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# Sections of convex bodies and splitting problem for selections

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**Abstract**

Let $F_1 : X \to Y_1$ and $F_2 : X \to Y_2$ be any convex-valued lower semicontinuous mappings and let $L : Y_1 \oplus Y_2 \to Y$ be any linear surjection. The *splitting problem* is the problem of representation of any continuous selection $f$ of the composite mapping $L(F_1; F_2)$ in the form $f = L(f_1; f_2)$, where $f_1$ and $f_2$ are some continuous selections of $F_1$ and $F_2$, respectively. We prove that the splitting problem always admits an approximate solution with $f_i$ being an $\varepsilon$-selection (Theorem 2.1). We also propose a special case of finding exact splittings, whose occurrence is stable with respect to continuous variations of the data (Theorem 3.1) and we show that, in general, exact splittings do not exist even for the finite-dimensional range.


*Keywords:* Convex-valued mapping; Continuous selection; Banach space; Lower semicontinuous map; Minkowski sum

## 1. Introduction

For any pair of mappings $f : \mathbb{R} \to [2, 3]$ and $g : \mathbb{R} \to [3, 7]$ the sum $h = f + g$ maps $\mathbb{R}$ into the segment $[5, 10]$. In the category of sets and mappings the converse statement is evidently true. Namely each $h : \mathbb{R} \to [5, 10]$ admits a splitting $h = f + g$ for some $f : \mathbb{R} \to [2, 3]$ and $g : \mathbb{R} \to [3, 7]$. The situation becomes more complicated for topological spaces and continuous

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mappings. Is it true that every *continuous*, map $h : X \to [5, 10]$ is a sum $h = f + g$ of some *continuous* maps $f : X \to [2, 3]$ and $g : X \to [3, 7]$? In fact, one can obtain the affirmative solution in the spirit of representation $h = h_+ - h_-$ (this is frequently used in the theory of summable functions [2, Section 25]).

However, what can one say about mappings not to the real line $\mathbb{R}$, but to more general range spaces? Or for example, is the analog of such a splitting into a sum of two maps valid for continuously differentiable functions? Also, one can consider single-valued mappings $f : X \to Y$ as a very special case of selections, namely, as selections of the constant mapping $F(\cdot) \equiv Y$. So passing to more complicated multivalued mappings we must deal with more general point of view on the splitting problem.

For another kind of such examples, let $A$ and $B$ be two convex subsets of a Banach space $Y$ and let $C = A + B$ be their pointwise sum (the Minkowski sum). So each $c \in C$ is a sum $c = a + b$ of two elements $a \in A$ and $b \in B$. Is it true that such a pair $(a; b)$ of items can be chosen in a continuous fashion with respect to $c \in C$?

Both examples above are very special cases of the general splitting problem in selection theory. To formulate it we introduce the following notation. For multivalued mappings $F_1 : X \to Y_1$, $F_2 : X \to Y_2$ and for a single-valued mapping $L : Y_1 \times Y_2 \to Y$ we denote by $L(F_1; F_2)$ the composite mapping, which associates to each $x \in X$ the set

$$\bigl\{y \in Y:\ y = L(y_1; y_2),\ y_1 \in F_1(x),\ y_2 \in F_2(x)\bigr\}.$$

**Splitting problem.** *Let $f$ be a continuous single-valued selection of $L(F_1; F_2)$. Is it true that $f = L(f_1; f_2)$ for some continuous single-valued selections $f_1$ and $f_2$ of $F_1$ and $F_2$, respectively*?

For $Y_1 = Y_2 = Y$ and $L(y_1; y_2) = y_1 + y_2$ we see the specific problem of splitting into sum of two items. More generally, for constant multivalued mappings, Splitting problem can be interpreted as the problem of continuous dependence of solutions of the linear equation $y = L(y_1; y_2)$ on the data $y$ and with constraints for $y_1$ and $y_2$.

Within the framework of the general theory of continuous selections it is quite natural to restrict ourselves to the case of paracompact domains $X$, Banach range spaces $Y_1$, $Y_2$, $Y$ and lower semicontinuous convex-valued, closed-valued mappings. In particular, instead of the Cartesian product $Y_1 \times Y_2$ we use below the direct sum $Y_1 \oplus Y_2$ of Banach spaces endowed with the *max*-norm: $\|(y_1; y_2)\| = \max\{\|y_1\|_1; \|y_2\|_2\}$.

Observe that the analogous splitting problem can be naturally stated for other types of selections: uniformly continuous, measurable, differentiable, Lipschitz selections, etc. See [1] for basic facts concerning such types of selections and [3,7,11] for results on Lipschitz selections. For example, let $F_1$ and $F_2$ be Lipschitz convex-valued and compact-valued mappings, let $L$ be linear and let $f$ be a Lipschitz single-valued selection of $L(F_1; F_2)$. Is it true that $f = L(f_1; f_2)$ for some Lipschitz single-valued selections $f_1$ and $f_2$ of $F_1$ and $F_2$, respectively?

Splitting problem has two "stability" versions: approximate- and continuity-type.

**Approximate Splitting problem.** *Let $f$ be a continuous single-valued selection of $L(F_1; F_2)$. Is it true that for every $\varepsilon > 0$ there exist continuous single-valued $\varepsilon$-selections $f_1$ and $f_2$ of $F_1$ and $F_2$ such that $f = L(f_1; f_2)$*?

**Continuity Splitting problem.** *Suppose that the Splitting problem admits an affirmative solution. Is it true, that the resulting pair $(f_1; f_2)$ continuously depends on the data, i.e. on the selection $f$ of $L(F_1; F_2)$*?

In Section 2 we present the affirmative solution of the Approximative Splitting problem for convex-valued mappings. Section 3 deals with the case $\dim Y_1 = \dim Y_2 = \dim Y = 1$ and other examples where exact splitting is possible. A collection of counterexamples is presented in Section 4.

To conclude the introduction we recall that single-valued mapping $f : X \to Y$ is said to be a selection of a multivalued mapping $F : X \to Y$ provided that $f(x) \in F(x)$, $x \in X$ and that lower semicontinuity of a multivalued mapping $F : X \to Y$ between topological spaces $X$ and $Y$ means that for each pair of points $x \in X$ and $y \in F(x)$, and each open neighborhood $U(y)$, there exists an open neighborhood $V(x)$ such that $F(x') \cap U(y) \neq \emptyset$, whenever $x' \in V(x)$.

Applying the Axiom of Choice to the family of nonempty intersections $F(x') \cap U(y)$, $x' \in V(x)$, we see that LSC mappings are exactly those, which admit local (noncontinuous) selections. In other words, the notion of lower semicontinuity is by definition very close to the notion of selection. For a metric range space $Y$ a single-valued mapping $f : X \to Y$ is said to be an $\varepsilon$-selection of a multivalued mapping $F : X \to Y$ provided that for each $x \in X$ the point $f(x)$ is $\varepsilon$-close to the set $F(x)$.

## 2. Approximative splittings

**Theorem 2.1.** *Let $F_1 : X \to Y_1$ and $F_2 : X \to Y_2$ be LSC convex-valued mappings from a paracompact domain $X$ into Banach spaces $Y_1$ and $Y_2$. Let $L : Y_1 \oplus Y_2 \to Y$ be a continuous linear surjection and let $f : X \to Y$ be a continuous single-valued selection of the mapping $L(F_1; F_2) : X \to Y$. Then for each $\varepsilon_0 > 0$ there exist two continuous single-valued $\varepsilon_0$-selections $f_1$ and $f_2$ of $F_1$ and $F_2$, respectively, such that*

$$f(x) = L\big(f_1(x); f_2(x)\big), \quad x \in X.$$

**Proof.** Pick an arbitrary $0 < \varepsilon < \varepsilon_0$ and define a new multivalued mapping $\Phi_\varepsilon : X \to Y_1 \oplus Y_2$ by setting

$$\Phi_\varepsilon(x) = \big\{(y_1; y_2)\colon \operatorname{dist}\big(y_1; F_1(x)\big) < \varepsilon,\ \operatorname{dist}\big(y_2; F_2(x)\big) < \varepsilon,\ L(y_1; y_2) = f(x)\big\},$$
$$x \in X.$$

By applying the convex-valued Michael selection theorem (see [4,5,9]) we shall show that the pointwise closure $\operatorname{Clos}\Phi_\varepsilon$ admits a selection, say $(f_1; f_2) : X \to Y_1 \oplus Y_2$. Then $\operatorname{dist}(f_i(x); F_i(x)) \leqslant \varepsilon < \varepsilon_0$, $i = 1, 2$, and $L(f_1(x); f_2(x)) = f(x)$, $x \in X$. So, $f_1$ and $f_2$ are the desired continuous single-valued mappings.

The nonemptiness of the set $\Phi_\varepsilon(x)$ follows merely from the assumption that $f$ is a selection of the mapping $L(F_1; F_2)$. The convexity of $\Phi_\varepsilon(x)$ is a direct corollary of convexity of the sets $F_1(x)$, $F_2(x)$ and linearity of the mapping $L : Y_1 \oplus Y_2 \to Y$. In fact, the set $\Phi_\varepsilon(x)$ is the intersection of two convex subsets of $Y_1 \oplus Y_2$:

$$\Phi_\varepsilon(x) = \big(\big(F_1(x) + \varepsilon D_1\big) \oplus \big(F_2(x) + \varepsilon D_2\big)\big) \cap L^{-1}\big(f(x)\big),$$

where $D_i$ is the open unit ball in the space $Y_i$. So, if we prove that the mapping $\Phi_\varepsilon : X \to Y_1 \oplus Y_2$ is lower semicontinuous, then for the mapping $\operatorname{Clos}\Phi_\varepsilon : X \to Y_1 \oplus Y_2$ all assumptions of the convex-valued Michael selection theorem will be satisfied, because the pointwise closure operation preserves nonemptiness, convexity and lower semicontinuity.

We now complete the proof in a straightforward way. So for fixed $x \in X$ and $y_1 \in Y_1$, $y_2 \in Y_2$ with $L(y_1; y_2) = f(x)$ and $\operatorname{dist}(y_1; F_1(x)) < \varepsilon$, $\operatorname{dist}(y_2; F_2(x)) < \varepsilon$ pick points $z_i \in F_i(x)$ such that $\|z_i - y_i\| < \varepsilon$, $i = 1, 2$. In other words, $y_i \in z_i + \varepsilon D_i$, $i = 1, 2$. Clearly, the last inclusions are stable with respect to "small" movements of the points $y_i$ and $z_i$. More precisely, there exists $\delta > 0$ with the property that

$$\operatorname{dist}(z_i'; z_i) < \delta \quad \Rightarrow \quad y_i + \delta D_i \subset z_i' + \varepsilon D_i, \quad i = 1, 2.$$

The surjection $L : Y_1 \oplus Y_2 \to Y$ is an open map, due to the Banach open mapping principle. Therefore the set $V = L(y_1 + \delta D_1; y_2 + \delta D_2)$ is an open neighborhood of the point $f(x) \in Y$. Lower semicontinuity of mappings $F_1$, $F_2$ and continuity of the mapping $f$ imply that the set

$$U = F_1^{-1}(z_1 + \delta D_1) \cap F_2^{-1}(z_2 + \delta D_2) \cap f^{-1}(V)$$

is an open neighborhood of the point $x \in X$.

So for each $x' \in U$ there are $z_1' \in F_1(x') \cap (z_1 + \delta D_1)$, $z_2' \in F_2(x') \cap (z_2 + \delta D_2)$ and there are $y_1' \in y_1 + \delta D_1$, $y_2' \in y_2 + \delta D_2$ such that $L(y_1'; y_2') = f(x')$. By construction we have

$$y_i' \in y_i + \delta D_i \subset z_i + \varepsilon D_i \subset F_i(x') + \varepsilon D_i, \quad i = 1, 2.$$

Hence the point $(y_1'; y_2')$ lies in the set $\Phi(x')$ and is $\delta$-close to the chosen point $(y_1; y_2) \in \Phi_\varepsilon(x)$. This is why the mapping $\Phi_\varepsilon$ is lower semicontinuous at the point $x$. □

**Remarks.** (1) The exact answer concerning topological problem for metric spaces can usually be obtained as a result of a convergent sequence of some approximate answers. Unfortunately, selections of the mapping $\Phi_\varepsilon$ from the proof of Theorem 2.1 are, in general, not $\varepsilon$-selections of the mapping $\Phi(x) = (F_1(x) \oplus F_2(x)) \cap L^{-1}(f(x))$. This is the main obstacle to proving theorems on exact splittings.

(2) Observe that one of the items $f_1$ or $f_2$ in Theorem 2.1 can be chosen to be a *genuine* selection of $F_1$ or $F_2$, provided that the equation $L(y_1; y_2) = y$ admits a resolution $y_1 = L_1(y_2; y)$ or $y_2 = L_2(y_1; y)$ for some continuous linear operators $L_1$ or $L_2$.

Indeed, let $f = L(f_1; f_2)$ with the accordance with Theorem 2.1. Assume that

$$y = L(y_1; y_2) \quad \Leftrightarrow \quad y_2 = L_2(y_1; y)$$

for some linear continuous operator $L_2 : Y_1 \oplus Y \to Y_2$. In particular, this means that $f_2(x) = L_2(f_1(x); f(x))$, $x \in X$. For the $\varepsilon$-selection $f_1$ of the mapping $F_1$ there exists a selection, say $g_1$, of $F_1$ such that $\|g_1(x) - f_1(x)\| < \varepsilon$, $x \in X$ (see [9]). Set $g_2 = L_2(g_1; f)$. Then $f = L(g_1; g_2)$ and

$$\big\|g_2(x) - f_2(x)\big\| = \big\|L_2(g_1; f) - L_2(f_1; f)\big\| < \|L_2\| \cdot \varepsilon, \quad x \in X.$$

Therefore $\operatorname{dist}(g_2(x); F_2(x)) \leqslant \|g_2(x) - f_2(x)\| + \operatorname{dist}(f_2(x); F_2(x)) < (1 + \|L_2\|) \cdot \varepsilon$. In other words, $g_2$ is $(1 + \|L_2\|)\varepsilon$-selection of $F_2$. The arbitrariness of $\varepsilon > 0$ completes the proof.

We are grateful to Umberto Marconi who turned our attention to the possibility of such strengthening of Theorem 2.1 in the case when $Y_1 = Y_2 = Y$ and $L(y_1; y_2) = y_1 + y_2$.

## 3. Exact splittings

We study the case $\dim Y_1 = \dim Y_2 = \dim Y = 1$. The key advantage of one-dimensional range spaces is that in such the case the kernel $\operatorname{Ker} L$ of linear surjection $L : Y_1 \oplus Y_2 \to Y$ separates the plane $Y_1 \oplus Y_2$.

**Theorem 3.1.** *Let $F_1 : X \to \mathbb{R}$ and $F_2 : X \to \mathbb{R}$ be two LSC convex-valued, closed-valued mappings from a paracompact domain $X$. Let $L : \mathbb{R}^2 \to \mathbb{R}$ be a continuous linear surjection and let $f : X \to \mathbb{R}$ be a continuous single-valued selection of the mapping $L(F_1; F_2) : X \to \mathbb{R}$. Then there are two continuous single-valued selections $f_1$ and $f_2$ of $F_1$ and $F_2$, respectively, such that*

$$f(x) = L\big(f_1(x); f_2(x)\big), \quad x \in X.$$

**Proof.** As in the proof of the previous theorem we shall use an auxiliary multivalued mapping $\Phi : X \to \mathbb{R}^2$, by setting

$$\begin{aligned}\Phi(x) &= \big\{(y; z)\colon\ y \in F_1(x),\ z \in F_2(x),\ L(y; z) = f(x)\big\}\\ &= \big(F_1(x) \oplus F_2(x)\big) \cap L^{-1}\big(f(x)\big).\end{aligned}$$

Clearly, all values of $\Phi : X \to \mathbb{R}^2$ are nonempty, convex, closed subsets of the plane.

Unfortunately, no general theorems on intersections of multivalued mappings can be directly applied. Simple examples show that in general, intersection of two (even Lipschitz) continuous, convex-valued, compact-valued mappings can fail to be lower semicontinuous (see Fig. 1).

For our purpose we principally use the "rectangular" structure of the sets $(F_1(x) \oplus F_2(x))$. We put

$$F(x) = \big(F_1(x) \oplus F_2(x)\big), \quad F : X \to \mathbb{R}^2.$$

Clearly, $F$ is a lower semicontinuous mapping. So, if $A \in F(x)$ and $B \in F(x)$ then the whole closed rectangle $\Pi(A; B)$ with diagonal $[A; B]$ and sides parallel to coordinate axises is a subset of the convex set $F(x)$.

It will be convenient to consider the plane $\mathbb{R}^2$ endowed with the *max*-norm: $\|(y; z)\| = \max\{|y|; |z|\}$. We shall denote the $\varepsilon$-neighborhood of a point $(y; z)$ by $D_\varepsilon(y; z)$. If $L : \mathbb{R}^2 \to \mathbb{R}$ is the projection $p_1$ onto the first factor, then $L(F_1; F_2) = F_1$ and it suffices to put $f_1 = f$ and take $f_2$ as an arbitrary selection of $F_2$. Analogously, for $L = p_2$. So, we can assume that $\operatorname{Ker} L$ is a skew line. For definiteness, let $\operatorname{Ker} L$ be the line $l$ which is given by the equation $z = ky$, $k \geqslant 1$. The cases $0 < k < 1$ and $k < 0$ can be verified analogously.

We temporarily say that parallel lines $l'$ and $l''$ are $\delta$-close provided that for some $\delta' < \delta$ the line $l'$ is parallel $\delta'$-shift of $l''$ with respect to the first coordinate axis.

**Lemma 3.2.** *For every $\varepsilon > 0$ there exists $\delta > 0$ such that if $l'$ is $\delta$-close to $l = \operatorname{Ker} L$, $B$ is a point on $l'$ and $A \in D_\delta(0; 0)$, then the intersection $\Pi(A; B) \cap D_\varepsilon(0; 0) \cap l'$ is nonempty.*

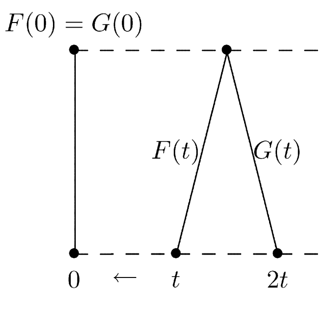


Fig. 1.

To complete the proof of the theorem pick any $x \in X$, $(y; z) \in F(x)$ and $\varepsilon > 0$. Find $\delta > 0$ as in Lemma 3.2 and apply Lemma 3.2 for the $\varepsilon$-neighborhood of the point $(y; z)$. Due to the continuity assumptions, the intersection

$$U = F^{-1}\big(D_\delta(y; z)\big) \cap f^{-1}\big(f(x) - \delta; f(x) + \delta\big)$$

is a nonempty open neighborhood of the point $x \in X$. Hence for each $x' \in X$, the smaller square $S = D_\delta(y; z)$ contains a point, say $A$, from the set $F(x')$ and, simultaneously, the line $l' = L^{-1}(f(x'))$ is $\delta$-close to the line $l = L^{-1}(f(x))$. The nonemptiness of the value $\Phi(x')$ implies that there exists a point, say $B$, in $l' \cap F(x')$. Due to the convexity of $F(x')$ and by Lemma 3.2 it follows that $\Pi(A; B) \subset F(x')$ and that the intersection $D_\varepsilon(y; z) \cap \Phi(x')$ contains the nonempty set $\Pi(A; B) \cap D_\varepsilon(y; z) \cap l'$. This means precisely that the mapping $\Phi$ is lower semicontinuous at $x$. $\square$

**Proof of Lemma 3.2.** If $B \in Q = D_\varepsilon(0; 0)$ then $B$ belongs to the intersection $\Pi(A; B) \cap D_\varepsilon(0; 0) \cap l'$. So let us consider the opposite case when $B(b_1; b_2) \notin Q$. For definiteness, let $b_2 \geqslant \varepsilon$, i.e. suppose that $B$ is displaced above the square $Q$ (see Fig. 2).

The ray $z = ky$, $y \geqslant 0$, intersects the boundary of $Q$ at the point $(\frac{\varepsilon}{k}; \varepsilon)$. Choose $\delta_1 > 0$ such that

$$\frac{\varepsilon}{k} + \delta_1 < \varepsilon \quad \text{and} \quad \frac{\varepsilon}{k} - \delta_1 > 0. \tag{1}$$

Draw the lines $l_-$ and $l_+$ through the points $B_-(\frac{\varepsilon}{k} - \delta_1; \varepsilon)$ and $B_+(\frac{\varepsilon}{k} + \delta_1; \varepsilon)$, respectively, parallel to $l = \operatorname{Ker} L$. Then the line $l'$ and the point $B$ are between lines $l_-$ and $l_+$.

Pick any $0 < \delta_2 < \frac{\varepsilon}{k} - \delta_1$ and consider the right-upper corner point $A_+(\delta_2; \delta_2)$ of the smaller square $S = D_\delta(0; 0)$. Then the rectangle $\Pi(A_+; B_-)$ is minimal in the sense that it is contained in any rectangle $\Pi(A; B)$, $A \in S$. Hence, it suffices to verify the nonemptiness of the intersection $\Pi(A_+; B_-) \cap D_\varepsilon(0; 0) \cap l'$. To achieve this we must assure that:

$$\text{The point } C\left(\frac{\varepsilon}{k} - \delta_1; \delta_2\right) \text{ lies under the line } l_+; \quad \text{and} \tag{2}$$

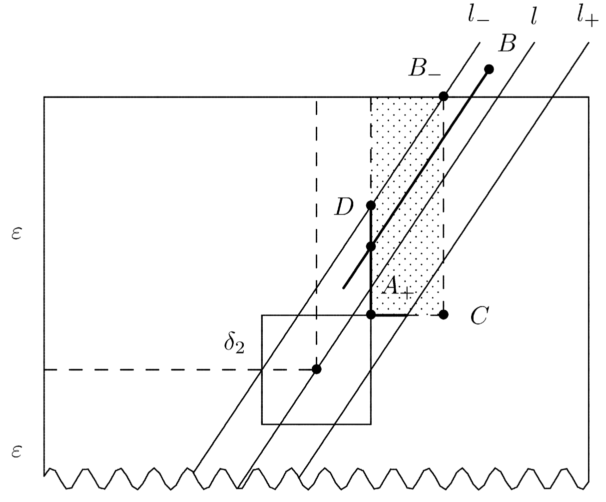


Fig. 2.

$$\text{The point } D = p_1^{-1}(\delta_2) \cap l_- \text{ lies in the square } Q. \tag{3}$$

A direct calculation shows that (2) holds if

$$k\left(\frac{\varepsilon}{k} - \delta_1\right) - k\delta_1 > \delta_2 \quad \text{or} \quad \delta_2 + 2k\delta_1 < \varepsilon, \tag{2$'$}$$

while (3) holds whenever

$$k\delta_2 + k\delta_1 < \varepsilon \quad \text{or} \quad \delta_2 < \frac{\varepsilon}{k} - \delta_1. \tag{3$'$}$$

Note that (3′) coincides with the choice of $\delta_2$ above.

Therefore, we need to find $\delta_1$ and $\delta_2$ such that

$$\delta_1 < \varepsilon\left(1 - \frac{1}{k}\right), \qquad \delta_1 < \frac{\varepsilon}{k}, \qquad \delta_2 < \frac{\varepsilon}{k} - \delta_1, \qquad \delta_2 + 2k\delta_1 < \varepsilon.$$

It easy to check that

$$\delta = \delta_1 = \delta_2 = \frac{1}{4}\min\left\{\frac{\varepsilon}{k}; \frac{\varepsilon(k-1)}{k}\right\}$$

is an appropriate choice. Thus lemma is proved. □

Note, that even in the case $\dim Y_1 = 2$, $\dim Y_2 = \dim Y = 1$ the mapping $\Phi(\cdot)$ from the proof of Theorem 3.1 can, in general, fail to be LSC and admit any exact selections (see Section 4).

**Remark.** The conclusion of Theorem 3.1 is stable with respect to the initial data $f$. Namely, to each selection $f$ of $L(F_1; F_2)$ one can associate the "second order" multivalued mapping $\Phi' : \mathrm{Sel}_{L(F_1;F_2)}(X; \mathbb{R}) \to \mathrm{Sel}_1(X; \mathbb{R}) \oplus \mathrm{Sel}_2(X; \mathbb{R})$ which associates to each selection $f$ of the mapping $L(F_1; F_2)$ the (nonempty!) set of all pairs $(f_1; f_2)$ of selections with the property that $f = L(f_1; f_2)$.

A selection of a multivalued mapping $\Phi'$ gives the continuous choice of the pair $(f_1; f_2)$ with respect to $f$. However, the classical selection theorems work in this situation either if domain $X$ is a compact space, or if $X$ is locally compact space. Then the space $C(X; \mathbb{R})$ of all real-valued continuous mappings is either a Banach space, or a Fréchet space. For arbitrary paracompact domains the above stability property holds for the space of all *bounded* real-valued continuous mappings.

## 4. Examples

We begin by showing that the hypotheses of Theorem 3.1 cannot be simply omitted.

**Example 4.1.** There exist LSC convex-valued, closed-valued mappings $F_1 : [0; \infty) \to \mathbb{R}$ and $F_2 : [0; \infty) \to \mathbb{R}^2$, a linear surjection $L : \mathbb{R}^3 \to \mathbb{R}$ and a selection $f$ of $L(F_1; F_2)$ such that the mapping $\Phi(x) = (F_1(x) \oplus F_2(x)) \cap L^{-1}(f(x))$ is not LSC.

**Proof.** We realize $\mathbb{R}$ as the subset $\{(x; 0; 0)\colon x \in \mathbb{R}\}$ of $\mathbb{R}^3$ and $\mathbb{R}^2$ as $\{(0; y; z)\colon y \in \mathbb{R},\ z \in \mathbb{R}\}$. For each $t \geqslant 0$, let $F_1(t) = \{(x; 0; 0)\colon x \geqslant 0\}$. For $t > 0$ let $F_2(t) = \{(0; y; t(y+1))\colon y \geqslant -1\}$ and let $F_2(0) = \{(0; y; 0)\colon y \geqslant -1\}$.

For the linear surjection $L(x; y; z) = x + z$ it easy to see that $L(F_1; F_2)(\cdot) \equiv \{(x; 0; 0)\colon\ x \geqslant 0\}$ and for its selection $f(t) \equiv (0; 0; 0)$ the mapping $\Phi$ looks as follows: $\Phi(0) = \{(0; y; 0)\colon\ y \geqslant -1\}$ and $\Phi(t) = \{(0; -1; 0)\}$ provided that $t > 0$. So $\Phi$ is not lower semicontinuous. □

Note that in spite of this, $\Phi$ admits a selection. However, a slight modification of Example 4.1 produces a situation when the mapping $\Phi$ admits no selections and Splitting problem has a negative solution.

**Example 4.2.** There exist two continuous compact-valued, convex-valued mappings $F_1 : [-1; 1] \to \mathbb{R}$ and $F_2 : [-1; 1] \to \mathbb{R}^2$, the linear surjection $L : \mathbb{R}^3 \to \mathbb{R}$ and the selection $f$ of $L(F_1; F_2)$ such that $f \neq L(f_1; f_2)$ for arbitrary selections $f_1$ and $f_2$ of $F_1$ and $F_2$, respectively.

**Proof.** We realize $\mathbb{R}$ and $\mathbb{R}^2$ as in Example 4.1. We also preserve the mapping $F_1(\cdot) \equiv \{(x; 0; 0)\colon\ 0 \leqslant x \leqslant 1\}$ and the linear surjection $L(x; y; z) = x + z$. Let $F_2(0) = [(0, -1, 0); (0, 1, 0)]$, $F_2(t) = [(0, -1, t); (0, 1, 0)]$ for $0 < t \leqslant 1$ and $F_2(t) = [(0, -1, 0); (0, 1, -t)]$ for $-1 \leqslant t < 0$ (see Fig. 3).

Then the composite mapping $L(F_1; F_2)$ associates to each $t \in [-1; 1]$ the segment $[0; 1 + |t|]$. This is why $f(t) \equiv 0$ is a selection of $L(F_1; F_2)$.

Suppose to the contrary, that the representation $f = L(f_1; f_2)$ holds for some selections $f_1$ and $f_2$ of $F_1$ and $F_2$, respectively. Then $0 \equiv x(t) + z(t)$ for $0 \leqslant x(t) \leqslant 1$ and $0 \leqslant z(t) \leqslant 1 + |t|$ and hence $x(t) \equiv 0 \equiv z(t)$. However, this means that for all $t > 0$ the selection $f_2$ always chooses the point $(0; 1; 0)$, while for all $t < 0$ selection $f_2$ always chooses the point $(0; -1; 0)$. Clearly, this contradicts the continuity of $f_2$. □

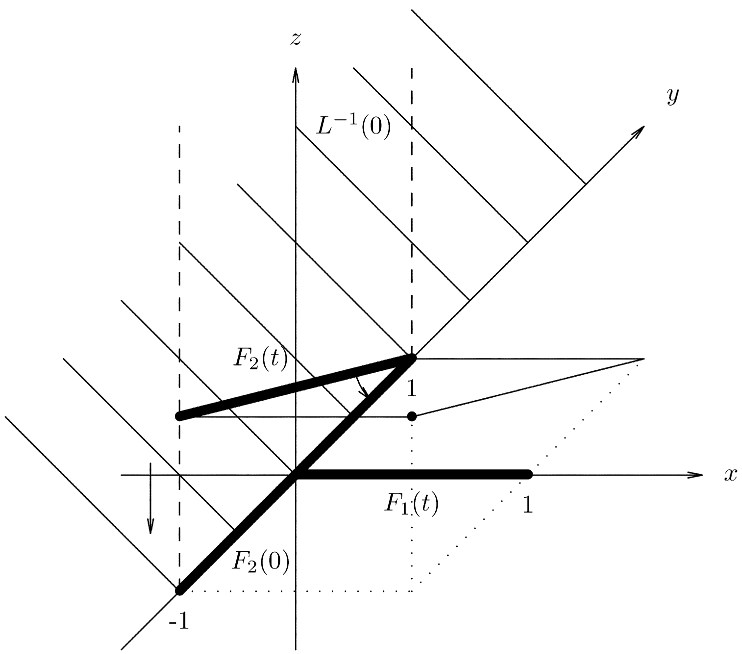


Fig. 3.

**Example 4.3.** One can assume in Example 4.2 that $F_1(\cdot)$ has a single value and that $F_2(\cdot)$ has countably many values.

**Proof.** Restrict $F_2$ in Example 4.2 to the countable subset $\{\pm\frac{1}{n}\} \cup \{0\}$ of the domain. $\square$

**Example 4.4.** In general, the Splitting problem for uniformly continuous and hence, for Lipschitz selections admits no solution.

**Proof.** The key reason is that uniformly continuous selections for $L(F_1; F_2)$ may exist, while there are Lipschitz mappings $F_2$ without any such selections. More precisely, let $X$ be the exponent $\exp_c Q$ of all compact, convex subsets of the Hilbert cube $Q$ endowed with the Hausdorff distance. One can assume that $X$ is linearly and isometrically embedded into a Fréchet space $Y_1$.

Let the mapping $F_1 : X \to Y_1$ be such an embedding. Let $Y_2$ be the Hilbert space $H$ and $F_2 : X \to Y_2$ the evaluation mapping $e$, i.e. $e(X) = X$, where $X$ on the left stands for the element of $\exp_c Q$ and $X$ on the right is a subcompactum of $Q \subset H$. Finally, for $L = p_1$ the composition $L(F_1; F_2) = F_1$ has the obvious selection $f = \mathrm{id}|_X$, while $f \neq L(f_1; f_2)$ because there are no Lipschitz [3] (there are even no uniformly continuous selections (see [7])) for the evaluation mapping $e : \exp_c Q \to Q$. $\square$

**Example 4.5.** Linearity of integrals for single-valued mappings implies linearity of integrals for multivalued mappings.

**Proof.** Recall that the (Aumann) integral for a multivalued mapping $G$ is defined as the set of integrals of all of its measurable single-valued selections (see [1]):

$$\int_X G\,d\mu = \left\{ \int_X g\,d\mu\colon\ g\text{-selection of } G \right\}.$$

So with the notations of Theorem 3.1 the inclusion

$$\int_X L(F_1, F_2)\,d\mu \supset L\left( \int_X F_1\,d\mu; \int_X F_2\,d\mu \right)$$

follows directly from linearity of integrals for single-valued mappings. To verify the inverse inclusion it suffices to represent each measurable selection $f$ of $L(F_1; F_2)$ as $f = L(f_1; f_2)$ for some measurable selections $f_i$ of $F_i$. In other words, we must resolve the Splitting problem for measurable maps. However, in this case the auxiliary multivalued mapping $\Phi : X \to Y$, where

$$\begin{aligned} \Phi(x) &= \big\{ (y; z)\colon\ y \in F_1(x), z \in F_2(x),\ L(y; z) = f(x) \big\} \\ &= \big( F_1(x) \oplus F_2(x) \big) \cap L^{-1}\big( f(x) \big), \end{aligned}$$

is clearly also measurable and it admits a measurable selection (due to the Kuratowski–Ryll–Nardzewski theorem). $\square$

**Question 4.6.** *Do there exist, for every convex sets $A$ and $B$, continuous single-valued mappings $s_1 : A + B \to A$ and $s_2 : A + B \to B$, defined on the Minkowski sum $A + B$, such that*

$$c = s_1(c) + s_2(c), \quad c \in A + B?$$

Observe that at present we have an affirmative answer only for those finite-dimensional sets $A$ and $B$ whose boundaries contain no parallel (non-degenerate) segments.

**Question 4.7.** *Is Theorem* 3.1 *true for exact splittings in the special case when* $Y_1 = Y_2 = Y$ *and* $L(y_1; y_2) = y_1 + y_2$?

The answer is again affirmative whenever $\dim Y < \infty$, $L : Y^2 \to Y$ is an arbitrary linear surjection and the boundary of the direct sum $A \oplus B$ contains no (non-degenerate) segments parallel to the kernel $\operatorname{Ker} L$ of $L$.

**Question 4.8.** *Do there exist analogues of Theorems* 2.1 *and* 3.1 *for nonconvex-valued mappings with controlled removal of the condition of convexity of their values, for example, in the spirit of paraconvexity* [6,8,10]?

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