1}(b_1), T^{-1}(b_2) \in \{a\}^{cc}$. Pick an arbitrary $x \in \{a\}^c$. Then x commutes with $T^{-1}(b_1)$ and $T^{-1}(b_2)$, and using now (4) we get

$$\pi\left(T\left(e^{-\lambda x}T^{-1}(b_1 b_2)e^{\lambda x}\right)\right) = \pi(b_1)\pi(b_2) \quad (\lambda \in \mathbb{C}).$$

Therefore, $\pi(T(e^{-\lambda x}T^{-1}(b_1 b_2)e^{\lambda x}) - b_1 b_2) = 0$. This equality holds for any finite-dimensional irreducible representation π. Since $\mathcal{B}$ has a separating family of such representations, it follows that $T(e^{-\lambda x}T^{-1}(b_1 b_2)e^{\lambda x}) = b_1 b_2$ for all $\lambda \in \mathbb{C}$. Developing with respect to λ and identifying the coefficients of λ this gives $T^{-1}(b_1 b_2)x = xT^{-1}(b_1 b_2)$. That is, $T^{-1}(b_1 b_2) \in \{a\}^{cc}$, and therefore $b_1 b_2 \in \mathcal{B}$. $\quad\square$

We shall also need the following lemma in the proof of Theorem 2.

Lemma 4. *Let $\mathcal{A}$ and $\mathcal{B}$ be complex, unital Banach algebras, $\mathcal{B}$ being commutative, and let $T : \mathcal{A} \to \mathcal{B}$ be unital, linear and bijective satisfying* (1)*. Then*

$$\sigma_{\mathcal{B}}\left(T(a)\right) = \sigma_{\mathcal{A}}(a) \quad (a \in \mathcal{A}).$$

Proof. Denote by $\mathrm{Rad}(\mathcal{A})$ the (Jacobson) radical of $\mathcal{A}$ and by $\mathrm{Rad}(\mathcal{B})$ the radical of $\mathcal{B}$, and let us first prove that $T(\mathrm{Rad}(\mathcal{A})) = \mathrm{Rad}(\mathcal{B})$. To see this, we shall use the characterization of the radical given by [2, Theorem 5.3.1]: we have $a \in \mathrm{Rad}(\mathcal{A})$ if and only if $\rho_{\mathcal{A}}(a + x) = 0$ for all x in $\mathcal{A}$ with $\rho_{\mathcal{A}}(x) = 0$. Using that T is bijective and spectral radius preserving, we have

$$
\begin{aligned}
a \in \mathrm{Rad}(\mathcal{A}) \quad &\Leftrightarrow \quad \rho_{\mathcal{A}}(a + x) = 0, \quad \forall x \in \mathcal{A}, \; \rho_{\mathcal{A}}(x) = 0 \\
&\Leftrightarrow \quad \rho_{\mathcal{B}}\left(T(a) + T(x)\right) = 0, \quad \forall x \in \mathcal{A}, \; \rho_{\mathcal{A}}(x) = 0 \\
&\Leftrightarrow \quad \rho_{\mathcal{B}}\left(T(a) + y\right) = 0, \quad \forall y \in \mathcal{B}, \; \rho_{\mathcal{B}}(y) = 0 \\
&\Leftrightarrow \quad T(a) \in \mathrm{Rad}(\mathcal{B}).
\end{aligned}
$$

By [2, Corollary 3.2.2], we have that $\mathcal{A}_1 := \mathcal{A}/\mathrm{Rad}(\mathcal{A})$ and $\mathcal{B}_1 := \mathcal{B}/\mathrm{Rad}(\mathcal{B})$ are unital semisimple Banach algebras. Also, by [2, Theorem 3.1.5] we also have $\sigma_{\mathcal{A}}(a) = \sigma_{\mathcal{A}_1}(\bar{a})$ for the coset $\bar{a}$ of $a \in \mathcal{A}$ in $\mathcal{A}/\mathrm{Rad}(\mathcal{A})$, and $\sigma_{\mathcal{B}}(b) = \sigma_{\mathcal{B}_1}(\bar{b})$ for all $b \in \mathcal{B}$. Since $T(\mathrm{Rad}(\mathcal{A})) = \mathrm{Rad}(\mathcal{B})$ then $\tilde{T} : \mathcal{A}_1 \to \mathcal{B}_1$ given by $\tilde{T}(\bar{a}) = \overline{T(a)}$ for all $\bar{a} \in \mathcal{A}_1$ is well defined. Clearly $\tilde{T}$ is linear and bijective, with $\tilde{T}(\bar{1}) = \bar{1}$. Also, (1) gives

$$
\begin{aligned}
\rho_{\mathcal{B}_1}\left(\tilde{T}(\bar{a})\right) &= \rho_{\mathcal{B}_1}\left(\overline{T(a)}\right) = \rho_{\mathcal{B}}\left(T(a)\right) = \rho_{\mathcal{A}}(a) \\
&= \rho_{\mathcal{A}_1}(\bar{a})
\end{aligned}
$$

for all $\bar{a} \in \mathcal{A}_1$. Since $\mathcal{B}$ is commutative, the same is also true for $\mathcal{B}_1$. Let us prove now that $\mathcal{A}_1$ must necessarily be commutative. So let $\bar{a}$ in $\mathcal{A}_1$. Since the spectral radius is subadditive on commuting elements, we have for all $\bar{x}$ in $\mathcal{A}_1$ that

$$
\begin{aligned}
\rho_{\mathcal{A}_1}(\bar{a} + \bar{x}) &= \rho_{\mathcal{B}_1}\left(\tilde{T}(\bar{a}) + \tilde{T}(\bar{x})\right) \leqslant \rho_{\mathcal{B}_1}\left(\tilde{T}(\bar{a})\right) + \rho_{\mathcal{B}_1}\left(\tilde{T}(\bar{x})\right) \\
&= \rho_{\mathcal{A}_1}(\bar{a}) + \rho_{\mathcal{A}_1}(\bar{x}) \leqslant M\left(1 + \rho_{\mathcal{A}_1}(\bar{x})\right),
\end{aligned}
$$

where $M = \max\{1, \rho_{\mathcal{A}_1}(\bar{a})\}$. Using [2, Theorem 5.2.2] and the fact that $\mathcal{A}_1$ is semisimple, we obtain that $\bar{a}$ belongs to the center of $\mathcal{A}_1$. That is, $\mathcal{A}_1$ is commutative.

Thus, $\mathcal{A}_1$ and $\mathcal{B}_1$ are unital, commutative and semisimple Banach algebras and $\tilde{T} : \mathcal{A}_1 \to \mathcal{B}_1$ is linear, unital and bijective having the property that $\rho_{\mathcal{B}_1}(\tilde{T}(\bar{a})) = \rho_{\mathcal{A}_1}(\bar{a})$ for all $\bar{a} \in \mathcal{A}_1$. The Nagasawa theorem [2, Theorem 4.1.17] implies that $\tilde{T}$ is an algebra isomorphism. In particular,

$$\sigma_{\mathcal{B}_1}\left(\tilde{T}(\bar{a})\right) = \sigma_{\mathcal{A}_1}(\bar{a}) \quad (\bar{a} \in \mathcal{A}_1).$$

Then

$$\sigma_{\mathcal{A}}(a) = \sigma_{\mathcal{A}_1}(\bar{a}) = \sigma_{\mathcal{B}_1}\left(\tilde{T}(\bar{a})\right) = \sigma_{\mathcal{B}_1}\left(\overline{T(a)}\right) = \sigma_{\mathcal{B}}\left(T(a)\right)$$

for all $a \in \mathcal{A}$. $\quad\square$

Lemma 4 is essentially known, though maybe not stated explicitly in this form in the literature. For the sake of completeness, we decided to include a proof here. It can be derived from [7] as follows. Since T is a bijective spectral isometry, by [7, Prop. 2.11] we have that the image under T of the Jacobson radical of $\mathcal{A}$ is exactly the Jacobson radical of $\mathcal{B}$. The induced mapping on the quotients by the radical is still a bijective spectral isometry from a semisimple Banach algebra into a semisimple commutative Banach algebra. By [7, Prop. 4.3] we have that its domain must be itself commutative. Then Nagasawa's theorem finishes the proof.

We are now ready for the proof of our main result.

Proof of Theorem 2. Fix $a \in \mathcal{A}$. Denote $\mathcal{A}_1 = \{a\}^{cc}$ and $\mathcal{B}_1 := T(\mathcal{A}_1)$. Then $\mathcal{A}_1$ is a commutative, unital Banach algebra. By Lemma 3 we have that $\mathcal{B}_1$ is a unital Banach algebra. Also, $T^{-1} : \mathcal{B}_1 \to \mathcal{A}_1$ satisfies

$$\rho_{\mathcal{A}_1}\big(T^{-1}(b)\big) = \rho_{\mathcal{A}}\big(T^{-1}(b)\big) = \rho_{\mathcal{B}}(b)$$
$$= \rho_{\mathcal{B}_1}(b)$$

for all $b \in \mathcal{B}_1$. Using Lemma 4 we obtain that $\sigma_{\mathcal{A}_1}(T^{-1}(b)) = \sigma_{\mathcal{B}_1}(b)$ for each $b \in \mathcal{B}_1$. For $b = T(a)$ this gives $\sigma_{\mathcal{A}_1}(a) = \sigma_{\mathcal{B}_1}(T(a))$. Now observe that $\sigma_{\mathcal{A}_1}(a) = \sigma_{\mathcal{A}}(a)$ and that $\sigma_{\mathcal{B}}(T(a)) \subseteq \sigma_{\mathcal{B}_1}(T(a))$, and therefore

$$\sigma_{\mathcal{B}}\big(T(a)\big) \subseteq \sigma_{\mathcal{A}}(a) \quad (a \in \mathcal{A}). \tag{5}$$

If a is invertible in $\mathcal{A}$, then $0 \notin \sigma_{\mathcal{A}}(a)$. Hence (5) implies that $0 \notin \sigma_{\mathcal{B}}(T(a))$, and therefore T preserves invertibility. We now use Theorem 1 to conclude that T is a Jordan morphism. $\square$

Acknowledgments

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