Available online at www.sciencedirect.com
ScienceDirect
Topology and its Applications 155 (2008) 1201–1206
Topology and its Applications
www.elsevier.com/locate/topol

Embeddability of multiple cones

D. Repovš [a,*], W. Rosicki [b], A. Zastrow [b], M. Željko [a]

[a] Institute of Mathematics, Physics and Mechanics, University of Ljubljana, PO Box 2964, Ljubljana 1001, Slovenia
[b] Institute of Mathematics, Gdańsk University, Ul. Wita Stwosza 57, 80-952 Gdańsk, Poland

Received 11 July 2007; received in revised form 21 February 2008; accepted 22 February 2008

Abstract

The main result of this paper is that if X is a Peano continuum such that its nth cone $C^n(X)$ embeds into $\mathbb{R}^{n+2}$ then X embeds into S^2. This solves a problem proposed by W. Rosicki.

MSC: primary 57Q35; secondary 54C25, 55S15, 57N35

Keywords: Embeddability; Polyhedron; Cone; Suspension; Peano continuum; Planar graph; Local planarity

1. Introduction

The classical Lefschetz–Nöbeling–Pontryagin Embedding Theorem [10] asserts that every compact metric space X of dimension n embedds into $\mathbb{R}^{2n+1}$. We are interested in the relationship between the embeddability of X and embeddability of its Cartesian product $X \times I^n$ with a cube I^n (respectively its cone $C(X)$, iterated cone $C^n(X) = C(\ldots(C(X))\ldots)$, suspension $\Sigma(X)$). Clearly, if X embeds in $\mathbb{R}^m$, then $X \times I^n$ and $C^n(X)$ embed into $\mathbb{R}^{n+m}$. However, sometimes they embed into lower-dimensional Euclidean space. Such is the case for the spheres S^n, where S^n, $C(S^n) \cong B^{n+1}$ and $S^n \times I$ all embed into $\mathbb{R}^{n+1}$.

Let X be a Peano continuum. It was proved in [14] that if the cone $C(X)$ of X embeds into $\mathbb{R}^3$, then X embeds into S^2. As a consequence, if the suspension $\Sigma(X)$ of X embeds into $\mathbb{R}^3$, then X is planar. Note that for each $n \geqslant 3$, there exists a Peano continuum X_n such that X_n is not embeddable in S^n, whereas the cone $C(X_n)$ of X_n is embeddable in $\mathbb{R}^{n+1}$ (see [14]).

The main result of this paper is Theorem 1.1 which solves a problem from [14]. Our proof is based on the methods of [4] and [14].

Theorem 1.1. *Let X be a Peano continuum. Suppose that for some $n \in \mathbb{N}$, $C^n(X)$ is embeddable in $\mathbb{R}^{n+2}$. Then X is embeddable in S^2.*

* Corresponding author.
E-mail addresses: dusan.repovs@guest.arnes.si (D. Repovš), wrosicki@math.univ.gda.pl (W. Rosicki), zastrow@math.univ.gda.pl (A. Zastrow), matjaz.zeljko@fmf.uni-lj.si (M. Željko).

0166-8641/$ – see front matter
doi:10.1016/j.topol.2008.02.007

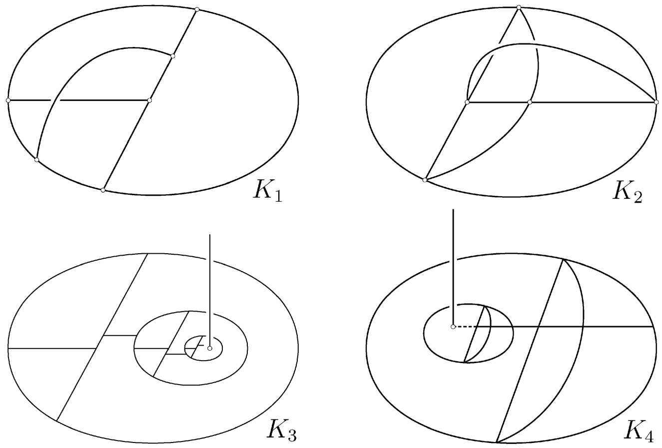

Fig. 1. Kuratowski curves K_1, K_2, K_3, K_4.

Let X be a Peano continuum. Claytor [7] proved that X is embeddable in S^2 if and only if X does not contain any of the Kuratowski curves K_1, K_2, K_3, K_4 (see Fig. 1).

2. Preliminaries

A space X is said to be *planar* if X is embeddable in $\mathbb{R}^2$. We say that X is *locally planar* if for every point $x \in X$ there exists a neighbourhood U_x of x in X such that U_x is embeddable in $\mathbb{R}^2$. Rosicki [13, Theorem 1.1] proved that if a Peano continuum X is embeddable in $\mathbb{R}^3$ and X is a nontrivial Cartesian product $X = Y \times Z$ then one of the factors is either an arc or a simple closed curve.

Rosicki [13] also proved that if a Peano continuum X is embeddable in $\mathbb{R}^3$ and is homeomorphic to the product $Y \times S^1$ then the factor Y must be planar. Alternatively, if $X = Y \times [0,1]$ is embeddable in $\mathbb{R}^3$ and $\check{H}^1(X) = \check{H}^2(X) = 0$ then Y must be planar. Cauty [4], generalizing Rosicki [13], proved that for every $n > 3$ and every Peano continuum X such that $X \times I^{n-2}$ is embeddable into an n-manifold, it follows that X must be locally planar. This theorem was stated earlier by Stubblefield [15]. However, Burgess [2] found a mistake in his proof.

Borsuk [1] constructed an example of a locally connected, locally planar continuum X which is not embeddable into any surface. This continuum contains a sequence (X_n) of subsets homeomorphic to Kuratowski curve K_1 which converge to an arc. Cauty [4] proved that $X \times I^{n-2}$ is not embeddable into any n-manifold so the converse to his theorem does not hold.

3. Local separation

We say that a subset $D \subset \mathbb{R}^n$ *locally separates* $\mathbb{R}^n$ at the point $x_0 \in D$ into $k \in \mathbb{N}$ components if there exists $\varepsilon > 0$ such that for all $0 < \delta < \varepsilon$, the set $B(x_0, \delta) \setminus D$ has exactly k components $A_1, \ldots, A_k$ for which $x_0 \in \overline{A_i}$, for all $i \in \{1, \ldots, k\}$.

It is easy to prove the following lemma using similar methods as in the proof of Lemma 1 in [14].

Lemma 3.1. *A homeomorphic image of any n-disk locally separates $\mathbb{R}^{n+1}$ at any point of its interior into two components.*

Note that $C^n(X) = \sigma^{n-1} * X = \{xt + y(1-t);\ x \in \sigma^{n-1},\ y \in X,\ t \in [0,1]\}$, where σ^{n-1} is an $(n-1)$-simplex. Then $\sigma^{n-1} * \{x\}$ is an n-ball and $\sigma^{n-1} * I$ is an $(n+1)$-ball. We consider σ^{n-1} as a subset of $\sigma^{n-1} * X$.

Lemma 3.2. *Let I_i, $i \in \{1, \ldots, k\}$, $k > 1$ be arcs with common endpoints and pairwise disjoint interiors and $C_k = C^n(\bigcup_{i=1}^k I_i) = \sigma^{n-1} * (\bigcup_{i=1}^k I_i)$. Let $h : C_k \to \mathbb{R}^{n+2}$ be an embedding. Then $h(C_k)$ locally separates $\mathbb{R}^{n+2}$ at any point $h(x_0)$, where x_0 is an interior point of σ^{n-1}, into k components (where σ^{n-1} is considered as a subset of C_k).*

Proof. The proof is by induction on k. If $k=2$, then $C_2 = \sigma^{n-1} * S^0 * S^0$ hence $h(C_2)$ locally separates $\mathbb{R}^{n+2}$ at $h(x_0)$ into two components, by Lemma 3.1.

Assume that Lemma 3.2 holds for $k-1$. Choose $\varepsilon > 0$ smaller than the distance between $h(x_0)$ and the image of $\partial\sigma^{n-1} * (\bigcup_{i=1}^{k} I_i)$. Let $\delta > 0$ be so small that

$$D_k = h\big(C_k \cap B(x_0,\delta)\big) \subset B\big(h(x_0),\varepsilon\big).$$

There exists an open connected set $U_k \subset \mathbb{R}^{n+2}$ such that $D_k = U_k \cap h(C_k)$. Consider the exact sequence of the pair $(U_k, U_k \setminus D_k)$:

$$\to H_1(U_k) \to H_1(U_k, U_k \setminus D_k) \to H_0(U_k \setminus D_k) \to H_0(U_k) \to H_0(U_k, U_k \setminus D_k) \to 0.$$

Since U_k is an open $(n+2)$-manifold, $H_1(U_k) \cong \check{H}_c^{n+1}(U_k)$ by the Poincaré duality, where $\check{H}_c$ denotes the Čech cohomology with compact supports. Also $H_1(U_k, U_k \setminus D_k) \cong \check{H}_c^{n+1}(D_k)$ (see [9, VIII, 7.14], where $L = \emptyset$, $K = D_k$ and $X = U_k$).

We know that $H_0(U_k, U_k \setminus D_k) = 0$ because U_k is arc-connected and $U_k \setminus D_k \neq \emptyset$. Therefore we can consider the exact sequence

$$\to \check{H}_c^{n+1}(U_k) \to \check{H}_c^{n+1}(D_k) \to H_0(U_k \setminus D_k) \to H_0(U_k) \to 0.$$

Next we show by induction that the map $\check{H}_c^{n+1}(U_k) \to \check{H}_c^{n+1}(D_k)$ is trivial. If $k=2$ then D_k is an open $(n+1)$-ball. Then $H_0(U_k \setminus D_k) \cong \mathbb{Z}^2$, by Lemma 3.1. Since $\check{H}_c^{n+1}(D_k) \cong \mathbb{Z}$ and $H_0(U_k) \cong \mathbb{Z}$, we obtain the exact sequence

$$\check{H}_c^{n+1}(U_k) \to \mathbb{Z} \to \mathbb{Z}^2 \to \mathbb{Z} \to 0.$$

Hence the map $\check{H}_c^{n+1}(U_k) \to \check{H}_c^{n+1}(D_k)$ is indeed trivial, as asserted.

Since $\check{H}_c^{n+1}(D_2) \cong \mathbb{Z}$, we obtain by induction that $\check{H}_c^{n+1}(D_k) \cong \check{H}_c^{n+1}(D_{k-1}) \oplus \check{H}_c^{n+1}(D_2') \cong \mathbb{Z}^{k-2} \oplus \mathbb{Z}$, where $D_2' = h(C^n(I_1 \cup I_k) \cap B(x_0,\delta))$.

The map $\check{H}_c^{n+1}(U_k) \to \check{H}_c^{n+1}(D_k) \cong \check{H}_c^{n+1}(h(D_{k-1})) \oplus \check{H}_c^{n+1}(D_2')$ is trivial because both of its coordinates are trivial, by inductive hypothesis.

Therefore the sequence

$$0 \to \check{H}_c^{n+1}(D_k) \to H_0(U_k \setminus D_k) \to H_0(U_k) \to 0$$

is exact. So the sequence

$$0 \to \mathbb{Z}^{k-1} \to H_0(U_k \setminus D_k) \to \mathbb{Z} \to 0$$

is also exact. Hence $H_0(U_k \setminus D_k) \cong \mathbb{Z}^k$ and $U_k \setminus D_k$ has k components.

The point $h(x_0)$ belongs to the closure of each of them. Indeed, if X_k is D_k with a small open neighbourhood of $h(x_0)$ removed then $\check{H}_c^{n+1}(X_k) \cong 0$ and the sequence

$$0 \to H_0(U_k \setminus X_k) \to H_0(U_k) \to 0$$

is exact, therefore $H_0(U_k \setminus X_k) \cong \mathbb{Z}$. □

4. Proof of Theorem 1.1

We shall need two more lemmata:

Lemma 4.1. *Consider the Kuratowski curve K_1 and let $n \in \mathbb{N}$. Then $C^n(K_1)$ is not embeddable in $\mathbb{R}^{n+2}$.*

Proof. Suppose to the contrary, that there exists an embedding $h : C^n(K_1) \to \mathbb{R}^{n+2}$. Consider $K_1 \subset \mathbb{R}^3$ and denote (see Fig. 2)

$$I_1 = [c,a] \cup [a,b], \quad I_2 = [c,p] \cup [p,b], \quad \text{and} \quad I_3 = [c,d] \cup [d,b].$$

If $X = \bigcup_i I_i$, then $\sigma^{n-1} * X = \bigcup_i (\sigma^{n-1} * I_i)$ is a union of $(n+1)$-disks. Let $x_0 \in \operatorname{Int}\sigma^{n-1}$ and choose $\varepsilon > 0$ so that (see Fig. 3)

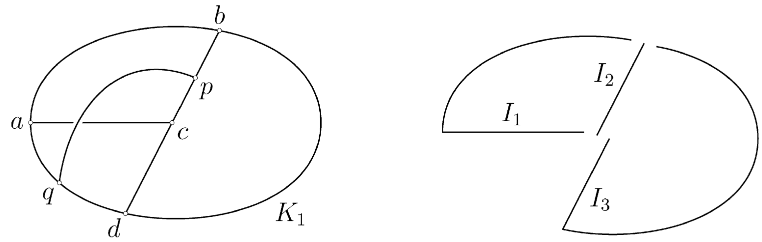

Fig. 2. Kuratowski curve K_1.

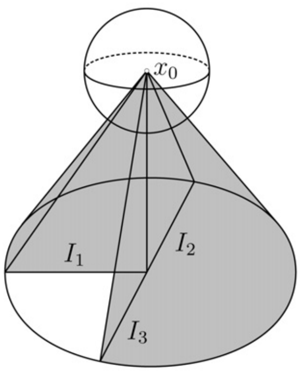

Fig. 3. Local separation at $h(x_0)$.

$$C_1 = h\big(\sigma^{n-1} * (I_1 \cup I_3)\big) \quad \text{locally separates } B\big(h(x_0), \varepsilon\big) \text{ into } B_1, A_1 \text{ at } h(x_0),$$
$$C_2 = h\big(\sigma^{n-1} * (I_1 \cup I_2)\big) \quad \text{locally separates } B\big(h(x_0), \varepsilon\big) \text{ into } B_2, A_2 \text{ at } h(x_0),$$
$$C_3 = h\big(\sigma^{n-1} * (I_2 \cup I_3)\big) \quad \text{locally separates } B\big(h(x_0), \varepsilon\big) \text{ into } B_3, A_3 \text{ at } h(x_0).$$

By Lemma 3.2 we have that $C = h(C^n(I_1 \cup I_2 \cup I_3)) = h(\sigma^{n-1} * \bigcup_{i=1}^3 I_i) = h(\bigcup_{i=1}^3 \sigma^{n-1} * I_i)$ locally separates $B(h(x_0), \varepsilon)$ into three components. We will show that we can adopt the notation for these three components to be B_1, A_2 and A_3.

We use abstract linear combinations for describing our joins, e.g.

$$\sigma^{n-1} * K_1 = \big\{xt + y(1-t);\ x \in \sigma^{n-1},\ y \in K_1,\ t \in [0,1]\big\}.$$

For $\sigma^{n-1} \subset \sigma^{n-1} * K_1$, we have that $h(\sigma^{n-1})$ is a subset of C_1, but that $h|_{\sigma^{n-1} * I_2}$ maps all linear combinations with $t \neq 1$, but sufficiently close to 1, to a subset that is connected but disjoint from C_1. Hence this subset can only be contained either in A_1 or in B_1. We may assume that it is in A_1. Since the entire neighbourhood of σ^{n-1} in $\sigma^{n-1} * I_2$ is mapped by h into A_1, we have $h(\sigma^{n-1} * I_2) \cap B_1 = \emptyset$, provided $\varepsilon > 0$ is small enough. Then B_1 is not divided by C, so it is one of the three components.

Analogously, by considering C_2 (respectively C_3) we can make sure that A_2 and A_3 are the other two components and that $h(\sigma^{n-1} * I_3) \cap A_2 = \emptyset$ and $h(\sigma^{n-1} * I_1) \cap A_3 = \emptyset$. Since $C \cup B_1 \cup A_2 \cup A_3$ and $C \cup A_1 \cup B_1$ are both disjoint decompositions of a neighbourhood of $h(x_0)$, the set $h(\sigma^{n-1} * I_2) \cup C_1$ separates the component A_1 into components A_2 and A_3.

Note that

$$x_0 * K_1 = \big\{x_0 t + x(1-t);\ x \in K_1,\ t \in [0,1]\big\} \subset C^n(K_1).$$

Choose t_0 near 1 so that

$$h\big(\big\{x_0 t + x(1-t);\ x \in K_1,\ t \geqslant t_0\big\}\big) \subset B\big(h(x_0), \varepsilon\big).$$

Let $p' = h(x_0 t_0 + p(1-t_0)) \in A_1$. The arc $H = h(\{x_0 t_0 + x(1-t_0);\ x \in (p,q)\})$ is contained in $B(h(x_0), \varepsilon) \setminus h(C)$. Therefore points p' and $q' = h(x_0 t_0 + q(1-t_0))$ are in the same component. Hence $q' \in A_2$ or $q' \in A_3$. So the arc

$I = h(\{x_0t_0 + x(1-t_0);\ x \in (a,q] \cup [q,d)\})$ is contained either in A_2 or in A_3. But this yields a contradiction since $a' = h(x_0t_0 + a(1-t_0)) \notin \overline{A_3}$ (so $I \not\subset A_3$) and $d' = h(x_0t_0 + a(1-t_0)) \notin \overline{A_2}$ (so $I \not\subset A_2$). □

The proof of the next lemma can be obtained by changing the proof of [14, Lemma 4] in the same way as we did it for the proof of Lemma 2.3 using the proof of [14, Lemma 3].

Lemma 4.2. *Consider the Kuratowski curve K_2 and let $n \in \mathbb{N}$. Then $C^n(K_2)$ is not embeddable in $\mathbb{R}^{n+2}$.*

Proof of Theorem 1.1. By Claytor's theorem (see [6,7]), it suffices to show that $C^n(K_i)$ is not embeddable into $\mathbb{R}^{n+2}$ for any $i \in \{1,2,3,4\}$. Now, Cauty [4] proved that $K_i \times I^n$ is not embeddable into $\mathbb{R}^{n+2}$ for any $i \in \{3,4\}$. Therefore also $C^n(K_i)$ is not embeddable into $\mathbb{R}^{n+2}$ for any $i \in \{3,4\}$. Hence we only have to consider the cases $i = 1$ and $i = 2$. The proof is now completed by application of Lemmata 4.1 and 4.2. □

5. Epilogue

Repovš, Skopenkov and Ščepin [12] proved that if $X \times I$ PL embeds into $\mathbb{R}^{n+1}$, where X is either an acyclic polyhedron and $\dim X \leqslant \frac{2n}{3} - 1$ or a homologically $(2\dim X - n - 1)$-connected manifold and $\dim X \leqslant \frac{2n}{3} - 1$ or a collapsible polyhedron, then X PL embeds into $\mathbb{R}^n$.

Question 5.1. What can one say about embeddability of X into Euclidean spaces if one considers $C(X)$ or $C^n(X)$ or $\Sigma(X)$ or $\Sigma^n(X)$ instead of $X \times I$ for X in [12]?

It follows by [12] that if X is a contractible polyhedron such that $X \times I$ embeds into $\mathbb{R}^{n+1}$ then X embeds into $\mathbb{R}^n$. So if X is contractible and $C(X) \subset \mathbb{R}^{n+1}$ then X embeds into $\mathbb{R}^n$.

Note that there exists a polyhedron P_n such that P_n is not embeddable into $\mathbb{R}^n$ but $C^2(P_n)$ is embeddable in $\mathbb{R}^{n+2}$. Namely, Cannon [3] proved that if H^n is a homology n-sphere then its double suspension $\Sigma^2(H^n)$is the $(n+2)$-sphere (see [8] and [11] for a far reaching generalization of this result by Edwards). So if $P_n = H^n \setminus B^n$ where B^n is an n-ball then the double cone $C^2(P_n)$ embeds in $\mathbb{R}^{n+2}$. The polyhedron P_n is acyclic but not contractible.

Question 5.2. Does there exist a contractible n-dimensional polyhedron X^n such that $C^k(X^n)$ embeds into $\mathbb{R}^{n+k}$, but X^n does not embed into $\mathbb{R}^n$?

In [14, Theorem 2] contractible continua X_n were constructed, such that X_n is not embeddable in $\mathbb{R}^n$, $C(X_n)$ is embeddable in $\mathbb{R}^{n+1}$, and X_n is not a polyhedron. By [12], if X is an n-polyhedron then $X \times I$ embeds into $\mathbb{R}^{2n+1}$. If X is an n-polyhedron then $C(X)$ need not embed into $\mathbb{R}^{2n+1}$. For example, the Kuratowski curves K_1 and K_2 are 1-polyhedra but the cones $C(K_1)$ and $C(K_2)$ do not embed into $\mathbb{R}^3$.

Question 5.3. Suppose that X is a compact contractible n-dimensional polyhedron. Does the cone $C(X)$ embed into $\mathbb{R}^{2n+1}$? Does the same hold if X is only acyclic?

Acknowledgements

This research was supported by Polish–Slovenian grant BI-PL 12/2004-2005, Polish grant N20100831/0524, SRA program P1-0292-0101-04, and SRA project J1-9643-0101. The authors acknowledge R. Cauty [5] for the hint communicated to the second author, and the referee for comments and suggestions.

References

[1] K. Borsuk, Über stetige Abbildungen der euklidischen Räume, Fund. Math. 21 (1933) 236–246.
[2] C.E. Burgess, Review of [6], Math. Rev. 26 (749) (1963) 144.
[3] J.W. Cannon, Shrinking cell-like decompositions of manifolds. Codimension three, Ann. of Math. (2) 110 (1) (1979) 83–112.
[4] R. Cauty, Sur le plongement de $X \times I^{n-2}$ dans une n-variété, Proc. Amer. Math. Soc. 94 (1985) 516–522.
[5] R. Cauty, personal communication, 1993.

[6] S. Claytor, Topological immersions of Peanian continua in a spherical surface, Ann. of Math. (2) 35 (1934) 809–835.
[7] S. Claytor, Peanian continua not embeddable in a spherical surface, Ann. of Math. (2) 38 (1937) 631–646.
[8] R.J. Daverman, Decompositions of Manifolds, Academic Press, New York, 1986.
[9] A. Dold, Lectures on Algebraic Topology, Springer-Verlag, Berlin, 1995.
[10] R. Engelking, Dimension Theory, North-Holland, Amsterdam, 1978.
[11] R.D. Edwards, Suspensions of homology spheres, preprint, math.GT/0610573.
[12] D. Repovš, A.B. Skopenkov, E.V. Ščepin, On embeddability of $X \times I$ into Euclidean space, Houston J. Math. 21 (1995) 199–204.
[13] W. Rosicki, On topological factors of 3-dimensional locally connected continuum embeddable in E^3, Fund. Math. 99 (1978) 141–154.
[14] W. Rosicki, On embeddability of cones in Euclidean spaces, Colloq. Math. 64 (1993) 141–147.
[15] B. Stubblefield, Some imbedding and non-imbedding theorems for N-manifolds, Trans. Amer. Math. Soc. 103 (1962) 403–420.