Topology and its Applications 157 (2010) 136–144

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Spaces of idempotent measures of compact metric spaces

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ARTICLE INFO

MSC:
primary 54C65, 52A30
secondary 28A33

Keywords:
Compact metric space
Hilbert cube bundle
Idempotent measure
Functor P of probability measures
Maslov integral
Absolute retract
Disjoint approximation property
Continuous selection
Multivalued map

ABSTRACT

We investigate certain geometric properties of the spaces of idempotent measures. In particular, we prove that the space of idempotent measures on an infinite compact metric space is homeomorphic to the Hilbert cube.

1. Introduction

The functor P of probability measures which acts on the category **Comp** of compact metrizable spaces has been investigated by many authors (see e.g. the survey [4]). Geometric properties of spaces of the form $P(X)$ were established, e.g. in [3]. In particular, it was proved in [3] that the map $P(f): P(X) \to P(Y)$ is a trivial Q-bundle (i.e. a trivial bundle whose fiber is the Hilbert cube Q) for an open map $f: X \to Y$ of finite-dimensional compact metric spaces with infinite fibers. Dranishnikov [2] constructed an example which shows that the condition of finite-dimensionality cannot be removed.

The space of idempotent measures was systematically studied in [16] (see also [15]), where it was proved in particular, that the space $I(X)$ of idempotent measures on a topological space X is compact Hausdorff if such is also X. The aim of this paper, which can be considered as a continuation of [16], is to establish certain geometric properties of the functor I.

In particular, we shall prove that $I(X)$ is homeomorphic to the Hilbert cube for every infinite compact metric space X. The construction of idempotent measures is functorial in the category of compact Hausdorff spaces and we also consider the geometry of the maps $I(f)$, for some maps f. In particular, we show that, much like in the case of probability measures, there exists an open map $f: X \to Y$ of compact metric spaces such that f has infinite fibers and the map $I(f)$ is not a trivial Q-bundle.

The paper is organized as follows. In Section 3 we provide the necessary information concerning the spaces of idempotent measures. Section 4 is devoted to the (pseudo)metrization of the spaces $I(X)$, for a metric space X. The main results on the topology of the spaces $I(X)$ for compact metric spaces X are given in Section 5. We also consider the geometry of

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0166-8641/$ – see front matter
doi:10.1016/j.topol.2009.04.040

the maps $I(f)$, for some maps f of compact metric spaces and this allows us to describe the topology of the spaces $I(X)$ for some nonmetrizable compact Hausdorff spaces X (cf. Section 6).

2. Preliminaries

The space $Q = \prod_{i=1}^{\infty}[0,1]_i$ is called the *Hilbert cube*. Recall that an *absolute retract* (AR) is a metrizable space which is a retract of every space in which it lies as a closed subset. The following characterization theorem was proved in [12].

Theorem 2.1 *(Toruńczyk's characterization theorem). A compact metric space X is homeomorphic to the Hilbert cube if the following two conditions are satisfied*:

(1) *X is an absolute retract*;
(2) *X satisfies the disjoint approximation property* (*DAP*)*, i.e. every two maps of a metric space into X can be approximated by maps with disjoint images.*

The following notion was introduced in [5]. A *c-structure* on a topological space X is an assignment to every nonempty finite subset A of X a contractible subspace $F(A)$ of X such that $F(A) \subset F(A')$ whenever $A \subset A'$. A pair (X, F), where F is a c-structure on X is called a *c-space*. A subset E of X is called an *F-set* if $F(A) \subset E$ for any finite $A \subset E$. A metric space (X, d) is said to be a *metric l.c.-space* if all the open balls are F-sets and all open r-neighborhoods of F-sets are also F-sets. In fact, it was proved in [6] that every compact metric *l.c.*-space is an AR.

A map $f : X \to Y$ is a *trivial Q-bundle* if f is homeomorphic to the projection map $p_1 : Y \times Q \to Y$. The following definition is due to Shchepin [10].

Definition 2.2. A map $f : X \to Y$ is said to be *soft* provided that for every commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & X \\ \downarrow & & \downarrow f \\ Z & \xrightarrow[\psi]{} & Y \end{array} \tag{1}$$

such that Z is a paracompact space and A is a closed subset of Z there exists a map $\Phi : Z \to X$ such that $f\Phi = \psi$ and $\Phi|A = \varphi$.

A map $f : X \to Y$ of compact metric spaces is said to satisfy the *fiberwise disjoint approximation property* if, for every $\varepsilon > 0$, there exist maps $g_1, g_2 : X \to X$ such that

(1) $fg_1 = fg_2 = f$;
(2) $d(1_X, g_i) < \varepsilon$, $i = 1, 2$; and
(3) $g_1(X) \cap g_2(X) = \emptyset$.

The following result was proved in [13].

Theorem 2.3 *(Toruńczyk–West characterization theorem for Q-manifold bundles). A map $f : X \to Y$ of compact metric ANR-spaces is a trivial Q-bundle if f is soft and f satisfies the fiberwise disjoint approximation property.*

The following is a generalization of the Michael selection theorem – see [6] for the proof. Recall that a multivalued map $F : X \to Y$ of topological spaces is called *lower semicontinuous* if for any open subset U of Y, the set $\{x \in X \mid F(x) \cap U \neq \emptyset\}$ is open in X. A *selection* of a multivalued map $F : X \to Y$ is a (single-valued) map $f : X \to Y$ such that $f(x) \in F(x)$, for every $x \in X$ (see e.g. [8]).

Theorem 2.4. *Let (X, d, F) be a complete metric l.c.-space. Then any lower semicontinuous multivalued map $T : Y \to X$ of a paracompact space Y whose values are nonempty closed F-sets has a continuous selection.*

3. Spaces of idempotent measures

In the sequel, all maps will be assumed to be continuous. Let X be a compact Hausdorff space. We shall denote the Banach space of continuous functions on X endowed with the sup-norm by $C(X)$. For any $c \in \mathbb{R}$ we shall denote the constant function on X taking the value c by c_X. We shall denote the weight of a topological space X by $w(X)$.

Let $\mathbb{R}_{\max} = \mathbb{R} \cup \{-\infty\}$ be the metric space endowed with the metric ϱ defined by $\varrho(x, y) = |e^x - e^y|$. Let also $\mathbb{R}^n_{\max} = (\mathbb{R}_{\max})^n$. Following the notation of idempotent mathematics (see e.g., [7]) we shall denote by $\odot : \mathbb{R} \times C(X) \to C(X)$ the map acting by $(\lambda, \varphi) \mapsto \lambda_X + \varphi$, and by $\oplus : C(X) \times C(X) \to C(X)$ the map acting by $(\varphi, \psi) \mapsto \max\{\varphi, \psi\}$. We also use the notations $\oplus$ and $\odot$ in $\mathbb{R}$ as alternatives for max and $+$ respectively. The convention $-\infty \odot x = -\infty$ allows us to extend $\odot$ over $\mathbb{R}_{\max}$.

Definition 3.1. A functional $\mu : C(X) \to \mathbb{R}$ is called an *idempotent measure* (a *Maslov measure*) if

(1) $\mu(c_X) = c$;
(2) $\mu(c \odot \varphi) = c \odot \mu(\varphi)$; and
(3) $\mu(\varphi \oplus \psi) = \mu(\varphi) \oplus \mu(\psi)$,

for every $\varphi, \psi \in C(X)$.

The number $\mu(\varphi)$ is the *Maslov integral* of $\varphi \in C(X)$ with respect to μ. It is pointed out in [16] (and is easy to see) that every idempotent measure μ is a nonexpanding functional in the sense that $|\mu(\varphi) - \mu(\psi)| \leqslant \|\varphi - \psi\|$, for every $\varphi, \psi \in C(X)$.

Let $I(X)$ denote the set of all idempotent probability measures on X. We endow $I(X)$ with the weak* topology. A basis of this topology is formed by the sets

$$\langle \mu; \varphi_1, \dots, \varphi_n; \varepsilon \rangle = \bigl\{ \nu \in I(X) \bigm| \bigl|\mu(\varphi_i) - \nu(\varphi_i)\bigr| < \varepsilon,\ i = 1, \dots, n \bigr\},$$

where $\mu \in I(X)$, $\varphi_i \in C(X)$, $i = 1, \dots, n$, and $\varepsilon > 0$.

The following is an example of an idempotent probability measure. Let $x_1, \dots, x_n \in X$ and $\lambda_1, \dots, \lambda_n \in \mathbb{R}_{\max}$ be numbers such that $\max\{\lambda_1, \dots, \lambda_n\} = 0$. Define $\mu : C(X) \to \mathbb{R}$ as follows: $\mu(\varphi) = \max\{\varphi(x_i) + \lambda_i \mid i = 1, \dots, n\}$. As usual, for every $x \in X$, we denote by δ_x (or $\delta(x)$) the functional on $C(X)$ defined as follows: $\delta_x(\varphi) = \varphi(x)$, $\varphi \in C(X)$ (the Dirac probability measure concentrated at x). Then one can write $\mu = \bigoplus_{i=1}^n (\lambda_i \odot \delta_{x_i})$. It was proved in [16] that $I(X)$ is a compact Hausdorff space if such is also X.

Given a map $f : X \to Y$ of compact Hausdorff spaces, the map $I(f) : I(X) \to I(Y)$ is defined by the formula $I(f)(\mu)(\varphi) = \mu(\varphi f)$, for every $\mu \in I(X)$ and $\varphi \in C(Y)$. That $I(f)$ is continuous and that I is a covariant functor acting in the category **Comp** was proved in [16]. Note that, if $\mu = \bigoplus_{i=1}^n (\lambda_i \odot \delta_{x_i}) \in I(X)$, then $I(f)(\mu) = \bigoplus_{i=1}^n (\lambda_i \odot \delta_{f(x_i)}) \in I(Y)$.

It follows from the general theory of covariant functors in the category **Comp** developed by Shchepin [10] that the notion of support can be defined for the idempotent measures. It was shown in [16] that for every $\mu \in I(X)$, where X is a compact Hausdorff space, the *support* of μ is the minimal closed set $\mathrm{supp}(\mu) \subset X$ satisfying the property: if $\varphi \in C(X)$ and $\varphi|\mathrm{supp}(\mu) \equiv 0$, then $\mu(\varphi) = 0$. A general fact concerning supports is that they are preserved by maps (see [10]). In the case of idempotent measures this means the following: $\mathrm{supp}(I(f)(\mu)) \subset f(\mathrm{supp}(\mu))$, for any map $f : X \to Y$ and $\mu \in I(X)$.

Note also that I preserves the class of embeddings (i.e. $I(f)$ is an embedding, for every embedding f); see [16].

3.1. Milyutin maps

The following result was proved in [16].

Theorem 3.2. *Let X be a compact metrizable space. Then there exists a zero-dimensional compact metrizable space X and a map $f : X \to Y$ for which there exists a map $s : Y \to I(X)$ such that* $\mathrm{supp}(s(y)) \subset f^{-1}(y)$, *for every $y \in Y$.*

A map f satisfying the conditions stated above is called a *Milyutin map* of idempotent measures. Note that it follows from the definition of the support that, in the notations of Theorem 3.2, we have $I(f)(s(y)) = \delta_y$, for every $y \in Y$.

3.2. Map ζ_X

Given $\varphi \in C(X)$, define $\bar{\varphi} : I(X) \to \mathbb{R}$ as follows: $\bar{\varphi}(\mu) = \mu(\varphi)$, $\mu \in I(X)$. It can be easily shown (see [16]) that $\bar{\varphi} \in C(I(X))$. Given $M \in I^2(X)$, define the map $\zeta_X(M) : C(X) \to \mathbb{R}$ as follows: $\zeta_X(M)(\varphi) = M(\bar{\varphi})$. It was proved in [16] that $\zeta_X(M) \in I(X)$ and the obtained map $\zeta_X : I^2(X) \to I(X)$ are continuous. Note also (see [16]) that $\zeta = (\zeta_X)$ is a natural transformation of the functor I^2 to the functor I, i.e. the diagram

$$\begin{array}{ccc} I^2(X) & \xrightarrow{I^2(f)} & I^2(Y) \\ \Big\downarrow{\scriptstyle \zeta_X} & & \Big\downarrow{\scriptstyle \zeta_Y} \\ I(X) & \xrightarrow[I(f)]{} & I(Y) \end{array}$$

is commutative, for every $f : X \to Y$.

4. Metrization

Let (X, d) be a compact metric space. By $\mathrm{LIP}_n = \mathrm{LIP}_n(X, d)$ we denote the set of Lipschitz functions with the Lipschitz constant $\leqslant n$ from $C(X)$. Fix $n \in \mathbb{N}$. For every $\mu, \nu \in I(X)$, let

$$\hat{d}_n(\mu, \nu) = \sup\{|\mu(\varphi) - \nu(\varphi)| \mid \varphi \in \mathrm{LIP}_n\}.$$

Theorem 4.1. *The function $\hat{d}_n$ is a continuous pseudometric on $I(X)$.*

Proof. We first remark that $\hat{d}_n$ is well defined. Indeed, $\sup\varphi - \inf\varphi \leqslant n \operatorname{diam} X$, for every $\varphi \in \mathrm{LIP}_n$, and, because of condition (1) from Definition 3.1, we obtain

$$\inf\varphi \leqslant \mu(\varphi) \leqslant \sup\varphi, \qquad \inf\varphi \leqslant \nu(\varphi) \leqslant \sup\varphi,$$

whence $|\mu(\varphi) - \nu(\varphi)| \leqslant n \operatorname{diam} X$.

Obviously, $\hat{d}_n(\mu, \mu) = 0$ and $\hat{d}_n(\mu, \nu) = \hat{d}_n(\nu, \mu)$, for every $\mu, \nu \in I(X)$.

We are going to prove that $\hat{d}_n$ satisfies the triangle inequality. For every $\varphi \in \mathrm{LIP}_n$ and $\mu, \nu, \tau \in I(X)$, we have

$$\hat{d}_n(\mu, \nu) + \hat{d}_n(\nu, \tau) \geqslant |\mu(\varphi) - \nu(\varphi)| + |\nu(\varphi) - \tau(\varphi)| \geqslant |\mu(\varphi) - \tau(\varphi)|,$$

whence, passing to sup in the right-hand side, we obtain $\hat{d}_n(\mu, \nu) + \hat{d}_n(\nu, \tau) \geqslant \hat{d}_n(\mu, \tau)$.

Now, we prove that $\hat{d}_n$ is continuous. Suppose to the contrary. Then one can find a sequence $(\mu_i)_{i=1}^{\infty}$ in $I(X)$ such that $\lim_{i\to\infty} \mu_i = \mu \in I(X)$ and $\hat{d}_n(\mu_i, \mu) \geqslant c'$, for some $c' > 0$. Then there exist $\varphi_i \in \mathrm{LIP}_n$, $i \in \mathbb{N}$, such that $|\mu_i(\varphi_i) - \mu(\varphi_i)| \geqslant c$, for some $c > 0$. Since the functionals in $I(X)$ are weakly additive, without loss of generality, one may assume that $\varphi_i(x_0) = 0$, for some base point $x_0 \in X$, $i \in \mathbb{N}$.

By the Arzelà–Ascoli theorem, there exists a limit point $\varphi \in \mathrm{LIP}_n$ of the sequence $(\varphi_i)_{i=1}^{\infty}$. Without loss of generality, we may assume that the sequence $(\varphi_i)_{i=1}^{\infty}$ converges to φ and $\|\varphi - \varphi_i\| \leqslant (c/3)$, for all $i \in \mathbb{N}$. Then, for all $i \in \mathbb{N}$,

$$\begin{aligned} c &\leqslant |\mu_i(\varphi_i) - \mu_i(\varphi)| + |\mu_i(\varphi) - \mu(\varphi)| + |\mu(\varphi) - \mu(\varphi_i)| \\ &\leqslant \frac{c}{3} + |\mu_i(\varphi) - \mu(\varphi)| + \frac{c}{3}, \end{aligned}$$

whence $|\mu_i(\varphi) - \mu(\varphi)| \geqslant (c/3)$, which contradicts the fact that $(\mu_i)_{i=1}^{\infty}$ converges to μ. □

Remark 4.2. Simple examples demonstrate that $\hat{d}_n$ cannot be a metric whenever X consists of more than one point.

Proposition 4.3. *The family of pseudometrics $\hat{d}_n$, $n \in \mathbb{N}$, separates the points in $I(X)$.*

Proof. Let $\mu, \nu \in I(X)$, $\mu \neq \nu$. There exists $\varphi \in C(X)$ such that $|\mu(\varphi) - \nu(\varphi)| > c$, for some $c > 0$. There exists $\psi \in \mathrm{LIP}_n$, for some $n \in \mathbb{N}$, such that $\|\varphi - \psi\| \leqslant (c/3)$. Then, similarly to the proof of Theorem 4.1, we can see that $|\mu(\psi) - \nu(\psi)| \geqslant (c/3)$ and therefore $\hat{d}_n(\mu, \nu) \geqslant (c/3)$. □

We let $\tilde{d}_n = (1/n)\hat{d}_n \leqslant \operatorname{diam} X$ and define a function $\tilde{d} : I(X) \times I(X) \to \mathbb{R}$ as follows:

$$\tilde{d}(\mu, \nu) = \sum_{i=1}^{\infty} \frac{\tilde{d}_i(\mu, \nu)}{2^i}.$$

It follows from what was proved above that $\tilde{d}$ is an admissible metric on the space $I(X)$.

Proposition 4.4. *The map $\delta = \delta_X$, $x \mapsto \delta_x : (X, d) \to (I(X), \tilde{d}_n)$, is an isometric embedding for every $n \in \mathbb{N}$.*

Proof. Let $x, y \in X$ and $\varphi \in \mathrm{LIP}_n$. Then $|\delta_x(\varphi) - \delta_y(\varphi)| \leqslant nd(x, y)$, therefore $\hat{d}_n(\delta_x, \delta_y) \leqslant nd(x, y)$. Thus $\tilde{d}_n(\delta_x, \delta_y) \leqslant d(x, y)$.

On the other hand, define $\varphi_x \in \mathrm{LIP}_n$ by the formula $\varphi_x(z) = nd(x, z)$, $z \in X$. Then $|\delta_x(\varphi_x) - \delta_y(\varphi_x)| = nd(x, y)$ and we are done. □

Corollary 4.5. *The map $\delta = \delta_X$, $x \mapsto \delta_x : (X, d) \to (I(X), \tilde{d})$, is an isometric embedding.*

Proposition 4.6. *Let $f : (X, d) \to (Y, \varrho)$ be a nonexpanding map of compact metric spaces. Then the map $I(f) : (I(X), \hat{d}_n) \to (I(Y), \hat{\varrho}_n)$ is also nonexpanding, for every $n \in \mathbb{N}$.*

Proof. Given $\varphi \in \mathrm{LIP}_n(Y)$, note that $\varphi f \in \mathrm{LIP}_n(X)$ and, for any $\mu, \nu \in I(X)$, we have

$$\left|I(f)(\mu)(\varphi) - I(f)(\nu)(\varphi)\right| = \left|\mu(\varphi f) - \nu(\varphi f)\right| \leqslant \hat{d}_n(\mu, \nu).$$

Passing to the limit in the left-hand side of the above formula, we are done. □

Corollary 4.7. *Let $f : (X, d) \to (Y, \varrho)$ be a nonexpanding map of compact metric spaces. The map $I(f) : (I(X), \tilde{d}) \to (I(Y), \tilde{\varrho})$ is nonexpanding.*

Note that the above constructions $d \mapsto \tilde{d}$ and $d \mapsto \tilde{d}_n$ can be applied not only to metrics but also to continuous pseudometrics. Proceeding in this way we obtain the iterations $(I(X), \tilde{d}_n)$, $(I^2(X), \tilde{\tilde{d}}_{nm} = (\tilde{d}_n)\tilde{}_m), \ldots$.

Proposition 4.8. *For a metric space (X, d), the map $\zeta_X : (I^2(X), \tilde{\tilde{d}}_{nn}) \to (I(X), \tilde{d}_n)$ is nonexpanding.*

Proof. We first prove that, for any $\varphi \in \mathrm{LIP}_n(X, d)$, we have $\bar{\varphi} \in \mathrm{LIP}_n(I(X), \tilde{d}_n)$. Indeed, given $\mu, \nu \in I(X)$, we see that

$$n\tilde{d}_n(\mu, \nu) = \hat{d}_n(\mu, \nu) \geqslant \left|\mu(\varphi) - \nu(\varphi)\right| = \left|\bar{\varphi}(\mu) - \bar{\varphi}(\nu)\right|$$

and we are done.

Suppose now that $M, N \in I^2(X)$, $\mu = \zeta_X(M)$, $\nu = \zeta_X(N)$. Given $\varphi \in \mathrm{LIP}_n(X, d)$, we obtain

$$(1/n)\left|\mu(\varphi) - \nu(\varphi)\right| = (1/n)\left|M(\bar{\varphi}) - N(\bar{\varphi})\right| \leqslant (1/n)\hat{\tilde{d}}_{nn}(M, N) = \tilde{\tilde{d}}_{nn}(M, N).$$

Passing to the limit in the left-hand side, we are done. □

Remark 4.9. Using the results on existence of the pseudometrics $\tilde{d}_n$, one can define the spaces of idempotent probability measures with compact support for metric and, more generally, uniform spaces. Indeed, let (X, d) be a metric space. By $\exp X$ we denote the family of nonempty compact subsets of X. We define the set $I(X)$ to be the direct limit of the direct system $\{I(A), I(\iota_{AB}); \exp X\}$ (here, for $A, B \in \exp X$ with $A \subset B$, we denote by $\iota_{AB} : A \to B$ the inclusion map).

For every $A \in \exp X$, we identify $I(A)$ with the corresponding subset of $I(X)$ along the map $I(\iota_A)$, where $\iota_A : A \to X$ is the limit inclusion map. For any $\mu \in I(X)$, there exists a unique minimal $A \in \exp X$ such that $\mu \in I(A)$. Then we say that A is the *support* of μ and write $\mathrm{supp}(\mu) = A$.

Now, define a family of pseudometrics $\hat{d}_n$, $n \in \mathbb{N}$, on $I(X)$ as follows. Given $\mu, \nu \in I(X)$, we let

$$\hat{d}_n(\mu, \nu) = \big(d\big|\big(\mathrm{supp}(\mu) \cup \mathrm{supp}(\nu)\big) \times \big(\mathrm{supp}(\mu) \cup \mathrm{supp}(\nu)\big)\big)\hat{}_n(\mu, \nu).$$

One can prove that, for any uniform space $(X, \mathcal{U})$, if the uniformity $\mathcal{U}$ is generated by a family $\{d^\alpha \mid \alpha \in A\}$ of pseudometrics, then the family $\{\tilde{d}^\alpha_n \mid \alpha \in A,\ n \in \mathbb{N}\}$ of pseudometrics on $I(X)$ generates a uniformity on $I(X)$.

5. Space of idempotent measures for metric compacta

Let X be a compact Hausdorff space. It was proved in [16] that the set $I(X)$ is homeomorphic to the $(n-1)$-dimensional simplex for any finite X with $|X| = n$.

For every $\mu, \nu \in I(X)$ and every $\alpha, \beta \in \mathbb{R}_{\max}$ with $\alpha \oplus \beta = 0$, we define $(\alpha \odot \mu) \oplus (\beta \odot \nu) : C(X) \to \mathbb{R}$ as follows:

$$\big((\alpha \odot \mu) \oplus (\beta \odot \nu)\big)(\varphi) = \big(\alpha \odot \mu(\varphi)\big) \oplus \big(\beta \odot \nu(\varphi)\big), \quad \text{for every } \varphi \in C(X).$$

Clearly, $(\alpha \odot \mu) \oplus (\beta \odot \nu) \in I(X)$. Note also that, given a map $f : X \to Y$, one has

$$I(f)\big((\alpha \odot \mu) \oplus (\beta \odot \nu)\big) = \big(\alpha \odot I(f)(\mu)\big) \oplus \big(\beta \odot I(f)(\nu)\big).$$

A set $A \subset I(X)$ is called *max-plus convex* if, for every $\mu, \nu \in A$ and every $\alpha, \beta \in \mathbb{R}_{\max}$ with $\alpha \oplus \beta = 0$, we have $(\alpha \odot \mu) \oplus (\beta \odot \nu) \in A$.

Lemma 5.1. *Let $\mu_0 \in I(X)$. The map $h : I(X) \times [-\infty, 0] \to I(X)$, $h(\mu, \lambda) = \mu \oplus (\lambda \odot \mu_0)$, is continuous.*

Proof. Let

$$(\mu, \lambda) \in I(X) \times [-\infty, 0], \quad \nu = h(\mu, \lambda), \quad \text{and} \quad \langle \nu; \varphi; \varepsilon \rangle$$

be a subbase neighborhood of ν.

Case (1). $h(\mu, \lambda) = \mu(\varphi)$. Then $\mu(\varphi) \geqslant \lambda + \mu_0(\varphi)$ and it is evident that, for any $\mu' \in \langle \mu; \varphi; \varepsilon \rangle$ and $\lambda' \in [-\infty, \lambda + \varepsilon) \cap [-\infty, 0]$, we have $h(\mu', \lambda') \in \langle \nu; \varphi; \varepsilon \rangle$.

Case (2). $h(\mu, \lambda) = \lambda + \mu_0(\varphi)$. Then necessarily $\lambda > -\infty$. For every $\mu' \in \langle \mu; \varphi; \varepsilon \rangle$ and $\lambda' \in (\lambda + \varepsilon, \lambda + \varepsilon) \cap [-\infty, 0]$, we have $h(\mu', \lambda') \in \langle \nu; \varphi; \varepsilon \rangle$. □

Lemma 5.2. *Let X be any compact metrizable space. Then every max-plus convex subset in $I(X)$ is contractible.*

Proof. First note that, since $I(X)$ is a compact semilattice with respect to the operation $\oplus$, there exists $\max A$ for every nonempty subset A of $I(X)$ (see [16]). In other words, $(\max A)(\varphi) = \max\{\mu(\varphi) \mid \mu \in A\}$, for every $\varphi \in C(X)$. Note that $\max A$ is well defined, because $\{\mu(\varphi) \mid \varphi \in A\}$ is compact, for every $\varphi \in C(X)$.

Now let $A \subset I(X)$ be a nonempty max-plus convex subset in $I(X)$. In order to show that $\max A \in A$, assume to the contrary and find $\varphi_1, \ldots, \varphi_n \in C(X)$ and $\varepsilon > 0$ such that

$$\langle \max A; \varphi_1, \ldots, \varphi_n; \varepsilon \rangle \cap A = \emptyset.$$

One can take $\mu_1, \ldots, \mu_n \in A$ so that $(\max A)(\varphi_i) = \mu_i(\varphi_i)$.

Because of max-plus convexity of A, we have

$$\mu_1 \oplus \cdots \oplus \mu_n = (0 \odot \mu_1) \oplus \cdots \oplus (0 \odot \mu_n) \in A.$$

For every $i = 1, \ldots, n$,

$$(\mu_1 \oplus \cdots \oplus \mu_n)(\varphi_i) = \mu_i(\varphi_i) = (\max A)(\varphi_i).$$

It now follows that

$$\mu_1 \oplus \cdots \oplus \mu_n \in \langle \max A; \varphi_1, \ldots, \varphi_n; \varepsilon \rangle \cap A,$$

which is a contradiction.

Define the map $H : A \times \mathbb{R}_{\max} \to A$ as follows: $H(\mu, \lambda) = \mu \oplus (\lambda \odot \max A)$. Then $H(\mu, -\infty) = \mu$ and $H(\mu, 0) = \max A$. This shows that the set A is indeed contractible. □

Theorem 5.3. *The space $I(X)$ is homeomorphic to the Hilbert cube for any infinite compact metrizable space X.*

Proof. We first show that $I(X)$ is an AR-space. Fix a metric d on X that generates its topology. Define a c-structure on the space $I(X)$ as follows. To every nonempty finite subset $A = \{\mu_1, \ldots, \mu_n\}$ of $I(X)$ assign a subspace

$$F(A) = \left\{ \bigoplus_{i=1}^{n} (\alpha_i \odot \mu_i) \;\middle|\; \alpha_1, \ldots, \alpha_n \in \mathbb{R}_{\max},\ \bigoplus_{i=1}^{n} \alpha_i = 0 \right\}. \tag{2}$$

It is easy to verify that the set $F(A)$ is max-plus convex and therefore contractible.

We are going to show that the c-structure F gives an $l.c.$-structure. We shall prove that every ball with respect to the metric $\tilde{d}$ in $I(X)$ is an F-set.

Let $\mu, \nu, \tau \in I(X)$, $\lambda \in [-\infty, 0]$, and $\varphi \in \mathrm{LIP}_n$. We are going to prove that

$$\left|\mu(\varphi) - \big((\lambda \odot \nu) \oplus \tau\big)(\varphi)\right| \leqslant \max\big\{\left|\mu(\varphi) - \nu(\varphi)\right|, \left|\mu(\varphi) - \tau(\varphi)\right|\big\}.$$

Indeed, if $((\lambda \odot \nu) \oplus \tau)(\varphi) = \tau(\varphi)$, there is nothing to prove. Otherwise, $(\lambda \odot \nu)(\varphi) > \tau(\varphi)$ and, assuming that

$$\left|\mu(\varphi) - \big((\lambda \odot \nu) \oplus \tau\big)(\varphi)\right| > \left|\mu(\varphi) - \tau(\varphi)\right|,$$

we see that $\mu(\varphi) < (\lambda \odot \nu)(\varphi)$ and therefore

$$\left|\mu(\varphi) - \big((\lambda \odot \nu) \oplus \tau\big)(\varphi)\right| \leqslant \left|\mu(\varphi) - \nu(\varphi)\right|.$$

Thus,

$$\hat{d}_n\big(\mu, (\lambda \odot \nu) \oplus \tau\big) \leqslant \max\big\{\hat{d}_n(\mu, \nu), \hat{d}_n(\mu, \tau)\big\}$$

and therefore, for any $\varepsilon > 0$, if $\tilde{d}(\mu, \nu) < \varepsilon$ and $\tilde{d}(\mu, \tau) < \varepsilon$, then $\tilde{d}(\mu, (\lambda \odot \nu) \oplus \tau) < \varepsilon$.

By using this fact inductively, we can see that every finite set A in the ε-ball at μ, the set $F(A)$ is also contained in the ε-ball at μ, which means that the ε-ball at μ is an F-set.

We now show that the space $I(X)$ satisfies DAP. Let $f : Y \to X$ be a Milyutin map of a zero-dimensional compact metrizable space Y. There exists a map $s : X \to I(Y)$ such that $I(f)s(x) = \delta_x$, for every $x \in X$. Let $r : Y \to Y'$ be a retraction of Y onto its finite subset Y'. Since Y is zero-dimensional, we may choose r as close to id_Y as we wish. Define $g_1 : I(X) \to I(X)$ as follows:

$$g_1(\mu) = I(fr)\zeta_Y I(s)(\mu) = \zeta_X I^2(fr) I(s)(\mu).$$

Then $\mathrm{supp}(g_1(\mu)) \subset f(Y')$, for every $\mu \in I(X)$. Note that, because of general results on preserving continuity by covariant functors in the category **Comp** (see [10]), the map g_1 can be made as close to $\mathrm{id}_{I(X)}$ as we wish.

In [16], for every nonempty closed subset A of X, it was proved that the map $j_X(A) : C(X) \to \mathbb{R}$ defined by $j_X(A)(\varphi) = \sup\{\varphi(x) \mid x \in A\}$ belongs to $I(X)$. In particular, $j_X(X) \in I(X)$ acts as follows: $j_X(X)(\varphi) = \max \varphi$, $\varphi \in C(X)$.

Define $g_2 : I(X) \to I(X)$ by the formula $g_2(\mu) = \mu \oplus (\lambda \odot j_X(X))$, where $\lambda \in (-\infty, 0]$ is fixed. If λ is small enough, the map g_2 is close to $\mathrm{id}_{I(X)}$.

Note that $\mathrm{supp}(g_2(\mu)) = X$, for every $\mu \in I(X)$, and therefore $g_1(I(X)) \cap g_2(I(X)) = \emptyset$. By Toruńczyk's theorem, $I(X)$ is homeomorphic to Q. □

Remark 5.4. Note that a version of the Michael selection theorem for finite-dimensional max-plus convex sets was proved in [14].

6. Maps of spaces of idempotent measures

Theorem 6.1. *Let $p_1 : X \times Y \to X$ denote the projection onto the first factor, where X, Y are compact metric spaces. Then the map $I(p_1) : I(X \times Y) \to I(X)$ is soft.*

Proof. It was proved in [16] that the map $I(p_1)$ is open. Given a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & I(X \times Y) \\ \downarrow & & \downarrow {\scriptstyle I(p_1)} \\ Z & \xrightarrow[\psi]{} & I(X) \end{array}$$

consider the multivalued map $\Psi : Z \to I(X \times Y)$ defined as follows:

$$\Psi(z) = \begin{cases} \{\varphi(z)\}, & \text{if } z \in A, \\ (I(p_1))^{-1}(\psi(z)), & \text{if } z \notin A. \end{cases}$$

Since A is a closed subset of Z and the map $I(p_1)$ is open, the map Ψ is lower semicontinuous. Also, it is easy to verify that the images of Ψ are F-sets in the metric $l.c.$-space $(I(X \times Y), \tilde{d}, F)$, for any fixed admissible metric d on $X \times Y$, where F is the c-structure defined by formula (2) in the proof of Theorem 5.3, for any finite subset $A = \{\mu_1, \ldots, \mu_n\} \in I(X \times Y)$. By Theorem 2.4, the map Ψ admits a selection $\Phi : Z \to I(X \times Y)$. Clearly, $\Phi|A = \varphi$ and $I(p_1)\Phi = \psi$. This completes the proof. □

Example 6.2. The following example demonstrates that, like in the case of probability measures, there exists an open map $f : X \to Y$ of metrizable compacta with infinite fibers such that the map $I(f) : I(X) \to I(Y)$ is not a trivial Q-bundle.

We exploit the construction from [2]. Let S^n denote the n-dimensional sphere and $\mathbb{R}P^n$ the n-dimensional real projective space. Let $\eta_n : S^n \to \mathbb{R}P^n$ denote the canonical map. The required map is

$$f = \prod_{i=1}^{\infty} \eta_{2^i-1} : X = \prod_{i=1}^{\infty} S^{2^i-1} \to Y = \prod_{i=1}^{\infty} \mathbb{R}P^{2^i-1}.$$

It was proved in [2] that the map

$$f_k = \prod_{i=1}^{k} \eta_{2^i-1} : \prod_{i=1}^{k} S^{2^i-1} \to \prod_{i=1}^{k} \mathbb{R}P^{2^i-1}$$

has the following property: the map

$$P_0(f_k) : P_0\left(\prod_{i=1}^{k} S^{2^i-1}\right) \to \prod_{i=1}^{k} \mathbb{R}P^{2^i-1},$$

where

$$P_0\left(\prod_{i=1}^{k} S^{2^i-1}\right) = \big(P(f_k)\big)^{-1}\left(\left\{\delta_x \,\middle|\, x \in \prod_{i=1}^{k} \mathbb{R}P^{2^i-1}\right\}\right) \subset P\left(\prod_{i=1}^{k} S^{2^i-1}\right)$$

and $P_0(f_k)$ sends every $\mu \in P_0(\prod_{i=1}^{k} S^{2^i-1})$ to the unique $x \in \prod_{i=1}^{k} \mathbb{R}P^{2^i-1}$ for which $\mathrm{supp}(\mu) \in f_k^{-1}(x)$, has no two disjoint selections.

Proceeding similarly as in [2] we reduce the problem of existence of two disjoint sections of the map $I(f)$ to that of existence of two disjoint selections of the map $I_0(f_k)$, for some k, where the map $I_0(f_k)$ is defined similarly as $P_0(f_k)$ with P replaced by I.

We only have to show that the maps $I_0(f_k)$ and $P_0(f_k)$ are homeomorphic in the sense that there exists a homeomorphism $h: I_0(f_k) \to P_0(f_k)$ making the diagram

$$\begin{array}{ccc} I_0(\prod_{i=1}^k S^{2^i-1}) & \xrightarrow{\quad h \quad} & P_0(\prod_{i=1}^k S^{2^i-1}) \\ & {\scriptstyle I_0(f_k)} \searrow \quad \swarrow {\scriptstyle P_0(f_k)} & \\ & \prod_{i=1}^k \mathbb{R}P^{2^i-1} & \end{array}$$

commutative. Let $\mu \in I_0(\prod_{i=1}^k S^{2^i-1})$. Since the fibers of the map f_k are finite, one has $\mu = \bigoplus_{i=1}^n (\lambda_i \odot \delta_{x_i})$, where $\max\{\lambda_1, \dots, \lambda_n\} = 0$ and all $x_1, \dots, x_n$ lie in the same fiber of the map f_k. Observe that

$$\left(e^{\lambda_1}, \dots, e^{\lambda_n}\right) \in \Gamma^{n-1} = \left\{(z_1, \dots, z_n) \in [0,1]^n \;\middle|\; z_1 \oplus \dots \oplus z_n = 1\right\}.$$

Then we can define

$$h(\mu) = \sum_{i=1}^n \frac{e^{\lambda_i}}{\sum_{j=1}^n e^{\lambda_j}} \delta_{x_i} \in P_0\left(\prod_{i=1}^k S^{2^i-1}\right)$$

and, clearly, $P_0(f_k)(h(\mu)) = I_0(f_k)(\mu)$.

The following reasoning demonstrates that h is bijective (we thank the referee for this argument): Let

$$\Delta^{n-1} = \left\{(t_1, \dots, t_n) \in [0,1]^n \;\middle|\; \sum_{i=1}^n t_i = 1\right\}$$

be the standard $(n-1)$-simplex. For each $(t_1, \dots, t_n) \in \Delta^{n-1}$, there exists a unique $(\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n_{\max}$ such that

$$\left(e^{\lambda_1}, \dots, e^{\lambda_n}\right) = \left(\frac{t_1}{t_1 \oplus \dots \oplus t_n}, \dots, \frac{t_n}{t_1 \oplus \dots \oplus t_n}\right) \in \Gamma^{n-1},$$

whence $\lambda_1 \oplus \dots \oplus \lambda_n = 0$. For each $i = 1, \dots, n$, we have

$$\frac{e^{\lambda_i}}{\sum_{j=1}^n e^{\lambda_j}} = \frac{t_i}{\sum_{i=1}^n t_j} = t_i.$$

A routine verification of continuity shows that h is indeed the desired homeomorphism.

7. Nonmetrizable case

The notion of normal functor in the category **Comp** of compact Hausdorff spaces was introduced in [10].

Theorem 7.1. *Let $\tau > \omega_1$. Then the set $I([0,1]^\tau)$ is not an AR.*

Proof. Shchepin [10] proved that, for any normal functor F which is not a power functor, any cardinal number $\tau > \omega_1$, and any compact metric space K with $|K| \geqslant 2$, the functor-power $F(K^\tau)$ is not an AR. It was proved in [16], that I is a normal functor, whence the result follows. □

Theorem 7.2. *Let X be an openly generated character-homogeneous compact Hausdorff space of weight ω_1. Then the space $I(X)$ is homeomorphic to I^{ω_1}.*

Proof. We can represent X as $\varprojlim \mathcal{S}$, where $\mathcal{S} = \{X_\alpha, p_{\alpha\beta}; \omega_1\}$ is an inverse system such that $p_{\alpha\beta}: X_\alpha \to X_\beta$ are open maps for all $\alpha, \beta < \omega_1$, $\alpha \geqslant \beta$, and X_α, $\alpha < \omega_1$, are infinite compact metric spaces. Since X is character-homogeneous, we may additionally assume that the maps $p_{\alpha\beta}$ do not contain singleton fibers.

Then we have $I(X) = \varprojlim I(\mathcal{S}) = \{I(X_\alpha), I(p_{\alpha\beta}); \omega_1\}$ (see [16]). By Theorem 6.1, the maps $I(p_{\alpha\beta})$ are soft. Applying arguments from [9] one can find a cofinal subset A of ω_1 such that, for every $\alpha, \beta \in A$, $\alpha > \beta$, the map $I(p_{\alpha\beta})$ satisfies the FDAP. Therefore, by the Toruńczyk–West theorem, the map $I(p_{\alpha\beta})$ is homeomorphic to the projection $\pi_1: Q \times Q \to Q$ onto the first factor. In turn, $I(X)$ is homeomorphic to $Q^A \simeq Q^{\omega_1} \simeq I^{\omega_1}$. □

8. Epilogue

One can also consider the spaces $I(K^\tau)$, for arbitrary τ and nondegenerate compact metrizable spaces K. The interesting results on autohomeomorphisms of the spaces $P(K^\tau)$, for $\tau > \omega_1$, were obtained by Smurov [11]. One can conjecture that these results have their analogues also in the case of spaces of idempotent measures.

In connection with the mentioned in the introduction result by Fedorchuk, the following question arises:

Question 8.1. Is the map $I(f):I(X) \to I(Y)$ a trivial Q-bundle, for any open map of finite-dimensional compact metric spaces with infinite fibers?

As it was remarked above, one can also consider the spaces $I(X)$ for noncompact metric spaces X. It looks plausible that the results on topology of spaces of probability measures proved in [1] should have their analogues also for the idempotent measures.

Acknowledgements

This research was supported by the Slovenian Research Agency grants P1-0292-0101-04, J1-9643-0101 and BI-UA/07-08-001. The authors thank the referee for many important remarks and suggestions.

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