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Examples of cohomology manifolds which are not homologically locally connected

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Abstract

Bredon has constructed a 2-dimensional compact cohomology manifold which is not homologically locally connected, with respect to the singular homology. In the present paper we construct infinitely many such examples (which are in addition metrizable spaces) in all remaining dimensions $n \geqslant 3$.

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1. Introduction

We begin by fixing terminology, notations and formulating some elementary facts. We shall use singular homology H_n, singular cohomology H^n and Čech cohomology $\check{H}^n$ groups with integer coefficients $\mathbb{Z}$. By *HLC* and *clc* spaces we shall denote homology and cohomology locally connected spaces with respect to singular homology and Čech cohomology, respectively. The general references for terms undefined in this paper will be [3–6,8,16].

Definition 1.1. (Cf. [3, Corollary 16.19, p. 377].) A finite-dimensional cohomology locally connected space X is an n-dimensional cohomology manifolds (n-cm) if

$$\check{H}^p\big(X, X \setminus \{x\}\big) = \begin{cases} \mathbb{Z}, & \text{for } p = n, \\ 0, & \text{for } p \neq n, \end{cases}$$

for all $x \in X$.

We denote the segment $[0, 1]$ by $\mathbb{I}$, the n-dimensional cube by $\mathbb{I}^n$, and the quotient space of a space X by its subset B by X/B. The double $d(X, B)$ of X with respect to B is the quotient space of the product $X \times \{0, 1\}$, where $\{0, 1\}$ is the two point set, by identification of the points $(x, 0)$ with $(x, 1)$ for every $x \in B$.

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Definition 1.2. (Cf. [3, Theorem 16.27, p. 385].) A space X is said to be a cohomology manifold with boundary B if B is a closed nowhere dense subspace of X and the double $d(X, B)$ is a cohomology n-manifold.

Bredon constructed a 2-dimensional compact cohomology manifold which is not homologically locally connected (non-*HLC*) space [3, Example 17.13, p. 131]. Every metrizable 2-dimensional locally compact cohomology manifold is a topological 2-manifold and therefore it is a *HLC* space (see e.g. [3, Theorem 16.32, p. 388]).

The main goal of the present paper is to construct for all remaining dimensions $n \geqslant 3$, infinitely many n-dimensional compact metrizable cohomology manifolds which are not homologically locally connected.

Main Theorem 1.3. *Let M^n be an n-dimensional compact manifold (possibly with boundary), $n \geqslant 3$. Then there exists a compact subset C of the interior $\operatorname{int}(M^n)$ of M^n such that the quotient space M^n/C is a non-homologically locally connected n-dimensional metrizable cohomology manifold.*

2. Preliminaries

Let G be any multiplicative (in general non-Abelian) group. By the *commutator* of two elements $a, b \in G$ we mean the following product $[a, b] = a^{-1}b^{-1}ab \in G$. Let G_n be the *lower central series* of G which is defined inductively: $G_1 = G$, $G_{n+1} = [G_n, G]$, where $[G_n, G]$ is the normal subgroup of G generated by the following set of commutators: $\{[a, b]\colon a \in G_n, b \in G\}$.

If F is a free group, the factor group F/F_n is called a *free nilpotent group*. Let $A * B$ be the free product of groups A and B. Every non-neutral element x of $A * B$ is then uniquely expressible in the *reduced form* as $x = u_1u_2 \cdots u_n$, where all $u_i \in A \cup B$ are non-neutral elements of the groups A and B, and if $u_i \in A$ then $u_{i+1} \in B$ (if $u_i \in B$ then $u_{i+1} \in A$, respectively), for $i \in \{1, 2, \ldots, n-1\}$.

Following Rhemtulla [14, p. 578], we define for an element $b \in B$ of order > 2 the mapping $w_b : A * B \to \mathbb{Z}$ as follows: Let (x, b) denote the multiplicity of b in x, i.e. the number of occurrences of b in the reduced form of x. Similarly, let (x, b^{-1}) be the multiplicity of b^{-1} in x. Write $w_b(x) = (x, b) - (x, b^{-1})$. For example, let a be an element of group A order of which is > 2. Then

$$w_b\big([ab, ba]\big) = w\big(b^{-1}a^{-1}a^{-1}b^{-1}ab^2a\big) = -2. \tag{1}$$

By the *commutator length* of $g \in G$, denoted by $cl(g)$, we denote the minimal number of the commutators of the group G whose product is equal to g. If such a number does not exist then we set $cl(g) = \infty$. We also set $cl(e) = 0$ for the neutral element e of the group $g \in G$. Obviously,

$$\begin{aligned} &cl(g) = \infty \quad \text{if and only if } g \notin G_2, \\ &cl(g_1g_2) \leqslant cl(g_1) + cl(g_2) \end{aligned} \tag{2}$$

or, equivalently (since $cl(g^{-1}) = cl(g)$)

$$cl(g_2) - cl(g_1) - cl(g_3) \leqslant cl(g_1g_2g_3). \tag{3}$$

If $\varphi : G \to G'$ is a homomorphism of groups then for every $g \in G$,

$$cl\big(\varphi(g)\big) \leqslant cl(g). \tag{4}$$

For any path connected space X, the fundamental group $\pi_1(X)$ does not depend on the choice of the base point and $H_1(X)$ is isomorphic to the factor group $\pi_1(X)/[\pi_1(X), \pi_1(X)]$. If g is a loop in the space X then by $[g]$ we denote the corresponding element of the fundamental group $\pi_1(X)$.

For the construction of examples and proofs of their asserted properties we shall need some general facts:

Proposition 2.1. *(See [10, Proposition 3.4].) The lower central series $\{F_n\}$ of any free group F has a trivial intersection, i.e. $\bigcap_{n=1}^{\infty} F_n = \{e\}$.*

Proposition 2.2. *(See [12, Theorem 1.5].) Every free nilpotent group is torsion-free.*

Proposition 2.3. *(See [11, Exercise 2.4.13].) Let F be any free group on two generators a, b. Then the subgroup F_2 is freely generated by the commutators $[a^n, b^m]$, where n and m are non-zero integers.*

Suppose that $b \in B$ and its order is > 2. The main properties of the function w_b of Rhemtulla which are necessary for our proofs are formulated in the following three propositions:

Proposition 2.4. *(See [14, p. 579].) For any elements $x \in A$ and $y \in B$*

$$\left|w_b(xy)\right| \leqslant \left|w_b(x)\right| + \left|w_b(y)\right| + 3.$$

Proposition 2.5. *(See [14, p. 579].) For any commutator $[x, y]$, $x \in A$ and $y \in B$, $|w_b([x, y])| \leqslant 9$.*

By induction, using Propositions 2.4 and 2.5, one gets the following:

Proposition 2.6. *(See [14, p. 579].) For any k commutators $[x_i, y_i]$, $x_i \in A$ and $y_i \in B$, $i = 1, 2, \ldots, k$,*

$$\left|w_b\left(\prod_{i=1}^{k}[x_i, y_i]\right)\right| \leqslant 12k - 3.$$

Proposition 2.7. *Let $A * B$ be the free product of groups A and B. Let $a \in A$ and $b \in B$ be arbitrary elements of order > 2. Then $\sup(cl([ab, ba]^n)\colon n \in \mathbb{N}) = \infty$.*

Proof. Suppose that $\sup(cl([ab, ba]^n)\colon n \in \mathbb{N}) = k$. Then for all n the element $[ab, ba]^n$ is a product of $\leqslant k$ commutators. It follows by Proposition 2.6 that $|w_b([ab, ba]^n)| \leqslant 12k - 3$. On the other hand since the word $[ab, ba]$ is cyclicaly reduced, we have the equality $w_b([ab, ba]^n) = -2n$ which generalizes the equality (1) (cf. [14, Proof of Lemma 2.28, p. 579] and [14, Proof of Lemma 2.26, Case 1, p. 578]). If n is large enough then $|-2n| > 12k - 3$ and we get a contradiction. □

Remark 2.8. If the order of the element b is 2 or, more generally, if $b^2a = ab^2$ in a group G then $[a, b]^{2n} = [[a, b]^{-n}, b]$ and $[a, b]^{2n+1} = [a[a, b]^{-n}, b]$, i.e. $cl([a, b]^{2n}) = 1 = cl([a, b]^{2n+1})$ for every n.

Proof. Since $[a, b] = b^{-1}[a, b]^{-1}b$ it follows that $[a, b]^n = b^{-1}[a, b]^{-n}b$. By multiplying both sides of the equality by $[a, b]^n$ we get $[a, b]^{2n} = [[a, b]^{-n}, b]$. Since $b[a, b]^n = [a, b]^{-n}b$ and $[a, b]^{2n+1} = [a, b]^n a^{-1}b^{-1}ab[a, b]^n$ we get $[a, b]^{2n+1} = [a[a, b]^{-n}, b]$. □

Proposition 2.9. *(Borsuk [1].) Consider a triple of continua $\widehat{W} \supset W \supset X$ where the space W is a strong deformation retract of $\widehat{W}$. Then the quotient space W/X is a strong deformation retract of $\widehat{W}/X$. In particular, W/X and $\widehat{W}/X$ then have the same homotopy type.*

3. Construction of compactum C

Let $\mathcal{P}$ be any inverse sequence of finite polyhedra with piecewise-linear bonding mappings:

$$P_0 \xleftarrow{f_0} P_1 \xleftarrow{f_1} P_2 \xleftarrow{f_2} \cdots.$$

We denote the *infinite mapping cylinder* of $\mathcal{P}$ by $C(f_0, f_1, f_2, \ldots)$ (cf. [15]) and we denote the natural compactification of this infinite mapping cylinder by $\varprojlim \mathcal{P}$ with the symbol $\widetilde{\mathcal{P}}$ (cf. [9]). Let $C(f_0, f_1, f_2, \ldots)^*$ be the one-point compactification of the infinite mapping cylinder. Denote the compactification point by p^*. Obviously, the quotient space $\widetilde{\mathcal{P}}$ by $\varprojlim \mathcal{P}$ is homeomorphic to $C(f_0, f_1, f_2, \ldots)^*$.

Proposition 3.1. *(See [9].) If P_0 is a point then $\widetilde{\mathcal{P}}$ is an absolute retract (AR).*

Proposition 3.2. *Suppose that in the inverse sequence $\mathcal{P}$ the dimensions of all polyhedra are $\leqslant n$ and let P_0 be a one-point space. Consider $\varprojlim \mathcal{P}$ as a subspace of the cube $\mathbb{I}^{2n+1}$. Then the quotient space $\mathbb{I}^{2n+1}$ by $\varprojlim \mathcal{P}$ is homotopy equivalent to $C(f_0, f_1, f_2, \ldots)^*$.*

Proof. Obviously $\dim \widetilde{\mathcal{P}} \leqslant n+1$ and therefore $\widetilde{\mathcal{P}}$ is embeddable into $\mathbb{I}^{2n+3}$. According to Proposition 3.1, $\widetilde{\mathcal{P}}$ is an *AR* and therefore a strong deformation retract of any *AR* which contains it. In particular, $\widetilde{\mathcal{P}}$ is a strong deformation retract of $\mathbb{I}^{2n+3}$.

Applying now Borsuk Theorem 2.9 to the triple $\mathbb{I}^{2n+3} \supset \widetilde{\mathcal{P}} \supset \varprojlim \mathcal{P}$, we can conclude that the quotient space of $\mathbb{I}^{2n+3}$ by $\varprojlim \mathcal{P}$ is homotopy equivalent to the quotient space $\widetilde{\mathcal{P}}$ by $\varprojlim \mathcal{P}$, i.e. to the space $C(f_0, f_1, f_2, \ldots)^*$.

The homotopy type of the quotient space $\mathbb{I}^{2n+3}/\varprojlim \mathcal{P}$ does not depend on the embedding of $\varprojlim \mathcal{P}$ to $\mathbb{I}^{2n+3}$, by Theorem 2.9 and West and Klee [2]. Therefore, applying again Theorem 2.9 to the triple $\mathbb{I}^{2n+3} \supset \mathbb{I}^{2n+1} \supset \varprojlim \mathcal{P}$, we can conclude that $\mathbb{I}^{2n+3}/\varprojlim \mathcal{P}$ is homotopy equivalent to $\mathbb{I}^{2n+1}/\varprojlim \mathcal{P}$ and thus $\mathbb{I}^{2n+1}/\varprojlim \mathcal{P} \simeq C(f_0, f_1, f_2, \ldots)^*$. □

Suppose that P_0 is a singleton and that for $n > 0$ P_n is a bouquet of 4 oriented circles with the base point p_n. The fundamental group $\pi_1(P_n)$ is a free group with natural generators $x_{n,1}, x_{n,2}, x_{n,3}, x_{n,4}$. Consider $\pi_1(P_n)$ as a free product of free groups $F(x_{n,1}; x_{n,2})$ and $F(x_{n,3}; x_{n,4})$. Let $y_{n,1}$, $y_{n,2}$ and $y_{n,3}$, $y_{n,4}$ be free generators of the commutator subgroups of the groups $F(x_{n,1}; x_{n,2})$ and $F(x_{n,3}; x_{n,4})$, respectively. For example, according to Proposition 2.3 we can suppose that: $y_{n,1} = [x_{n,1}, x_{n,2}]$, $y_{n,2} = [x_{n,1}^2, x_{n,2}^2]$, $y_{n,3} = [x_{n,3}, x_{n,4}]$, $y_{n,4} = [x_{n,3}^2, x_{n,4}^2]$.

Suppose that f_0 is a trivial mapping and that for $n > 0$, the mapping $f_n : P_{n+1} \to P_n$ is piecewise-linear and such that $f_n(p_{n+1}) = (p_n)$ and $f_{n\sharp}(x_{n+1,i}) = y_{n,i}$, for $i = 1, 2, 3, 4$, where $f_{n\sharp}$ is a homomorphism of the corresponding fundamental groups induced by f_n. All homomorphisms $f_{n\sharp}$ are monomorphisms since by our choice the elements $y_{n,1}$, $y_{n,2}$ and $y_{n,3}$, $y_{n,4}$ are free generators. Therefore we can consider the elements $x_{n,i}$ for $i = 1, 2, 3, 4$ as elements of the group

$$F = F_{x_{1,1}; x_{1,2}; x_{1,3}; x_{1,4}}. \tag{5}$$

The 1-dimensional compactum $C = \varprojlim \mathcal{P}$ is embeddable into the interior $(0, 1)^3$ of the cube $\mathbb{I}^3$. Hereafter we shall fix such an embedding. Since $y_{n,i}$ belongs to the commutator subgroup of $\pi_1(P_n)$ and since $\dim P_n = 1$ it follows that $f_{n*} : H_*(P_{n+1}) \to H_*(P_n)$ is a trivial mapping. By the Universal Coefficient Theorem the mapping $H^*(P_n) \to H^*(P_{n+1})$ is also trivial and the following holds:

$$\check{H}^*(C) \cong \check{H}^*(pt). \tag{6}$$

4. Proof of Main Theorem

To prove Main Theorem 1.3 we must first prove the following:

Theorem 4.1. *The* 1*-dimensional singular homology group of the quotient space $\mathbb{I}^3/C$ is uncountable.*

Proof. Let I_n be the segment connecting p_{n+1} and p_n in the space $C(f_0, f_1, f_2, \ldots)^*$. Since $f_{n\sharp}$ maps the groups $F(x_{n+1,1}, x_{n+1,2})$ and $F(x_{n+1,3}, x_{n+1,4})$ to $F(x_{n,1}, x_{n,2})$ and $F(x_{n,3}, x_{n,4})$, respectively, the space $C(f_0, f_1, f_2, \ldots)^*$ splits, i.e. it is the union of two closed subspaces, the intersection of which is the segment $\{p^*\} \cup (\bigcup_{i=0}^{\infty} I_n)$.

Suppose that $H_1(\mathbb{I}^3/C)$ were countable. Then $H_1(C(f_0, f_1, f_2, \ldots)^*)$ would also be countable, according to Proposition 3.2. From the following Mayer–Vietoris exact sequence:

$$H_1(P_1) \to H_1\big(C(f_0)\big) \oplus H_1\big(C(f_1, f_2, f_3, \ldots)^*\big) \to H_1\big(C(f_0, f_1, f_2, \ldots)^*\big) \to 0$$

it would then follow that the group $H_1(C(f_1, f_2, f_3, \ldots)^*)$ is countable.

Define inductively a sequence of loops g_n with the base point $p_1 \in C(f_1, f_2, f_3, \ldots)^*$. Let g_1 be any loop in P_1 representing a commutator of the group $\pi_1(P_1)$. Suppose that the loops g_i are defined for $i \leqslant n-1$. Then define the loop g_n in the following way. Consider the set of loops $g_1^{\varepsilon_1} g_2^{\varepsilon_2} g_3^{\varepsilon_3} \cdots g_{n-1}^{\varepsilon_{n-1}}$, where every ε_i is equal to 0 or ± 1. This set is finite and therefore there exists the maximum of the commutator length of its elements. Call this number K_n. We have

$$cl\big(\big[g_1^{\varepsilon_1} g_2^{\varepsilon_2} g_3^{\varepsilon_3} \cdots g_{n-1}^{\varepsilon_{n-1}}\big]\big) \leqslant K_n. \tag{7}$$

Consider the elements $x_{n,1}, x_{n,3}$ as elements of the group F (see (5)). These are non-neutral elements of the commutator subgroup and therefore by Proposition 2.1 there exists a finite number m_n such that $x_{n,1} \notin F_{m_n}$ and $x_{n,3} \notin F_{m_n}$. Let $Y_{m_n} = C(f_1, f_2, f_3, \ldots)^*/C(f_{m_n+1}, f_{m_n+2}, f_{m_n+3} \ldots)^*$. There is a homomorphism $\pi_1(Y_{m_n}) \to F/F_{m_n}$. Since by Proposition 2.2 the free nilpotent group F/F_{m_n} has no torsion, it follows that the order of any element which does not lie in the kernel of this homomorphism must be infinite. Therefore the orders of the natural elements $\widetilde{x_{n,1}}$ and $\widetilde{x_{n,3}}$ which correspond to $x_{n,1}$ and $x_{n,3}$ in $\pi_1(Y_{m_n})$ are infinite.

The polyhedron Y_{m_n} splits in two finite polyhedra, therefore the group $\pi_1(Y_{m_n})$ is a free product of two groups. According to Proposition 2.7 there exists a number L_n such that

$$cl\big([\widetilde{x_{n,1}}\widetilde{x_{n,3}}, \widetilde{x_{n,3}}\widetilde{x_{n,1}}]^{L_n}\big) > 2K_n + n. \tag{8}$$

Let g_n be a loop in $P_{m_n} \cup (\bigcup_{i=1}^{m_n-1} I_i) \subset C(f_1, f_2, f_3, \ldots)^*$ which represents the element $[\widetilde{x_{n,1}}\widetilde{x_{n,3}}, \widetilde{x_{n,3}}\widetilde{x_{n,1}}]^{L_n}$. Then by an inductive procedure we get a sequence of loops $\{g_n\colon n \in \mathbb{N}\}$.

Consider now the sequence ε of units and zeros: $\varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots)$. To every such sequence there corresponds an element $[g^\varepsilon] = [g_1^{\varepsilon_1} g_2^{\varepsilon_2} g_3^{\varepsilon_3} \cdots]$ of the group $\pi_1(C(f_0, f_1, f_2, \ldots)^*)$. (Note that the infinite product of loops is not always defined but in our case obviously there exists a loop g^ε such that its projection to the space Y_{m_n} is homotopy equivalent to the projection of $g_1^{\varepsilon_1} g_2^{\varepsilon_2} g_3^{\varepsilon_3} \cdots g_k^{\varepsilon_k}$ for $k \geqslant m_n$, and the element $[g^\varepsilon]$ is well-defined.)

Since the set of all sequences of units and zeros is uncountable whereas by our hypothesis the group $H_1(C(f_1, f_2, f_3, \ldots)^*)$ is countable, it follows that there exist two elements $[g^\varepsilon]$ and $[g^{\varepsilon'}]$ generating the same element in the homology group and such that $\varepsilon_i \neq \varepsilon_i'$ for an infinite set of indices $\{i\}$. The element $[g^{\varepsilon'}][(g^\varepsilon)]^{-1}$ belongs to the commutator subgroup of $\pi_1(C(f_1, f_2, f_3, \ldots)^*)$ or equivalently, it has a finite commutator length $cl([g^{\varepsilon'}][g^\varepsilon]^{-1}) = k < \infty$. Let n be a number such that

$$n > k \tag{9}$$

and $\varepsilon_n \neq \varepsilon_n'$. We can suppose that $\varepsilon_n = 1$ and $\varepsilon_n' = 0$. Consider the mapping $\Pi_n : \pi_1(C(f_1, f_2, f_3, \ldots)^*) \to \pi_1(Y_n)$. By (4) we have the following:

$$cl\big(\Pi_n\big(\big[g^{\varepsilon'}\big]\big[g^\varepsilon\big]^{-1}\big)\big) \leqslant k. \tag{10}$$

On the other hand

$$\Pi_n\big(\big[g^{\varepsilon'}\big]\big[g^\varepsilon\big]^{-1}\big) = \big[g_1^{\varepsilon_1'} g_2^{\varepsilon_2'} \cdots g_{n-1}^{\varepsilon_{n-1}'}\big][g_n]^{-1}\big[g_{n-1}^{-\varepsilon_{n-1}} \cdots g_2^{-\varepsilon_2} g_1^{-\varepsilon_1}\big].$$

By equality (3) we therefore have the following:

$$\begin{aligned} &cl\big([g_n]\big) - cl\big(\big[g_1^{\varepsilon_1'} g_2^{\varepsilon_2'} \cdots g_{n-1}^{\varepsilon_{n-1}'}\big]\big) - cl\big(\big[g_{n-1}^{-\varepsilon_{n-1}} \cdots g_2^{-\varepsilon_2} g_1^{-\varepsilon_1}\big]\big) \\ &\quad \leqslant cl\big(\Pi_n\big(\big[g^{\varepsilon'}\big]\big[g^\varepsilon\big]^{-1}\big)\big). \end{aligned} \tag{11}$$

By (8) we have that $cl([g_n]) > 2K_n + n$. By (7) we have also that

$$cl\big(\big[g_1^{\varepsilon_1'} g_2^{\varepsilon_2'} \cdots g_{n-1}^{\varepsilon_{n-1}'}\big]\big) \leqslant K_n$$

and

$$cl\big(\big[g_{n-1}^{-\varepsilon_{n-1}} \cdots g_2^{-\varepsilon_2} g_1^{-\varepsilon_1}\big]\big) \leqslant K_n.$$

It follows that the left side of the inequality (11) is not less than $(2K_n + n) - K_n - K_n = n$. However, by (10) the part on the right does not exceed k. We thus get a contradiction to (9). □

Corollary 4.2. *The singular cohomology group H^* of the space $\mathbb{I}^3/C$ is non-trivial.*

Proof. Suppose that singular cohomology groups H^1 and H^2 of the space $\mathbb{I}^3/C$ were trivial. Then by the Universal Coefficient Theorem for singular homology and cohomology it would follow that

$$\mathrm{Hom}\big(H_1\big(\mathbb{I}^3/C\big), \mathbb{Z}\big) = 0 \quad \text{and} \quad \mathrm{Ext}\big(H_1\big(\mathbb{I}^3/C\big), \mathbb{Z}\big) = 0.$$

Therefore by the Nunke Theorem (cf. [3, Theorem 15.6, p. 372] or [13]) it would follow that $H_1(\mathbb{I}^3/C) = 0$. This would contradict Theorem 4.1. □

Theorem 4.3. *The space $\mathbb{I}^3/C$ is a non-homologically locally connected (non-HLC) 3-dimensional compact metrizable cohomology manifold with boundary.*

Proof. For *HLC* paracompact spaces the singular cohomology and Čech cohomology are naturally isomorphic [3, Theorem 1.1, p. 184]. The Čech cohomology of the space $\mathbb{I}^3/C$ is trivial—by the Vietoris–Begle Theorem and since the Čech cohomology of the space C is trivial (see (6)), but singular cohomology is non-trivial, as it was shown in Corollary 4.2. Therefore $\mathbb{I}^3/C$ is not an *HLC* space.

Other assertions of the theorem immediately follow by Wilder's monotone mapping theorem (see [3, Theorem 16.33, p. 389]). □

Remark 4.4. Theorem 4.3 shows that the statement of the first author in the paper [7] preceding theorem on page 531 (page 113 in English) should be corrected as follows: Every homologically locally connected (HLC) acyclic with respect to Cech cohomology compactum is an acyclic space with respect to the singular, Borel–Moore, Steenrod–Sitnikov, Vietoris and Čech homology.

Proof of Main Theorem. Since we can suppose that $\mathbb{I}^n/C \subset M^n/C$ and since the boundary of $\mathbb{I}^n/C$ is an $(n-1)$-dimensional sphere, $n \geqslant 3$, it follows by the Mayer–Vietoris exact sequence that $H_1(\mathbb{I}^n/C)$ is isomorphic to the subgroup of $H_1(M^n/C)$.

By Theorem 4.1, the group $H_1(\mathbb{I}^n/C)$ is uncountable. Therefore

$$\text{the group } H_1\big(M^n/C\big) \text{ is uncountable.} \tag{12}$$

Suppose that $H_1(M^n/C)$ were an *HLC* space. Then its Čech cohomology would be finitely generated, by Wilder's theorem (see e.g. [3, Theorem 17.4, p. 127]). Since for *HLC* spaces Čech and singular cohomology are isomorphic, it would follow that $H^1(M^n/C)$ and $H^2(M^n/C)$ must be finitely generated. Then by the Universal Coefficient Formula for singular homology and cohomology it would follow that the groups $\mathrm{Hom}(H_1(M^n/C), \mathbb{Z})$ and $\mathrm{Ext}(H_1(M^n/C), \mathbb{Z})$ are finitely generated. It would then follow, by Bredon's Theorem [3, Proposition 14.7, p. 367], that the group $H_1(M^n/C)$ must be finitely generated and therefore countable. This contradicts (12). Hence the proof of Theorem 1.3 is completed. □

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