



On the splitting problem for selections

Maxim V. Balashov^a, Dušan Repovš^{b,c,*}

^a Department of Higher Mathematics, Moscow Institute of Physics and Technology, Institutski Str. 9, Dolgoprudny, Moscow region 141700, Russia

^b Faculty of Mathematics and Physics, University of Ljubljana, Jadranska 19, Ljubljana 1000, Slovenia

^c Faculty of Education, University of Ljubljana, Kardeljeva ploščad 16, Ljubljana 1000, Slovenia

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ABSTRACT

We investigate when does the Repovš–Semenov splitting problem for selections have an affirmative solution for continuous set-valued mappings in finite-dimensional Banach spaces. We prove that this happens when images of set-valued mappings or even their graphs are P -sets (in the sense of Balashov) or strictly convex sets. We also consider an example which shows that there is no affirmative solution of this problem even in the simplest case in \mathbb{R}^3 . We also obtain affirmative solution of the approximate splitting problem for Lipschitz continuous selections in the Hilbert space.

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1. Introduction

The splitting problem for selections was recently stated in [10]. Let $F_i : X \rightarrow 2^{Y_i}$, $i = 1, 2$, be any (lower semi) continuous mappings with closed convex images and let $L : Y_1 \oplus Y_2 \rightarrow Y$ be any linear surjection. The splitting problem is the problem of representation of any continuous selection $f(x) \in L(F_1(x), F_2(x))$ in the form $f(x) = L(f_1(x), f_2(x))$, where $f_i(x) \in F_i(x)$ are some continuous selections, $i = 1, 2$.

This problem is related to some classical problems of set-valued analysis. First, it is a special case of the selection problem which is sufficiently common for various applications [11]. In particular it is quite close to the celebrated problem of parametrization of set-valued mappings [2,6,8].

Second, every affirmative solution of this problem is in fact, an answer to the following question: When does the operation of intersection of two (continuous) set-valued mappings yield a continuous (or lower semicontinuous) set-valued mapping with respect to the Hausdorff metric? This question is also quite common for certain branches of set-valued and nonsmooth analysis.

It is well known that the intersection of two continuous set-valued mappings is not necessarily continuous [2]. We shall first consider the extreme example which demonstrates the last assertion.

* Address for correspondence: Faculty of Mathematics and Physics, University of Ljubljana, Jadranska 19, Ljubljana 1000, Slovenia.

E-mail addresses: balashov@mail.mipt.ru (M.V. Balashov), dusan.repovs@guest.arnes.si (D. Repovš).

Consider Question 4.6 from [10]: Do there exist for every closed convex sets A, B and $C = A + B$, continuous functions $a : C \rightarrow A$ and $b : C \rightarrow B$ with the property that $a(c) + b(c) = c$, for all $c \in C$? In a space of dimension ≥ 3 the answer is negative.

Example 1.1. Consider the following sets in the 3-dimensional Euclidean space \mathbb{R}^3 (where $\text{co}(X)$ denotes the convex hull of X):

$$D_0 = \{(\cos t, \sin t, 0) \mid t \in [0, \pi]\}, \quad A_0 = \text{co}(D_0 \cup \{(1, 0, 1)\} \cup \{(-1, 0, 1)\}),$$

$$D_1 = \{(\cos t, \sin t, 1) \mid t \in [-\pi, 0]\}, \quad A_1 = \text{co}(D_1 \cup \{(1, 0, 0)\} \cup \{(-1, 0, 0)\}),$$

and $A = A_0 \cup A_1$. It is easy to see that A is a convex compact set. We also define the set $B = \text{co}((0, 0, 0), (0, 0, 1))$ and the set $C = A + B$.

Let $\Gamma = \{(\cos t, \sin t, 1 - \frac{2}{\pi}t) \mid t \in [-\frac{\pi}{2}, \frac{\pi}{2}]\}$, $\Gamma \subset \partial C$. Let

$$\Gamma_1 = \left\{ \left(\cos t, \sin t, 1 - \frac{2}{\pi}t \right) \mid t \in (0, \pi/2) \right\},$$

$$\Gamma_2 = \left\{ \left(\cos t, \sin t, 1 - \frac{2}{\pi}t \right) \mid t \in [-\pi/2, 0) \right\}$$

and $c_0 = (1, 0, 1)$.

Suppose that $c \in \Gamma_1$. In this case there exists only one pair of points $a(c) \in A$ and $b(c) \in B$ with the property $a(c) + b(c) = c$. Indeed, if $t_c \in (0, \frac{\pi}{2}]$ satisfies $c = (\cos t_c, \sin t_c, 1 - \frac{2}{\pi}t_c)$ then $a(c) = (\cos t_c, \sin t_c, 0)$, $b(c) = (0, 0, 1 - \frac{2}{\pi}t_c)$. The point $a(c)$ is unique because it is an exposed point of the set A for the vector $p_c = (\cos t_c, \sin t_c, 0)$. Clearly, the point $b(c)$ is also unique. So we have

$$\lim_{c \rightarrow c_0, c \in \Gamma_1} a(c) = (1, 0, 0). \tag{1.1}$$

In the case when $c \in \Gamma_2$, similar considerations show that there exists only one pair of points $a(c) = (\cos t_c, \sin t_c, 1) \in A$ and $b(c) = (0, 0, -\frac{2}{\pi}t_c) \in B$ with $a(c) + b(c) = c$ and such that

$$\lim_{c \rightarrow c_0, c \in \Gamma_2} a(c) = (1, 0, 1). \tag{1.2}$$

Formulae (1.1) and (1.2) show that $a(c)$ is not continuous at the point $c = c_0$.

Simultaneously, we want to emphasize that the set-valued mappings

$$C \ni c \rightarrow (c - B) \cap A \tag{1.3}$$

and

$$C \ni c \rightarrow (A, B) \cap L^{-1}(c) \tag{1.4}$$

do not allow any continuous (on $c \in C$) selections. Here $L^{-1}(c) = \{(y_1, y_2) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid y_1 + y_2 = c\}$.

Indeed, otherwise in the case (1.3) we could choose this selection as $a(c) \in (c - B) \cap A \subset A$ and set $b(c) = c - a(c) \in B$. In the case (1.4) we could choose $(a(c), b(c)) \in (A, B) \cap L^{-1}(c)$. In both cases we would have $a(c) + b(c) = c$, for all $c \in C$. This would contradict the fact that function $a(c)$, as it follows by Example 1.1, is not continuous at the point $c = c_0 = (1, 0, 1)$.

We shall further obtain an affirmative solution of the splitting problem for selections for some special cases in finite-dimensional Banach spaces. It suffices to solve this problem in Euclidean space \mathbb{R}^n (with the inner product $\langle \cdot, \cdot \rangle$) because all norms in any finite-dimensional Banach space are equivalent.

Our general idea is to prove continuity of the intersection $(F_1(x), F_2(x)) \cap L^{-1}(f(x))$. This is the key idea. When this is done we can choose some continuous selection (e.g. the Steiner point) of the map $x \rightarrow (F_1(x), F_2(x)) \cap L^{-1}(f(x))$ and solve the problem.

Unfortunately, this map is not continuous even in the simplest cases (as we can see from Example 1.1). So we need to invoke some special geometrical properties of maps F_i or surjection L .

The main results of Section 2 are Theorem 2.4 with Corollary 2.5, Theorems 2.6 and 2.10. The key geometric objects used in Section 2 are P -sets (see Definition 2.1) and strictly convex sets.

In Section 3 we shall consider situation when in infinite-dimensional Hilbert spaces X, Y_1, Y_2 there exist for every $\varepsilon > 0$ and any Lipschitz continuous selection $f(x) \in L(F_1(x), F_2(x))$, Lipschitz continuous selections $f_i(x) \in F_i(x) + B_\varepsilon^{Y_i}(0)$, $i = 1, 2$, with the property $f(x) = L(f_1(x), f_2(x))$, for all x . Here, $B_\varepsilon^{Y_i}(0) = \{y \in Y_i \mid \|y\| \leq \varepsilon\}$. The main results of Section 3 are Theorems 3.1 and 3.7.

We need to give some definitions for further explanation. We shall say that the subspace $\mathcal{L} \subset \mathcal{L}_1 \oplus \mathcal{L}_2$ is *not parallel* to the subspaces \mathcal{L}_1 and \mathcal{L}_2 if for any pair of distinct points $w_1, w_2 \in \mathcal{L}$, the projections of w_1 and w_2 onto \mathcal{L}_1 (resp. \mathcal{L}_2) parallel to \mathcal{L}_2 (resp. \mathcal{L}_1) yield different points.

Let h be the Hausdorff distance. For any bounded subsets A, B of a Banach space X we have

$$h(A, B) = \inf\{h > 0 \mid A \subset B + B_h^X(0), B \subset A + B_h^X(0)\}.$$

For any subsets A, B of a linear space X the operation

$$A \stackrel{*}{\ominus} B = \{x \in X \mid x + B \subset A\} = \bigcap_{b \in B} (A - b)$$

is called the *geometric difference* (or the *Minkowski–Pontryagin difference*) of sets A and B . A direct consequence of the definition of geometric difference is that $(A \stackrel{*}{\ominus} B) + B \subset A$.

The *Chebyshev center* $c(A)$ of a convex closed bounded subset A of a Hilbert space X is the point

$$c(A) = \arg \inf_{x \in X} \left(\sup_{a \in A} \|x - a\| \right).$$

Chebyshev center always exists and it is unique.

Let X, Y be any Banach spaces. We say that a continuous linear surjection $L : X \rightarrow Y$ has the *Lip-property* if the set-valued mapping $L^{-1}(y) = \{x \in X \mid Lx = y\}$ has a Lipschitz (at y) selection $l(y) \in L^{-1}(y)$.

For example, if $Y_1 = Y_2 = Y$ and $L : Y_1 \oplus Y_2 \rightarrow Y, L(y_1, y_2) = y_1 + y_2$, then $l(y) = (\frac{1}{2}y, \frac{1}{2}y)$.

2. P-sets and the splitting problem for selections

We shall obtain an affirmative solution in certain cases when the images of the set-valued mapping are P -sets [3]. Let $q \in \mathbb{R}^n$ be an arbitrary unit vector and $L(q) = \{z \in \mathbb{R}^n \mid \langle q, z \rangle = 0\}, l(q) = \{\lambda q \mid \lambda \in \mathbb{R}\}$. The space \mathbb{R}^n is the orthogonal sum of sets $L(q)$ and $l(q) : \mathbb{R}^n = L(q) \oplus l(q)$. Any point $z \in \mathbb{R}^n$ can be expressed in the form $z = x + \mu q$, where $\mu \in \mathbb{R}, x \in L(q)$, or $z = (x, \mu)$. Let $P_q : \mathbb{R}^n \rightarrow L(q)$ be the operator of orthogonal projection: for any $z \in \mathbb{R}^n, P_q z = x$, where $z = (x, \mu)$.

Let $A \subset \mathbb{R}^n$ be any convex compact set. Let us define the function $f_{A,q} : P_q A \rightarrow \mathbb{R}$ by

$$f_{A,q}(x) = \min\{\mu \mid (x, \mu) \in A\}, \quad \text{for all } x \in P_q A. \tag{2.5}$$

Definition 2.1. (See [3].) A convex compact subset $A \subset \mathbb{R}^n$ is called a P -set, if for any unit vector q , the function $f_{A,q}$ (2.5) is continuous on the set $P_q A$.

Proposition 2.2. (See [3,9].) Any convex compact subset of \mathbb{R}^2 is a P -set. In \mathbb{R}^n any convex polyhedron is a P -set, any strictly convex compact subset is a P -set, and any finite Minkowski sum of P -sets is a P -set. If $L : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a linear operator and $A \subset \mathbb{R}^n$ is a P -set then $LA \subset \mathbb{R}^m$ is also a P -set. Moreover, the map $L : A \rightarrow LA$ is open in induced topologies.

We emphasize that a P -set is not necessarily a polyhedron or strictly convex.

Example 2.3. The cylinder $\{(x_1, x_2, x_3) \mid x_1^2 + x_2^2 \leq 1, 0 \leq x_3 \leq 1\}$ is a P -set as the Minkowski sum of two P -sets

$$\{(x_1, x_2, 0) \mid x_1^2 + x_2^2 \leq 1\} + \{(0, 0, x_3) \mid 0 \leq x_3 \leq 1\}.$$

On the other hand, the subset $A_0 \subset \mathbb{R}^3$,

$$A_0 = \text{co}(\{(x_1 - 1)^2 + x_2^2 = 1, x_3 = 1\} \cup \{(0, 0, 0)\}),$$

is not a P -set.

Indeed, for $q = (0, 0, 1)$ it is easy to see that $f_{A_0,q}$ is not upper semicontinuous at the point $(0, 0) \in P_q A_0$. Note that the sum $A_0 + A_1$ (where A_1 is an arbitrary convex compact set) is not a P -set [3].

It was proved in [3] that if the subset $A \subset \mathbb{R}^n$ is convex and compact then the function $f_{A,q}$ (2.5) is lower semicontinuous on $P_q A$. This is quite obvious. Therefore the question about continuity of the function $f_{A,q}$ is the question about its upper semicontinuity.

The domain of the set-valued mapping $F : \mathbb{R}^n \rightarrow 2^{\mathbb{R}^m}$ is the set $\text{dom } F = \{x \in \mathbb{R}^n \mid F(x) \neq \emptyset\}$.

Theorem 2.4. Let $A \subset \mathbb{R}^n$ be any convex compact subset and $\mathcal{L} \subset \mathbb{R}^n$ any subspace. Suppose that one of the following properties is satisfied:

- (1) $\dim \mathcal{L} = n - 1$, or
- (2) A is a P -set.

Then the set-valued mapping $F(z) = (z + \mathcal{L}) \cap A$ is continuous in the Hausdorff metric.

Remark. It is easy to see that $\text{dom } F = A + \mathcal{L}$.

Proof. (1) Part (1) is a well-known fact, but we prove it for completeness. Let $A \subset \mathbb{R}^n$ be an arbitrary convex compact subset and $\dim \mathcal{L} = n - 1$. Consider $F(z) = (z + \mathcal{L}) \cap A$, for all $z \in \text{dom } F$. Using the closed graph theorem for set-valued mappings [1, Theorem 8.3.1] we conclude that map F is upper semicontinuous at any point $z_0 \in \text{dom } F$.

This means that for any sequence $\{z_k\} \subset \text{dom } F$, $z_k \rightarrow z_0$, and any $\varepsilon > 0$, there exists a number k_ε such that for any $k > k_\varepsilon$ the following holds:

$$F(z_k) \subset F(z_0) + B_\varepsilon(0). \tag{2.6}$$

Suppose that lower semicontinuity fails at some point $z_0 \in \text{dom } F$. Then there exist a number $\varepsilon_0 > 0$ and a sequence $\{z_k\} \subset \text{dom } F$ such that $\lim z_k = z_0$ and

$$F(z_0) \not\subset F(z_k) + B_{\varepsilon_0}(0). \tag{2.7}$$

This implies that $z_k \notin z_0 + \mathcal{L}$, for all k . From the condition (2.7) we conclude that for any k , there exists a point $w_k \in F(z_0)$ for which $w_k \notin F(z_k) + B_{\varepsilon_0}(0)$.

We may assume that $w_k \rightarrow w_0 \in F(z_0)$, due to the compactness of the set $F(z_0)$ and since

$$w_0 \notin F(z_k) + B_{\varepsilon_0/2}(0). \tag{2.8}$$

Without loss of generality we may assume that $z_1 \in A$. Otherwise, we can choose an arbitrary point $\tilde{z}_1 \in F(z_1)$ instead of z_1 . We can also suppose that $\|z_1 - w_0\| > \frac{\varepsilon_0}{2}$, otherwise we could reduce ε_0 .

Let φ be the angle between the segment $[w_0, z_1]$ and the hyperplane \mathcal{L} , $\varphi \in (0, \frac{\pi}{2}]$ (note that the segment $[w_0, z_1]$ is not parallel to \mathcal{L}). Without loss of generality we may assume that the halfspace with the bound $z_0 + \mathcal{L}$, which contains z_1 , contains the entire sequence $\{z_k\}$.

Let us choose a number k for which the distance from the point z_k to the hyperplane $z_0 + \mathcal{L}$ is sufficiently small:

$$\varrho(z_k, z_0 + \mathcal{L}) < \frac{\varepsilon_0}{2} \sin \varphi.$$

Define the point $w = w_0 + \frac{z_1 - w_0}{\|z_1 - w_0\|} \frac{\varepsilon_0}{2}$. We have

$$\varrho(w, z_0 + \mathcal{L}) = \frac{\varepsilon_0}{2} \sin \varphi > \varrho(z_k, z_0 + \mathcal{L}).$$

We can now conclude that the points w_0 and w lie on the opposite sides of the hyperplane $z_k + \mathcal{L}$. This follows from the last estimate, the inclusion $w_0 \in z_0 + \mathcal{L}$ and the fact that the points z_k, z_1 (and consequently w) lie on the same side of the hyperplane $z_0 + \mathcal{L}$. This means that the following holds:

$$[w_0, w] \cap (z_k + \mathcal{L}) \neq \emptyset. \tag{2.9}$$

From the inclusions $w_0 \in F(z_0)$, $z_1 \in A$ we obtain that $[w_0, w] \subset [w_0, z_1] \subset A$. We can conclude from this inclusion and the equality $\|w_0 - w\| = \frac{\varepsilon_0}{2}$ that $[w_0, w] \subset A \cap B_{\varepsilon_0/2}(w_0)$. According to (2.9), we have

$$A \cap B_{\varepsilon_0/2}(w_0) \cap (z_k + \mathcal{L}) \neq \emptyset,$$

i.e. $w_0 \in (A \cap (z_k + \mathcal{L})) + B_{\varepsilon_0/2}(0)$. This contradicts the existence of the inclusion (2.8).

The upper and lower semicontinuity imply continuity in the Hausdorff metric.

(2) Suppose now that $\dim \mathcal{L} = m$, $1 \leq m \leq n$, and that $A \subset \mathbb{R}^n$ is a P -set. Upper semicontinuity can be proved in the same way as in (1) above.

Assuming to the contrary, as in (1), we conclude that there exist $\varepsilon_0 > 0$, $w_0 \in F(z_0)$ and a sequence $\{z_k\} \subset \text{dom } F$, $\lim z_k = z_0$ such that

$$w_0 \notin F(z_k) + B_{\varepsilon_0/2}(0).$$

The map F is upper semicontinuous and not continuous. Hence

$$F_0 = \limsup_{k \rightarrow \infty} F(z_k) = \bigcap_{\varepsilon > 0} \bigcup_{\delta > 0} \bigcap_{k > \delta} (F(z_k) + B_\varepsilon(0)) \subset F(z_0).$$

Let us fix any point $w \in F_0$. Obviously, $w \neq w_0$. Let $q = \frac{w-w_0}{\|w-w_0\|}$. Suppose that $v_{k_n} \in F(z_{k_n})$ is a sequence such that $v_{k_n} \rightarrow w$. We have $F(z_{k_n}) = F(v_{k_n})$, $F(z_0) = F(w_0) = F(w)$.

Let $w_0 = (x_0, \mu_0)$. Note that $x_0 = P_q w$. Let $x_{k_n} = P_q v_{k_n}$. For sufficiently large n (when $\|w - v_{k_n}\| < \frac{\varepsilon_0}{4}$) we have $(v_{k_n} + l(q)) \cap B_{\varepsilon_0/4}(w_0) \neq \emptyset$, $l(q) = \{\lambda(w - w_0) \mid \lambda \in \mathbb{R}\} \subset \mathcal{L}$ and consequently, $f_{A,q}(x_{k_n}) \geq \mu_0 + \frac{\varepsilon_0}{4}$. Together with $f_{A,q}(x_0) \leq \mu_0$, this contradicts the definition of a P -set. \square

The graph of a set-valued mapping $F : \mathbb{R}^n \rightarrow 2^{\mathbb{R}^m}$ is the set $\text{graph } F = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m \mid x \in \text{dom } F, y \in F(x)\}$.

Corollary 2.5. *Let $F : \mathbb{R}^n \rightarrow 2^{\mathbb{R}^m}$ be any set-valued mapping with a convex closed graph.*

- (1) *If $n = 1$ and graph F is convex and compact then F is continuous.*
- (2) *If graph F is a P -set then F is continuous.*

Proof. Proof of Corollary 2.5 follows from Theorem 2.4 and the equality

$$\{x_0\} \times F(x_0) = \text{graph } F \cap \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m \mid x = x_0\}. \quad \square$$

Corollary 2.5 is false if graph F is not a P -set. Let q be the vector $(0, 0, 1) \in \mathbb{R}^3$. Let $\text{graph } F = A_0$, A_0 be as in Example 2.3, $\text{dom } F = \{(x_1 - 1)^2 + x_2^2 \leq 1\}$ and $F(x_1, x_2) = \{\lambda \mid (x_1, x_2, \lambda) \in A_0\}$. Then the set-valued mapping F is not lower semicontinuous at the point $(0, 0)$.

The Steiner point of a convex compact subset $A \subset \mathbb{R}^n$ is the point

$$s(A) = \frac{1}{\mu_n B_1(0)} \int_{\|p\|=1} s(p, A) p \, dp,$$

$s(p, A) = \sup_{x \in A} \langle p, x \rangle$, where μ_n is the Lebesgue measure in \mathbb{R}^n . For any convex compact subsets $A, B \subset \mathbb{R}^n$, we have:

$$\|s(A) - s(B)\| \leq L_n h(A, B), \quad L_n = \frac{2}{\sqrt{\pi}} \frac{\Gamma(\frac{n}{2} + 1)}{\Gamma(\frac{n+1}{2})},$$

and $s(A) \in A$. The Lipschitz constant L_n above is the best possible [2,9].

Theorem 2.6. *Let A, B be any closed convex subsets and let $C = A + B$. If C is a P -set then there exist continuous functions $a : C \rightarrow A$ and $b : C \rightarrow B$ with the property that $a(c) + b(c) = c$, for all $c \in C$.*

Proof. Let $L : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a linear operator, $L(x_1, x_2) = x_1 + x_2$. Then L is a surjection. The set-valued mapping $\mathbb{R}^n \ni c \rightarrow L^{-1}(c) = \{(x_1, x_2) \mid x_1 + x_2 = c\}$ is Lipschitz continuous in the Hausdorff metric [1, Corollary 3.3.6] and its values are parallel affine planes of the same dimension. The set C is a P -set and consequently, A and B are P -sets, too [3]. Using Corollary 2.5, we conclude that the set-valued mapping $C \ni c \rightarrow (A, B) \cap L^{-1}(c)$ is continuous. Taking the Steiner point $s(\cdot)$ of the latter set-valued mapping, we get the following:

$$(a(c), b(c)) = s((A, B) \cap L^{-1}(c)). \quad \square$$

Theorem 2.7. *Consider any set-valued mappings $F_1 : \mathbb{R}^n \rightarrow 2^{\mathbb{R}^{m_1}}$ and $F_2 : \mathbb{R}^n \rightarrow 2^{\mathbb{R}^{m_2}}$. Suppose that $\text{graph}(F_1, F_2)$ is a P -set. Suppose that $L : \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \rightarrow \mathbb{R}^k$ is a linear surjection. Then for any continuous selection $f(x) \in L(F_1(x), F_2(x))$, there exist continuous selections $f_i(x) \in F_i(x)$, $i = 1, 2$, with $f(x) = L(f_1(x), f_2(x))$.*

Proof. We can take

$$(x, f_1(x), f_2(x)) = s((\text{graph}(F_1, F_2)) \cap (x, L^{-1}(f(x)))).$$

The map $x \rightarrow (x, L^{-1}(f(x)))$ is continuous, due to [1, Corollary 3.3.6] and the intersection is continuous by Corollary 2.5. \square

Theorem 2.8. *Suppose that a compact subset $A \subset \mathbb{R}^n$ is strictly convex and that $B \subset \mathbb{R}^n$ is an arbitrary convex closed subset. In this case there exist continuous functions $a : C \rightarrow A$ and $b : C \rightarrow B$ with the property that $a(c) + b(c) = c$, for all $c \in C$.*

Proof. We shall consider $H(c) = (c - A) \cap B$, $c \in C$. Note that $H(c) \neq \emptyset$, for all $c \in C$. If $\text{ri } H(c_0) \neq \emptyset$, for some $c_0 \in C$, the condition of nonempty interior yields the continuity of $H(c)$ at the point $c = c_0$ [2,9].

Note that $\text{ri } H(c_0)$ is the relative interior of the set $H(c_0)$, i.e. the interior of the set $H(c_0)$ in the affine hull of the set B .

If $\text{ri} H(c_0) = \emptyset$ then $H(c_0)$ is a single point, due to the strict convexity of A . The intersection of $c - A$ and B is upper semicontinuous, by the closed graph theorem [1, Theorem 3.1.8], i.e.

$$H(c) \subset H(c_0) + B_\varepsilon(0), \quad \forall c \in B_\delta(c_0) \cap C.$$

But this implies

$$H(c_0) \in H(c) + B_\varepsilon(0), \quad \forall c \in B_\delta(c_0) \cap C.$$

Both formulae yield the continuity (in the Hausdorff metric) at the point $c = c_0$. Thus $H(c)$ is continuous at any point $c \in C$ and $b(c) = s(H(c))$, $a(c) = c - b(c)$. \square

Theorem 2.9. *Let X be any metric space. Consider any set-valued mappings $F_1 : X \rightarrow 2^{\mathbb{R}^{m_1}}$ and $F_2 : X \rightarrow 2^{\mathbb{R}^{m_2}}$ which are continuous in the Hausdorff metric. Suppose that F_1 has strictly convex compact images and that F_2 has closed convex images. Suppose also that $f(x) \in F_1(x) + F_2(x)$ is a continuous selection. Then there exist continuous selections $f_i(x) \in F_i(x)$ with $f(x) = f_1(x) + f_2(x)$, for all $x \in X$.*

Proof. Proof is similar to that of Theorem 2.8. The set-valued mapping

$$H(x) = (f(x) - F_1(x)) \cap F_2(x)$$

is continuous and $f_2(x) = s(H(x))$, $f_1(x) = f(x) - f_2(x)$. \square

Theorem 2.10. *Let X be any metric space. Let $F_i : X \rightarrow 2^{\mathbb{R}^{m_i}}$, $i = 1, 2$, be set-valued mappings with strictly convex compact or single-point images, which are continuous in the Hausdorff metric. Let $\mathcal{L}_i = \mathbb{R}^{m_i}$, $i = 1, 2$, and let $L : \mathcal{L}_1 \oplus \mathcal{L}_2 \rightarrow \mathbb{R}^k$ be a linear surjection such that $\mathcal{L} = \ker L$ is not parallel to \mathcal{L}_i , $i = 1, 2$. Then for any continuous selection $f(x) \in L(F_1(x), F_2(x))$, there exist continuous selections $f_i(x) \in F_i(x)$, $i = 1, 2$, such that $f(x) = L(f_1(x), f_2(x))$, for all $x \in X$.*

Proof. By the closed graph theorem [1, Theorem 3.1.8] and [1, Corollary 3.3.6], the set-valued mapping

$$H(x) = (F_1(x), F_2(x)) \cap L^{-1}(f(x))$$

is upper semicontinuous, for all $x \in X$.

Note that the mapping which associates to each convex compact subset of \mathbb{R}^n its nearest (with respect to zero) point, is a continuous selection of sets in the Euclidean space, in the Hausdorff metric [2,5] and hence

$$L^{-1}(f(x)) = w(x) + \mathcal{L},$$

where $w(x) = (w_1(x), w_2(x))$ is a continuous (at $x \in X$) projection of the zero onto $L^{-1}(f(x))$ in the Euclidean space $\mathcal{L}_1 \oplus \mathcal{L}_2$.

Thus we can write

$$H(x) = w(x) + (F_1(x) - w_1(x), F_2(x) - w_2(x)) \cap \mathcal{L}.$$

Consequently, we can assume that $w(x) = 0$, $H(x) = (F_1(x), F_2(x)) \cap \mathcal{L} \neq \emptyset$ and prove the continuity of the last map. We shall assume that $H(x_0)$ is not a single point, otherwise H would be continuous at the point $x = x_0$ for the same reason as in Theorem 2.8.

Suppose that H is not lower semicontinuous at the point x_0 . This means that

$$\exists w_0 \in H(x_0), \quad \varepsilon_0 > 0, \quad x_k \rightarrow x_0, \quad \text{such that } w_0 \notin H(x_k) + B_{\varepsilon_0}(0), \quad \forall k. \tag{2.10}$$

Let $w \in \limsup_{k \rightarrow \infty} H(x_k) \subset H(x_0)$; $w \neq w_0$, $w, w_0 \in \mathcal{L}$. Let $[u, u_0] \subset F_1(x_0) \cap \mathcal{L}_1$ be a projection of the segment $[w, w_0]$ onto \mathcal{L}_1 parallel to \mathcal{L}_2 and $[v, v_0] \subset F_2(x_0) \cap \mathcal{L}_2$ a projection of the segment $[w, w_0]$ onto \mathcal{L}_2 parallel to \mathcal{L}_1 .

By hypothesis, we have $u \neq u_0$, $v \neq v_0$. Sets $F_i(x_0)$, $i = 1, 2$, are strictly convex and $\frac{u+u_0}{2} \in \text{int} F_1(x_0)$, $\frac{v+v_0}{2} \in \text{int} F_2(x_0)$. Thus we can find $\alpha > 0$ such that:

$$B_\alpha^1\left(\frac{u+u_0}{2}\right) \subset F_1(x_0), \quad B_\alpha^2\left(\frac{v+v_0}{2}\right) \subset F_2(x_0)$$

and

$$B_\alpha\left(\frac{w+w_0}{2}\right) \subset (F_1(x_0), F_2(x_0)).$$

In the last inclusion we considered the ball of norm $\max\{\|u\|_{\mathcal{L}_1}, \|v\|_{\mathcal{L}_2}\}$, $(u, v) \in \mathcal{L}_1 \oplus \mathcal{L}_2$.

Without loss of generality we may assume that $\|w - w_0\| > \varepsilon_0$ (otherwise we reduce $\varepsilon_0 > 0$). Let $t = \frac{\varepsilon_0}{\|w - w_0\|} \in (0, 1)$. By a homothety with center w_0 and the coefficient t we get that for $\tilde{w} = w_0 + \frac{t}{2}(w - w_0)$ the inclusion $B_{t\varepsilon}(\tilde{w}) \subset (F_1(x_0), F_2(x_0))$ holds and $\|\tilde{w} - w_0\| = \frac{\varepsilon_0}{2}$.

By continuity of F_i , $i = 1, 2$, we get that there exists k_0 such that $B_{\frac{1}{2}t\varepsilon}(\tilde{w}) \subset (F_1(x_k), F_2(x_k))$, for all $k > k_0$ and hence we have:

$$\tilde{w} \in (F_1(x_k), F_2(x_k)) \cap \mathcal{L} \cap B_{\varepsilon_0}(w_0), \quad \forall k > k_0,$$

i.e. $H(x_k) \cap B_{\varepsilon_0}(w_0) \neq \emptyset$ for all $k > k_0$. This contradicts (2.10).

So we have proved that $H(x)$ is continuous in the Hausdorff metric, for all $x \in X$. Taking the Steiner point of $H(x) = (F_1(x), F_2(x)) \cap L^{-1}(f(x))$ we obtain that $(f_1(x), f_2(x)) = s(H(x))$. \square

Corollary 2.11. *Let X be any metric space. Let $F_i : X \rightarrow 2^{\mathbb{R}^{m_i}}$, $i = 1, 2$, be $(\varepsilon - \delta)$ -lower semicontinuous set-valued mappings with strictly convex compact images. Let $\mathcal{L}_i = \mathbb{R}^{m_i}$, $i = 1, 2$, and let $L : \mathcal{L}_1 \oplus \mathcal{L}_2 \rightarrow \mathbb{R}^k$ be a linear surjection such that $\mathcal{L} = \ker L$ is not parallel to \mathcal{L}_i , $i = 1, 2$. Suppose that for any $x \in X$ and any point $f \in L(F_1(x), F_2(x))$ there exists a pair of distinct points $w_1, w_2 \in (F_1(x), F_2(x))$ such that $f = Lw_i$, $i = 1, 2$. Then for any continuous selection $f(x) \in L(F_1(x), F_2(x))$ there exist continuous selections $f_i(x) \in F_i(x)$, $i = 1, 2$, such that $f(x) = L(f_1(x), f_2(x))$, for all $x \in X$.*

Proof. We can repeat word-by-word the proof from Theorem 2.10 of the lower semicontinuity of H at any point x_0 . The only difference is when we choose the point w as an arbitrary point of the set $H(x_0) \setminus \{w_0\}$. Using the Michael selection theorem [7] we can choose a continuous selection $(f_1(x), f_2(x)) \in H(x)$. \square

Finally, we shall prove that the exact solution of the splitting problem for selections takes place on the dense subset of arguments of the G_δ -type.

Theorem 2.12. *Let X be any metric space and Y, Y_i , $i = 1, 2$, any Banach spaces. Let $F_i : X \rightarrow 2^{Y_i}$, $i = 1, 2$, be upper semicontinuous set-valued mappings with convex compact images and suppose that the set $\text{cl}(F_1(X), F_2(X))$ is compact. Let $L : Y_1 \oplus Y_2 \rightarrow Y$ be any continuous linear surjection. Then for any continuous selection $f(x) \in L(F_1(x), F_2(x))$ there exist a dense G_δ -set $X_f \subset X$ and selections $f_i(x) \in F_i(x)$, $i = 1, 2$, continuous on the set X_f , such that $f(x) = L(f_1(x), f_2(x))$, for all $x \in X_f$.*

Proof. The intersection $H(x)$ of the continuous mapping $L^{-1}(f(x))$ and the upper semicontinuous mapping $(F_1(x), F_2(x))$ with compact images is upper semicontinuous [2,9].

Moreover, the set $\text{graph } H$ is closed. By [1, Theorem 3.1.10], $H(x)$ is continuous on some dense G_δ -set $X_f \subset X$. Note that X_f is also a metric space. Applying the Michael selection theorem [7] to the set-valued mapping $H : X_f \rightarrow 2^{Y_1 \oplus Y_2}$, we obtain continuous selections $(f_1(x), f_2(x)) \in H(x)$, for all $x \in X_f$. \square

We conclude this section by some final remarks concerning P -sets.

Lemma 2.13. *Let $A \subset \mathbb{R}^n$ be any convex compact subsets and suppose that in terms of Definition 2.1, for any unit vector q the operator P_q is an open map $P_q : A \rightarrow P_q A$, in the induced topologies. Then A is a P -set.*

Proof. Suppose that A is not a P -set. Then for some unit vector q there exists a sequence $\{x_k\} \subset P_q A$ such that $\lim x_k = x_0$ and $\lim f_{A,q}(x_k) = \mu_0 > f(x_0)$. Let $z_0 = (x_0, f_{A,q}(x_0))$, $\tilde{z}_0 = (x_0, \mu_0)$, $z = \frac{1}{2}(\tilde{z}_0 + z_0) = (x_0, \frac{1}{2}(\mu_0 + f_{A,q}(x_0)))$, and $\varepsilon = \frac{1}{4}\|\tilde{z}_0 - z_0\| = \frac{1}{4}(\mu_0 - f_{A,q}(x_0))$. Then $x_k \notin P_q(B_\varepsilon(z) \cap A)$, for all k . This contradicts the openness of P_q . \square

Theorem 2.14. *Any convex compact subset $A \subset \mathbb{R}^n$ is a P -set if and only if for any natural number m and any linear map $L : \mathbb{R}^n \rightarrow \mathbb{R}^m$, the map $L : A \rightarrow LA$ is open in the induced topology.*

Proof. The openness $L : A \rightarrow LA$ was proved in [3]. By Lemma 2.13, we get the converse statement, it suffices to take $L = P_q$. \square

Corollary 2.15. *Let E be any Banach space, $\dim E = n$, and $E = L \oplus l$, $\dim L = n - 1$, $\dim l = 1$. A set $A \subset E$ is a P -set if and only if the projection A onto L , parallel to l , is an open map and this property holds for any pairs of subspaces L, l with $E = L \oplus l$, $\dim L = n - 1$, $\dim l = 1$.*

Proof. Proof follows from Theorem 2.14 and the equivalence of the Euclidean and the given norm. \square

3. Solution of the approximate splitting problem for Lipschitz selections in a Hilbert space

Let $X, Y, Y_i, i = 1, 2$, be infinite-dimensional Hilbert spaces. Define $B_\varepsilon^X(x) = \{y \in X \mid \|x - y\| \leq \varepsilon\}$. Let $L : Y_1 \oplus Y_2 \rightarrow Y$ be any continuous linear surjection. We shall consider the following problem: When can an arbitrary Lipschitz continuous (resp. simply Lipschitz) selection $f(x) \in L(F_1(x), F_2(x))$ be represented in the form $f(x) = L(f_1(x), f_2(x))$, where $f_i(x) \in F_i(x)$ are some Lipschitz selections, $i = 1, 2$.

Clearly, Example 1.1 shows that there is no affirmative solution of this problem in such a formulation.

We shall prove that there exists an approximate solution of the Lipschitz splitting problem, namely that for any $\varepsilon > 0$, any pair of uniformly continuous set-valued mappings $F_i, i = 1, 2$, with closed convex bounded images, and any Lipschitz selection $f(x) \in L(F_1(x), F_2(x))$, there exist Lipschitz selections $f_i(x) \in F_i(x) + B_\varepsilon^{Y_i}(0)$ such that $f(x) = L(f_1(x), f_2(x))$.

Theorem 3.1. *Let (X, ϱ) be any metric space and $(Y, \|\cdot\|)$ any Banach space. Let $F_i : X \rightarrow 2^Y, i = 1, 2$, be any set-valued mappings with closed convex images. Let $\omega_i : [0, +\infty) \rightarrow [0, +\infty), i = 1, 2$, be the modulus of continuity for F_i , i.e. $\lim_{t \rightarrow +0} \omega_i(t) = 0$ and*

$$\forall x_1, x_2 \in X: h(F_i(x_1), F_i(x_2)) \leq \omega_i(\varrho(x_1, x_2)), \quad i = 1, 2.$$

Let $d(x) = \min_{1 \leq i \leq 2} \text{diam } F_i(x) < +\infty$ for all $x \in X$. Suppose there exist a function $\gamma : X \rightarrow [0, +\infty)$ and $\alpha > 0$ such that:

$$\gamma(x)B_1^Y(0) \subset F_1(x) - F_2(x), \quad \forall x \in X, \quad (3.11)$$

$$d(x) \leq \alpha\gamma(x), \quad \forall x \in X. \quad (3.12)$$

Then the set-valued mapping $G(x) = F_1(x) \cap F_2(x)$ is uniformly continuous with the modulus $\omega(t) = \max\{\omega_1(t), \omega_2(t)\} + \alpha(\omega_1(t) + \omega_2(t))$.

Proof. We shall use ideas from Theorem 2.2.1 from [9]. Note that $G(x) \neq \emptyset$ follows by inclusion (3.11). Fix $t > 1$. Choose any pair of points $x_1, x_2 \in X$ and $y_1 \in G(x_1)$.

We shall prove that there exists a point $y_2 \in G(x_2)$ with

$$\|y_1 - y_2\| \leq t\omega(\varrho(x_1, x_2)). \quad (3.13)$$

Define $\omega = \omega(\varrho(x_1, x_2)), \omega_i = \omega_i(\varrho(x_1, x_2))$. By the uniform continuity of F_i it follows that:

$$y_1 \in F_1(x_2) + t\omega_1 B_1^Y(0), \quad (3.14)$$

$$y_1 \in F_2(x_2) + t\omega_2 B_1^Y(0). \quad (3.15)$$

Set $d(x_2) = \text{diam } F_1(x_2)$. By inclusion (3.14) it follows that there exists a point $y \in F_1(x_2)$ such that $\|y - y_1\| \leq t\omega_1$.

From this and by the inclusion (3.15) we conclude that

$$y \in F_2(x_2) + t(\omega_1 + \omega_2)B_1^Y(0).$$

Let

$$\theta = \frac{\gamma(x_2)}{\gamma(x_2) + t(\omega_1 + \omega_2)} \in [0, 1).$$

From the previous inclusion we get the inclusion

$$\theta y \in \theta F_2(x_2) + \theta t(\omega_1 + \omega_2)B_1^Y(0).$$

Keeping in mind that $\theta t(\omega_1 + \omega_2) = (1 - \theta)\gamma(x_2)$, we get

$$(1 - \theta)\gamma(x_2)B_1^Y(0) \subset (1 - \theta)F_2(x_2) - (1 - \theta)F_1(x_2),$$

and

$$\theta y \in \theta F_2(x_2) + (1 - \theta)F_2(x_2) - (1 - \theta)F_1(x_2) = F_2(x_2) - (1 - \theta)F_1(x_2).$$

Hence there exists a point $z \in F_1(x_2)$ with

$$\theta y + (1 - \theta)z \in F_2(x_2).$$

Let $y_2 = \theta y + (1 - \theta)z$. Then $y_2 \in F_1(x_2)$ since $y, z \in F_1(x_2)$. Thus $y_2 \in G(x_2)$.

From the equality $y_1 - y_2 = (y_1 - y) + (1 - \theta)(y - z)$ we conclude that:

$$\|y_1 - y_2\| \leq \|y_1 - y\| + (1 - \theta)\|y - z\| \leq t\omega_1 + (1 - \theta)d(x_2). \tag{3.16}$$

If $\gamma(x_2) = 0$ then $d(x_2) = 0$ and $\|y_1 - y_2\| \leq t\omega_1$.

If $\gamma(x_2) > 0$ then from the definition of θ and from (3.12) we get

$$(1 - \theta)d(x_2) \leq \frac{t(\omega_1 + \omega_2)}{\gamma(x_2) + t(\omega_1 + \omega_2)}\alpha\gamma(x_2) \leq t\alpha(\omega_1 + \omega_2).$$

So by inequality (3.16),

$$\|y_1 - y_2\| \leq t(\omega_1 + \alpha(\omega_1 + \omega_2)).$$

By taking the limit $t \rightarrow 1 + 0$ we obtain

$$h(G(x_1), G(x_2)) \leq (\omega_1 + \alpha(\omega_1 + \omega_2)).$$

Finally, note that in the general case we must take $\max\{\omega_1, \omega_2\}$ instead of ω_1 , because it may happen that $d(x_2) = \text{diam } F_2(x_2)$. \square

Propositions 3.2, 3.3, and 3.4 are well known:

Proposition 3.2. (See [6, Lemma 5].) Let X be a Banach space and $A_1, A_2 \subset X$ any convex closed bounded sets, $d = \max\{\text{diam } A_1, \text{diam } A_2\}$. Let $B_\alpha^X(x_i) \subset A_i, i = 1, 2$. Then for any $\beta \in (0, \alpha)$ the following holds:

$$h(A_1 * B_\beta^X(0), A_2 * B_\beta^X(0)) \leq \frac{d}{\alpha - \beta}h(A_1, A_2).$$

Proposition 3.3. (See [6, Lemma 4].) Let X be a Hilbert space and $A_1, A_2 \subset X$ any convex closed bounded sets, $A_i \subset B_r^X(a_i), i = 1, 2$. Then $c(A_i) \in A_i, i = 1, 2$, and

$$\|c(A_1) - c(A_2)\| \leq 2\sqrt{6rh(A_1, A_2)} + h(A_1, A_2).$$

The next proposition follows by the well-known Valentine extension theorem [14]:

Proposition 3.4. (See [6, Lemma 3].) Let X, Y be Hilbert spaces and $X_1 \subset X$ any convex subset of X . Let $w : X_1 \rightarrow Y$ be a uniformly continuous function. Then for any $\varepsilon > 0$, there exists a Lipschitz continuous function $v : X_1 \rightarrow Y$ with $\|v(x) - w(x)\| < \varepsilon$, for all $x \in X_1$.

Lemma 3.5. Let X be a Hilbert space, Y a Banach space and $L : X \rightarrow Y$ a continuous linear surjection. Then the operator L has the Lip-property.

Proof. Let $\ker L = \mathcal{L}$ and \mathcal{L}^\perp be the orthogonal subspace of \mathcal{L} . The set-valued mapping $L^{-1}(y)$ is Lipschitz continuous with respect to the Hausdorff distance [1, Corollary 3.3.6]. Let $R(y) = \inf_{x \in L^{-1}(y)} \|x\|, l(y) \in L^{-1}(y): \|l(y)\| = R(y)$, and

$$G(y) = B_{2R(y)}^X(0) \cap L^{-1}(y).$$

It is well known ([9, Theorem 2.2.2], see also [2,4,8]), that $G(y)$ is a Lipschitz set-valued mapping with respect to the Hausdorff distance. This also follows by Theorem 3.1.

Now consider $H(y) = G(y) \cap \mathcal{L}^\perp$. The point $l(y)$ is a metric projection of zero onto $L^{-1}(y)$, hence $l(y) \in G(y)$ and (because of $l(y) \in \mathcal{L}^\perp$) $l(y) \in H(y)$. Moreover, from the fact that some shift of \mathcal{L} contains $G(y)$, we can deduce that $H(y) = \{l(y)\}$.

From the properties

$$B_{\frac{2}{\sqrt{3}}R(y)}^X(0) \subset G(y) - \mathcal{L}^\perp, \quad \text{diam } G(y) \leq \frac{2}{\sqrt{3}}\sqrt{3}R(y)$$

and from Theorem 3.1 we now obtain that $H(y) = \{l(y)\}$ is Lipschitz continuous in the Hausdorff distance, hence $l(y)$ is a Lipschitz function. \square

Remark 3.6. We gave a purely geometric proof of Lemma 3.5. Note that this lemma can also be proved with the help of the Implicit function theorem [12].

Theorem 3.7. Let $X, Y, Y_i, i = 1, 2$, be Hilbert spaces and $X_1 \subset X$ a convex subset of X . Let $L : Y_1 \oplus Y_2 \rightarrow Y$ be a continuous linear surjection. Let $F_i : X_1 \rightarrow 2^{Y_i}, i = 1, 2$, be uniformly continuous (with modulus ω_i) set-valued mappings with convex closed bounded images and $d = \sup_{x \in X_1} \text{diam}(F_1(x), F_2(x)) < +\infty$. Suppose that for all $x \in X_1$ $f(x) \in L(F_1(x), F_2(x))$ is a Lipschitz selection. Then for any $\varepsilon > 0$ there exist Lipschitz selections $f_i(x) \in F_i(x) + B_\varepsilon^{Y_i}(0)$ with $f(x) = L(f_1(x), f_2(x))$, for all $x \in X_1$.

Proof. Fix $\varepsilon > 0$. Let $f(x) \in L(F_1(x), F_2(x))$ be a Lipschitz selection. The set-valued mapping $L^{-1}(f(x))$ is Lipschitz continuous in Hausdorff metric [1, Corollary 3.3.6]. Let $w(x) = L(f(x)) \in L^{-1}(f(x))$. The function $w(x)$ is Lipschitz continuous as a superposition of two Lipschitz functions: $l(y)$ (Lemma 3.5) and $f(x)$.

Hence $L^{-1}(f(x)) = w(x) + \mathcal{L}$, $\mathcal{L} = \ker L$. Consider

$$H(x) = ((F_1(x), F_2(x)) + B_\varepsilon^{Y_1 \oplus Y_2}(0)) \cap L^{-1}(f(x)) = w(x) + ((F_1(x), F_2(x)) - w(x) + B_\varepsilon^{Y_1 \oplus Y_2}(0)) \cap \mathcal{L}.$$

Without loss of generality we may assume that $w(x) = 0$ and $H(x) = ((F_1(x), F_2(x)) + B_\varepsilon^{Y_1 \oplus Y_2}(0)) \cap \mathcal{L}$. The mappings $x \rightarrow (F_1(x), F_2(x)) + B_\varepsilon^{Y_1 \oplus Y_2}(0)$ and $x \rightarrow \mathcal{L}$ are uniformly continuous ($x \in X_1$),

$$B_\varepsilon^{Y_1 \oplus Y_2}(0) \subset ((F_1(x), F_2(x)) + B_\varepsilon^{Y_1 \oplus Y_2}(0)) - \mathcal{L}$$

and

$$\text{diam}((F_1(x), F_2(x)) + B_\varepsilon^{Y_1 \oplus Y_2}(0)) \leq d + 2\varepsilon \leq \frac{d + 2\varepsilon}{\varepsilon} \varepsilon.$$

Using Theorem 3.1 we obtain that $H(x), x \in X_1$, is a uniformly continuous set-valued mapping.

Consider \mathcal{L} with the induced topology: the ball $B_r^{\mathcal{L}}(w) \subset \mathcal{L}$ ($w \in \mathcal{L}$) is $B_r^{\mathcal{L}}(w) = B_r^{Y_1 \oplus Y_2}(w) \cap \mathcal{L}$. Obviously, \mathcal{L} is a Hilbert space. The set-valued mapping $H(x) \subset \mathcal{L}$ has a nonempty interior in \mathcal{L} , moreover $H(x) \neq B_\varepsilon^{\mathcal{L}}(0) \neq \emptyset$, for all $x \in X_1$.

Let $H_1(x) = H(x) \neq B_{\varepsilon/2}^{\mathcal{L}}(0)$. By Proposition 3.2 we have that $H_1(x)$ is a uniformly continuous set-valued mapping with convex closed images. By Proposition 3.3 we have that the Chebyshev center $c(H_1(x))$ of the set-valued mapping $H_1(x)$ is a uniformly continuous function and $c(H_1(x)) \in H_1(x)$.

By Proposition 3.4 there exists a Lipschitz continuous function $v(x) \in c(H_1(x)) + B_{\varepsilon/2}^{\mathcal{L}}(0)$, for all $x \in X_1$. Hence

$$v(x) \in c(H_1(x)) + B_{\varepsilon/2}^{\mathcal{L}}(0) \subset H_1(x) + B_{\varepsilon/2}^{\mathcal{L}}(0) \subset H(x).$$

We can now choose $(f_1(x), f_2(x)) = v(x)$. \square

Remark 3.8. In the finite-dimension case (when $\dim Y_i < \infty, i = 1, 2$) Lip-property of L follows from results [4,8] (see also [2,6,9]). Let $R(y) = \inf\{\|l\| \mid l \in L^{-1}(y)\}$, for all $y \in Y$. The set-valued mapping

$$G(y) = B_{2R(y)}^{Y_1 \oplus Y_2}(0) \cap L^{-1}(y)$$

is Lipschitz continuous on y (this also follows by Theorem 3.1). We can choose $l(y) = s(G(y))$, where $s(\cdot)$ is the Steiner point.

Remark 3.9. It is easy to see that the proof of Theorem 3.7 can be given in any uniformly convex Banach (not necessarily Hilbert) spaces Y_1, Y_2 , where every uniformly continuous function can be approximated (with arbitrary precision) by a Lipschitz function, for any continuous linear surjection with Lip-property.

Example 3.10. An exact solution of the splitting problem does not exist for Lipschitz selections. Besides Example 1.1 we can demonstrate one more example. Tsar'kov proved [13] that there exists a Lipschitz (with respect to the Hausdorff distance) set-valued mapping $F : [0, 1] \rightarrow 2^Y$ (Y an infinite dimension Hilbert space) with convex closed bounded images, such that the mapping F has no Lipschitz selection. Thus for $L : Y \oplus Y \rightarrow Y, L(y_1, y_2) = y_1 - y_2$, we have $f(x) = 0 \in F(x) - F(x)$, but the Lipschitz function $f(x) = 0$ cannot be represented in the form $0 = f_1(x) - f_2(x)$, where $f_i(x) \in F(x)$ is a Lipschitz selection.

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