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Topology of manifolds modeled on countable direct limits of Menger compacta

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Abstract

We construct n-dimensional counterparts of manifolds modeled on the space ℓ^2 equipped by the bounded weak topology (μ_n^∞-manifolds). For μ_n^∞-manifolds we prove the characterization, triangulation and classification theorems. In addition, a universal map of μ_n^∞ onto Q^∞ (the countable direct limit of Hilbert cubes and Z-embeddings) is constructed and characterized.

MSC: 54E45; 57N20; 58B05

Keywords: Menger manifold; Universal Dranishnikov map; Countable direct limit

1. Introduction

Theory of manifolds modeled on universal n-dimensional Menger compacta μ_n (Menger manifolds; μ_n-manifolds), whose background was created by Bestvina [4], has been widely developed in the papers of Dranishnikov [9], Chigogidze [6,7], Sakai [16], Ageev and Repovš [1] and others. As the results demonstrate, the Menger manifolds are closer to the Q-manifolds (i.e. the manifolds modeled on the Hilbert cube Q; see [5]) than to the finite-dimensional Euclidean manifolds.

In this paper we consider manifolds modeled on the countable direct limits μ_n^∞ of Menger compacta. These manifolds can be considered as n-dimensional counterparts of the manifolds modeled on the countable direct limits Q^∞ of sequences of Hilbert cubes (a series of papers [14,15,19] is devoted to the latter). Note that the model space Q^∞ naturally appears in functional analysis as a separable Hilbert space ℓ^2 endowed with the bounded weak (bw) topology: a set in (ℓ^2, bw) is closed if and only if its intersection with every closed ball is closed in the weak topology [11]. Therefore, the space μ_n^∞ can serve as an n-dimensional counterpart of the space (ℓ^2, bw).

The theory of μ_n^∞-manifolds can be pursued slightly further than that of μ_n-manifolds. To the universal Dranishnikov map, which plays an important role in formulations (as well as proofs) of the stability theorem and triangulation theorem, there corresponds, in the case of μ_n^∞-manifolds, a map $\varphi_n : \mu_n^\infty \to Q^\infty$, which can be uniquely, up to home-

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omorphisms, characterized by means of its fundamental properties. Note that there is no characterization theorem for the universal Dranishnikov map $F_n : \mu_n \to Q$ (see [2] for the properties of F_n).

The paper is organized as follows. Section 3 is devoted to the characterization theorem. In Section 4 we construct the universal map from μ_n^∞ onto Q^∞ and in Section 5 we use the universal map to formulate and prove the triangulation and stability theorem.

2. Preliminaries

2.1. *n-invertible and n-soft maps*

The notions of n-invertible and n-soft maps were introduced by Shchepin [17]. A map $f : X \to Y$ is said to be *n-invertible* provided that for every map $g : Z \to Y$, where Z is a paracompact space with $\dim Z \leqslant n$ there exists a map $h : Z \to X$ such that $fh = g$.

A map $f : X \to Y$ is said to be *(m,n)-soft* provided that for every commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & X \\ \big\downarrow & & \big\downarrow f \\ Z & \xrightarrow[\psi]{} & Y \end{array}$$

such that Z is a paracompact space with $\dim Z \leqslant n$ and A is a closed subset of Z with $\dim A \leqslant m$ there exists a map $\Phi : Z \to X$ such that $f\Phi = \psi$ and $\Phi|A = \varphi$. The (n,n)-soft maps are called *n-soft*.

Note that the $(-1,n)$-soft maps are precisely n-invertible maps.

If in the definition of n-soft map we require that Z is a polyhedron and A its subpolyhedron, then f is said to be a *polyhedrally n-soft* map.

We say that two maps $f_1, f_2 : X \to Y$ are *n-homotopic* (written $f_1 \simeq_n f_2$) if for any paracompact space Z with $\dim Z \leqslant n$ and any map $g : Z \to X$ the maps $f_1 g$ and $f_2 g$ are homotopic (see e.g. [9]).

Lemma 2.1. *Let $f, g : A \to X$ be n-homotopic maps of a metrizable compactum A into a topological space X. Then there exists a metrizable compactum $C \subset X$ such that $C \supset f(A) \cup g(A)$ and the maps $f, g : A \to C$ are n-homotopic.*

Proof. There exists an n-dimensional compactum B and an n-invertible map $h : B \to A$ (see [8, Theorem 1.2]). Then the maps fh and gh are homotopic; denote by $H : B \times I \to X$ the homotopy which connects them and let $C = H(B \times I)$.

If B' is a paracompact space with $\dim B' \leqslant n$ and a map $h' : B' \to A$ is given, then there exists a map $\alpha : B' \to B$ such that $h\alpha = h'$. Then $H(\alpha \times \mathrm{id}_I)$ is a homotopy of the maps fh' and gh'. Thus, the maps $f, g : A \to C$ are n-homotopic.

Lemma 2.2. *Suppose that a map $f : X \to Y$ of metrizable compacta induces isomorphisms of the homotopy groups of dimension $\leqslant n-1$, $Y \in \mathrm{LC}^{n-1}$, (P, L) is a polyhedral pair, $\dim P \leqslant n-1$ and $\alpha : P \to Y$, $\beta : L \to X$ are maps such that $f\beta = \alpha|L$. Then there exists a map $\hat{\beta} : P \to X$ such that $\hat{\beta}|L = \beta$ and $f\hat{\beta} \simeq_{n-1} \alpha$.*

Proof. This is essentially Lemma 2.8.7 from [4]. Here we only use the notion of $(n-1)$-homotopy instead of μ-homotopy in [4].

2.2. *μ_n-manifolds*

Recall the construction of the standard universal n-dimensional Menger compactum μ_n (see e.g. [10]). Let $\mathcal{F}_i$, $i = 0, 1, 2, \ldots$, be the family of 3^{mi} congruent cubes obtained by means of partition of the unit m-dimensional cube I^m, $m \geqslant n$, by $(m-1)$-dimensional affine subspaces in $\mathbb{R}^m$ given by the equations $x_j = k/3^i$, $j = 1, 2, \ldots, m$ and $0 \leqslant k \leqslant 3^i$. For a collection $\mathcal{K}$ of cubes, denote by $S_n(\mathcal{K})$ the union of all faces of dimension $\leqslant n$ of the cubes in $\mathcal{K}$. Taking $\mathcal{F}_0 = \{I^m\}$ and $F_0 = \bigcup \mathcal{F}_0$ and assuming that $\mathcal{F}_i$ and F_i are already defined for all $i < k$, set

$$\mathcal{F}_k = \Big\{K \in \mathcal{F}_{k-1} \mid K \cap \Big(\bigcup S_n(\mathcal{F}_{k-1})\Big) \neq \emptyset\Big\}, \qquad F_k = \bigcup \mathcal{F}_k.$$

Finally, let $\mu_n^m = \bigcap_{i=0}^{\infty} F_i \subset I^m$.

For $m \geqslant 2n+1$ and n fixed, all spaces μ_n^m are homeomorphic [4]. Let $\mu_n = \mu_n^{2n+1}$.

A paracompact space X is said to be a *μ_n-manifold* if there exists a base of the topology of X consisting of sets homeomorphic to open subsets of μ_n. We assume that the μ_n-manifolds under consideration are separable.

Recall that a closed embedding $f : X \to Y$ is said to be a *Z-embedding* if the image $f(X)$ is a Z-set in Y; the latter means that the identity map 1_Y can be approximated by the maps whose image misses $f(X)$ (see e.g. [3]).

Theorem 2.3 *(Z-embedding extension theorem [4]). Let (A, B) be a compact metrizable pair,* $\dim A \leqslant n$. *For every Z-embedding $f : B \to \mu_n$ there exists an extension to a Z-embedding $\bar{f} : A \to \mu_n$.*

2.3. By $\mathcal{MC}$ (respectively $\mathcal{MC}(n)$) we will denote the class of metrizable compacta (respectively the class of metrizable compacta of dimension $\leqslant n$). Given a class $\mathcal{C}$ of topological spaces, we denote by $\mathcal{C}^\infty$ the class of spaces which can be represented as countable direct limits of sequences of spaces $X_1 \hookrightarrow X_2 \hookrightarrow \cdots$, where $X_i \in \mathcal{C}$ and X_i is a closed subset of X_{i+1}, for every i.

By Q we will denote the Hilbert cube, $Q = \prod_{i=1}^{\infty}[-1, 1]_i$. Let Q^∞ denote the direct limit of the sequence

$$Q \to Q \times \{0\} \hookrightarrow Q \times Q \to Q \times Q \times \{0\} \hookrightarrow Q \times Q \times Q \to \cdots.$$

By $\mathbb{R}^\infty$ we denote the direct limit of the sequence

$$\mathbb{R} \to \mathbb{R} \times \{0\} \hookrightarrow \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R} \times \{0\} \hookrightarrow \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \cdots.$$

3. A characterization theorem

3.1. μ_n^∞-manifolds

Denote by μ_n^∞ the direct limit of the sequence

$$\mu_n^{(1)} \hookrightarrow \mu_n^{(2)} \hookrightarrow \mu_n^{(3)} \hookrightarrow \cdots, \tag{3.1}$$

in which all spaces $\mu_n^{(i)}$ are topological copies of μ_n and all embeddings are Z-embeddings.

A paracompact space X is said to be a *μ_n^∞-manifold* if there exists an open cover of the space X with each element homeomorphic to an open subset in μ_n^∞. We assume that all μ_n^∞-manifolds under consideration are separable.

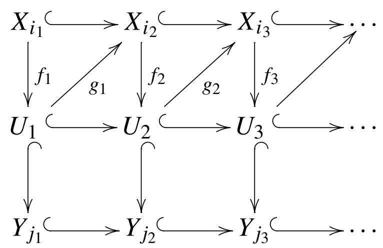
A space Y is said to be *strongly* (*neighborhood*) *n-universal* if for every compact metrizable pair (A, B), where $\dim A \leqslant n$, and every embedding $f : B \to Y$ there exists an embedding $\bar{f} : A \to Y$ (respectively an embedding $\bar{f} : U \to Y$ of some neighborhood U of the set B in A) which extends f.

Theorem 3.1. *A space X is homeomorphic to μ_n^∞* (*respectively is a μ_n^∞-manifold*) *if and only if $X \in \mathcal{MC}(n)^\infty$ and X is strongly n-universal* (*respectively strongly neighborhood n-universal*).

Proof. To prove the "if" part, let $X = \varinjlim X_i$, where X_i are compact metrizable spaces with $\dim X_i \leqslant n$, and assume that X is strongly neighborhood n-universal. Write $\mu_n^\infty = \varinjlim Y_j$, where Y_j are homeomorphic to μ_n and every embedding $Y_j \hookrightarrow Y_{j+1}$ is a Z-embedding.

As in [14], we apply the "back and forth" argument. Set $i_1 = j_1 = 1$. There exists an embedding $f_1 : X_{i_1} \to Y_{j_1}$. By the strong n-universality property of X, there exists an embedding $g_1 : U_1 \to X$ of a closed neighborhood U_1 of $f_1(X_{i_1})$ in Y_{j_1} such that $g_1|f_1(X_{i_1}) = f_1^{-1}$.

Since U_1 is compact, there exists $i_2 > i_1$ such that $g_1(U_1) \subset X_{i_2}$. Proceeding similarly, one obtains the commutative diagram



in which the maps $f_k : X_{i_k} \to U_k$, $g_k : U_k \to X_{i_{k+1}}$ are embeddings and U_{k+1} is a closed neighborhood of $f_k(X_{i_k})$ in Y_{i_k}. Then

$$X = \varinjlim X_i \cong \varinjlim X_{i_k} \cong \varinjlim\{X_{i_1} \xrightarrow{f_1} U_1 \xrightarrow{g_1} X_{i_2} \xrightarrow{f_2} U_2 \xrightarrow{g_2} \cdots\} \cong \varinjlim U_k$$

(hereafter $\cong$ means 'is homeomorphic to') and the latter set $U = \varinjlim U_k$ is an open subset of $\varinjlim Y_{j_k} \cong \mu_n^\infty$. Therefore, X is a μ_n^∞-manifold. In case X is strongly n-universal, we can take $U_k = Y_{j_k}$, hence $X \cong \varinjlim Y_{j_k} \cong \mu_n^\infty$.

As for the "only if" part, the μ_n^∞-manifold case needs a proof. Let us prove that every μ_n^∞-manifold is strongly neighborhood n-universal.

Obviously, if a space is strongly neighborhood n-universal then such is also every one of its open subspaces. Also, if a space is the discrete union of its open strongly neighborhood n-universal subspaces, then this space is strongly neighborhood n-universal.

Now, we are going to prove that if a space M is the union of two of its open subspaces M_1, M_2 satisfying the strong neighborhood n-universality property, then M satisfies this property as well.

Given a compact metrizable pair (A, B) with $\dim A \leqslant n$ and an embedding $g : B \to M$, choose open sets U_1, U_2, V_1 and V_2 in A so that $\mathrm{cl}\, U_1 \cap \mathrm{cl}\, U_2 = \emptyset$, $g^{-1}(M \setminus M_2) \subset V_1 \subset \mathrm{cl}\, V_1 \subset U_1$ and $g^{-1}(M \setminus M_1) \subset V_2 \subset \mathrm{cl}\, V_2 \subset U_2$. Using the strong neighborhood n-universality of $M_1 \cap M_2$ (see the remark above), we have a closed neighborhood W_0' of $B \setminus (V_1 \cup V_2)$ in $A \setminus (V_1 \cup V_2)$ and an embedding $f : W_0' \to M_1 \cap M_2$ which is an extension of $g|(B \setminus (V_1 \cup V_2))$. Since $(g(B \setminus (U_1 \cup U_2))) \cap (g(B \cap \mathrm{cl}\, V_1) \cup g(B \cap \mathrm{cl}\, V_2)) = \emptyset$, there is a closed neighborhood W_0 of $B \setminus (U_1 \cup U_2)$ in $A \setminus (U_1 \cup U_2)$ such that $W_0 \subset W_0'$ and $f(W_0)$ misses $g(B \cap \mathrm{cl}\, V_1) \cup g(B \cap \mathrm{cl}\, V_2)$. Then, we obtain an embedding $g_0 : B \cup W_0 \to M$ defined by $g_0|B = g$ and $g_0|W_0 = f|W_0$. Now, using the strong neighborhood n-universality of M_i, $i = 1, 2$, we have a closed neighborhood W_i' of $(B \setminus U_{3-i}) \cup W_0$ in A and an embedding $f_i : W_i' \to M_i$ which is an extension of $g_0|(B \setminus U_{3-i}) \cup W_0$. Choose W_1 and W_2 so that W_i is a closed neighborhood of $B \cap \mathrm{cl}\, U_i$, $W_i \subset W_i'$, $i = 1, 2$, and $f_1(W_1) \cap f_2(W_2) = \emptyset$. Then, $W = W_0 \cup W_1 \cup W_2$ is a closed neighborhood of B in A and the map $f : W \to M$ defined by $f|(W_0 \cup W_i) = f_i|(W_0 \cup W_i)$, $i = 1, 2$, is an embedding.

Together with the strong n-universality property of μ_n^∞ and [13] this gives us the strong neighborhood n-universality of every μ_n^∞-manifold.

Finally, note that every μ_n^∞-manifold X belongs to the class $\mathcal{MC}(n)^\infty$. Indeed, let $\{U_i \mid i \in \mathbb{N}\}$ be a locally finite open cover of X by sets homeomorphic to open subsets of μ_n^∞. Then $U_i = \varinjlim_j K_{ij}$, where $K_{i1} \subset K_{i2} \subset \cdots$ is a sequence of compact subsets of X. Obviously, $X = \varinjlim_j (\bigcup_{i=1}^j K_{ij})$ with $\dim K_{ij} \leqslant n$, hence $X \in \mathcal{MC}(n)^\infty$. □

Note that, in the proof of the "if" part, each X_{i_k} is a Z-set in $X_{i_{k+1}}$. Indeed, U_k is a Z-set in U_{k+1} because Y_{j_k} is a Z-set in $Y_{j_{k+1}}$.

In the above proof, we have actually demonstrated the following:

Theorem 3.2 *(Open embedding theorem). Every μ_n^∞-manifold admits an open embedding into μ_n^∞.*

The following is a consequence of the characterization theorem.

Theorem 3.3. *Every μ_n^∞-manifold is homeomorphic to the countable direct limit of μ_n-manifolds and Z-embeddings.*

Proof. Let X be a μ_n^∞-manifold and $X = \varinjlim X_i$, where X_i are compacta. It can be assumed that each X_i is a Z-set in X_{i+1} (see the remark after the proof of Theorem 3.1). There exists an embedding $i_1 : X_1 \to \mu_n$. By the strong neighborhood n-universality property, there exists a closed neighborhood U_1 of the set $i_1(X_1)$ in μ_n such that the embedding $i_1^{-1} : i_1(X_1) \to X_1 \subset X$ can be extended to an embedding $j_1 : U_1 \to X$. Without loss of generality, one can assume that U_1 is a μ_n-manifold. Indeed, in the standard construction of μ_n, one can cover the set $i_1(X_1)$ in μ_n by a finite subfamily $\mathcal{A} \subset \mathcal{F}_k$, for sufficiently large k, so that $\bigcup \mathcal{A}$ is a $(2n+1)$-manifold with boundary and $U_1 = (\bigcup \mathcal{A}) \cap \mu_n$. The technique of [4, Section 1.1.2] allows us to show that U_1 is a μ_n-manifold. Thus, we have a compact μ_n-manifold $V_1 = j_1(U_1)$ such that $X_1 \subset V_1$.

Suppose that compact μ_n-manifolds $V_1 \subset V_2 \subset \cdots \subset V_k \subset X$ are chosen so that $X_i \subset V_i$ and V_i is a Z-set in V_{i+1}, for every $i = 1, 2, \ldots, k-1$. There exists $l \geqslant k+1$ such that $V_k \subset X_l$. There exists a Z-embedding $i_l : X_l \to \mu_n$.

Similarly as above, it follows from the strong neighborhood n-universality property that there exists a closed neighborhood U_{k+1} of the set $i_l(X_l)$ in μ_n such that U_{k+1} is a μ_n-manifold and the embedding $i_l^{-1}: i_l(X_l) \to X_l \subset X$ can be extended to an embedding $j_l : U_{k+1} \to X$. Put $V_{k+1} = j_l(U_{k+1})$. It follows from the properties of Z-sets in μ_n that V_k is a Z-set in V_{k+1}. Obviously, $X = \varinjlim V_i$. $\square$

Theorem 3.4. *Every $X \in \mathcal{MC}(n)^\infty$ admits a Z-embedding into μ_n^∞.*

Proof. Let $X = \varinjlim X_i$, where X_i are compacta with $\dim X_i \leqslant n$. Let $i_1 : X_1 \to \mu_n^{(1)}$ be a Z-embedding (recall that, as in (3.1), $\mu_n^\infty = \varinjlim \mu_n^{(i)}$). Suppose that, for every $j < k$, Z-embeddings $i_j : X_j \to \mu_n^{(j)}$ are defined so that the following conditions hold:

(i) $i_{j+1}|X_j = i_j$ for every $j < k - 1$;
(ii) $i_{j+1}(X_j) \cap \mu_n^{(j)} = i_j(X_j)$ for every $j < k - 1$.

In order to construct a Z-embedding i_k, note that, since $\mu_n^{(k)}$ is an $AE(n)$-space, there is an extension, $\tilde{i}_k$, of the map $X_{k-1} \xrightarrow{i_{k-1}} \mu_n^{(k-1)} \hookrightarrow \mu_n^{(k)}$ over X_k. Applying the Z-set approximation theorem for μ_n-manifolds [4, Theorem 2.3.8], one can approximate $\tilde{i}_k$ by a Z-embedding i_k so that $i_k(X_k) \cap \mu_n^{(k-1)} = i_k(X_{k-1})$.

It is easy to see that the map $\varinjlim i_k$ is a Z-embedding of X into μ_n^∞. $\square$

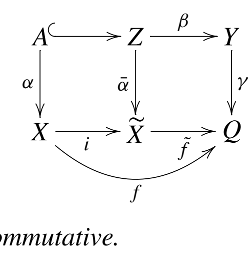
4. Universal maps

Recall that an *embedding* of a map $f : X \to Y$ into a map $f' : X' \to Y'$ consists of a pair of embeddings $g : X \to X'$, $h : Y \to Y'$ such that $f'g = hf$. If both g, h are homeomorphisms, we say that a *homeomorphism* of a map f onto a map f' is given.

Dranishnikov constructed in [9] $(n-1, n)$-soft polyhedrally n-soft maps $F_n : \mu_n \to Q$ and $G_n : \mu_n \to \mu_n$ which, in addition to other properties, satisfy also the following universality property: every map of metrizable compacta $h : X \to Y$, where $\dim X \leqslant n$ (respectively $\dim X \leqslant n$, $\dim Y \leqslant n$) can be embedded into the map F_n (respectively into G_n).

Lemma 4.1. *For any map $f : X \to Q$, where X is a metrizable compactum with* $\dim X \leqslant n$, *there exists a map $\tilde{f} : \widetilde{X} \to Q$, where $\widetilde{X}$ is a metrizable compactum with* $\dim \widetilde{X} \leqslant n$, *and an embedding $i : X \to \widetilde{X}$ such that $\tilde{f} \circ i = f$ and the following condition holds*:

$(*)$ *for every compact metrizable pair (Z, A), where* $\dim Z \leqslant n$, *every metrizable compactum Y, an embedding $\alpha : A \to X$ and maps $\beta : Z \to Y$, $\gamma : Y \to Q$ such that $f \circ \alpha = \gamma \circ \beta|A$, there exists an embedding $\bar{\alpha} : Z \to \widetilde{X}$ for which the diagram*



is commutative.

Proof. Denote by $\mathfrak{A}$ the set of all possible sextuples $S = (Z, A, Y, \alpha, \beta, \gamma)$, in which Z, Y are metrizable compacta lying in the Hilbert cube Q, $\dim Z \leqslant n$, A is a closed subset in Z, $\alpha : A \to X$ is an embedding, and $\beta : Z \to Y$, $\gamma : Y \to Q$ are maps such that $f \circ \alpha = \gamma \circ \beta|A$.

For every $S \in \mathfrak{A}$, choose an n-invertible map $h_S : K_S \to Q$, where K_S is an n-dimensional metrizable compactum [9]. Fix a map $g_S : Z \to K_S$ such that $h_S \circ g_S = \gamma \circ \beta$.

In the space $T = X \sqcup (\bigsqcup\{K_S \mid S \in \mathfrak{A}\})$ consider the equivalence relation $\sim$ defined by the condition $\alpha(a) \sim g_S(a)$ for every $S = (Z, A, Y, \alpha, \beta, \gamma) \in \mathfrak{A}$ and every $a \in A$. Denote by H the quotient space of the space T, and by $q : T \to H$ the quotient map.

It is easy to see that the map q is closed and thus H is a normal space. It follows from the Dowker theorem [10] that $\dim H = n$, therefore $\dim \beta H = n$ (see [10]; as usual, by βH we denote the Stone–Čech compact extension of a space H).

Denote by $j : X \to H$ and $j_S : K_S \to H$, $S \in \mathfrak{A}$, the natural embeddings. There exists a map $h : H \to Q$ such that $h \circ j = f$ and $h \circ j_S = h_S$ for every $S \in \mathfrak{A}$. Denote by $\hat{h} : \beta H \to Q$ the unique extension of the map h.

By the Mardešić factorization theorem [12], there exists an n-dimensional metrizable compactum X_1 and maps $h_1 : \beta H \to X_1$, $F : X_1 \to Q$ such that $\hat{h} = F \circ h_1$. Let $s : X_1 \times Q \to Q$ be an embedding and

$$f_n' = f_n | f_n^{-1}\big(s(X_1 \times Q)\big) : f_n^{-1}\big(s(X_1 \times Q)\big) \to s(X_1 \times Q)$$

(here $F_n : \mu_n \to Q$ is the universal Dranishnikov map [9]). Denote by $\mathcal{R}$ the partition of the space $F_n^{-1}(s(X_1 \times Q))$, whose only nontrivial elements are the sets of the form $F_n^{-1}(s(x, 0))$, $x \in X$. Let $\widetilde{X} = F_n^{-1}(s(X_1 \times Q))/\mathcal{R}$ and denote by $q_1 : F_n^{-1}(s(X_1 \times Q)) \to \widetilde{X}$ the quotient map. Let $g : \widetilde{X} \to X_1 \times Q$ be a map such that $s \circ g \circ q_1 = f_n'$.

Let $\tilde{f} = F \circ \mathrm{pr}_1 \circ g$ and define an embedding $i_1 : X_1 \to \widetilde{X}$ by the formula $i_1(x) = q_1(F_n^{-1}(s(x, 0)))$, $x \in X_1$. Let $i = i_1 \circ h_1 \circ j$. Then

$$\begin{aligned}
\tilde{f} \circ i(x) &= F \circ \mathrm{pr}_1 \circ g \circ i_1 \circ h_1 \circ j(x) \\
&= F \circ \mathrm{pr}_1 \circ s^{-1} \circ s \circ g \circ q_1 \circ F_n^{-1}\big(s\big(h_1 \circ j(x), 0\big)\big) \\
&= F \circ \mathrm{pr}_1 \circ s^{-1} \circ f_n' \circ F_n^{-1} \circ s\big(h_1 \circ j(x), 0\big) = F \circ \mathrm{pr}_1\big(h_1 \circ j(x), 0\big) \\
&= F \circ h_1 \circ j(x) = \hat{h} \circ j(x) = f(x).
\end{aligned}$$

To show condition $(*)$, let $S = (Z, A, Y, \alpha, \beta, \gamma) \in \mathfrak{A}$. Define a map $\alpha_1 : Z \to X_1$ as $\alpha_1 = h_1 \circ j_S \circ g_S$. Let $p : Z \to Z/A$ be the quotient map and let $\eta : Z/A \to Q$ be an embedding such that $\eta(\{A\}) = 0$.

Define an embedding $\theta : Z \to X_1 \times Q$ by the formula $\theta(z) = (\alpha_1(z), \eta \circ p(z))$, $z \in Z$. From the n-invertibility of the map f_n' it follows that there exists a map $\bar{\theta} : Z \to F_n^{-1}(s(X_1 \times Q))$ such that $f_n' \circ \bar{\theta} = s \circ \theta$. Set $\bar{\alpha} = q_1 \circ \bar{\theta}$.

First of all, it is obvious that the map $\bar{\alpha}$ is an embedding. If $a \in A$, then

$$\begin{aligned}
\bar{\alpha}(a) &= q_1 \circ \bar{\theta}(a) = q_1\big(F_n^{-1}\big(s\big(\alpha_1(a), 0\big)\big)\big) = i_1 \circ \alpha_1(a) \\
&= i_1 \circ h_1 \circ j_S \circ g_S(a) = i_1 \circ h_1 \circ j \circ \alpha(a) = i \circ \alpha(a).
\end{aligned}$$

We have also

$$\begin{aligned}
\tilde{f} \circ \bar{\alpha} &= F \circ \mathrm{pr}_1 \circ g \circ q_1 \circ \bar{\theta} = F \circ \mathrm{pr}_1 \circ s^{-1} \circ f_n' \circ \bar{\theta} = F \circ \mathrm{pr}_1 \circ \theta \\
&= F \circ \alpha_1 = F \circ h_1 \circ j_S \circ g_S = \hat{h} \circ j_S \circ g_S = h_S \circ g_S = \gamma \circ \beta. \qquad \square
\end{aligned}$$

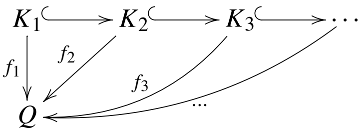
Definition 4.2. A map $f : X \to Y$ is said to be *strongly* (n, ∞)*-universal* (respectively, *strongly* (n, n)*-universal*, *strongly* (n, ω)*-universal*), if for every compact metrizable pair (Z, A), where $\dim Z \leqslant n$, and a metrizable compactum C (respectively metrizable compactum C of dimension $\leqslant n$, finite-dimensional metrizable compactum C), every embedding $\alpha : A \to X$ and maps $\beta : Z \to C$, $\gamma : C \to Y$ such that $f \circ \alpha = \gamma \circ \beta | A$, there exists an embedding $\bar{\alpha} : Z \to X$ such that $\bar{\alpha} | A = \alpha$ and $f \circ \bar{\alpha} = \gamma \circ \beta$.

Theorem 4.3. *There exists a unique* (*up to homeomorphisms*) *strongly* (n, ∞)*-universal map* $\varphi_n : \mu_n^\infty \to Q^\infty$.

Proof. Let $F_n : \mu_n \to Q$ denote the universal Dranishnikov map (see [9]). Using Lemma 4.1, define a sequence of maps $f_i : K_i \to Q$ and embeddings $K_i \hookrightarrow K_{i+1}$ for which the following conditions hold:

(1) $K_1 = \mu_n$ and $f_1 = F_n$;

(2) the diagram



is commutative;

(3) for every compact metrizable pair (Z, A), where $\dim Z \leqslant n$, every metrizable compactum Y, and maps $\alpha : Z \to Y$, $\psi : Y \to Q$ and embedding $\varphi : A \to K_i$ such that $\psi \circ \alpha | A = f_i \circ \varphi$, there exists an embedding $\bar{\varphi} : Z \to K_{i+1}$ such that $\bar{\varphi}|A = \varphi$ and $f_{i+1} \circ \bar{\varphi} = \psi \circ \alpha$.

In order to construct such a sequence, we proceed inductively. Suppose that f_i is already constructed. Apply Lemma 4.1 to the map $f_i : K_i \to Q$ and obtain a map $f_{i+1} : K_{i+1} \to Q$ of a metrizable compactum K_{i+1} with $\dim K_{i+1} \leqslant n$ and an embedding $j : K_i \hookrightarrow K_{i+1}$ so that condition (3) is nothing but condition $(*)$ from Lemma 4.1.

Let

$$Q = \prod_{j=1}^{\infty} [-1, 1]_j, \qquad Q^{(i)} = \prod_{j=1}^{\infty} \left[-1 + \frac{1}{i+1}, 1 - \frac{1}{i+1} \right]_j, \quad i \geqslant 1.$$

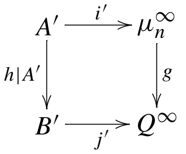
The set $Y = \operatorname{rint} Q = \bigcup \{ Q^{(i)} \mid i \geqslant 1 \}$ is called the *radial interior* of the Hilbert cube Q.

Let $X_i = f_i^{-1}(Q^{(i)})$, $X = \bigcup \{ X_i \mid i \geqslant 1 \}$, and let $\varphi_n : X \to Y$ be a map such that $\varphi_n | X_i = f_i | X_i$, $i \geqslant 1$. Topologize the sets X and Y as the countable direct limits $\varinjlim \{X_i\}$ and $\varinjlim \{Q^{(i)}\}$; the resulting spaces are denoted by $\widehat{X}$ and $\widehat{Y}$, respectively. Then the map $\varphi_n : \widehat{X} \to \widehat{Y}$ is continuous. It follows from the characterization theorem 3.1 and the Sakai characterization theorem [14] that $\widehat{X} \cong \mu_n^\infty$ and $\widehat{Y} \cong Q^\infty$.

The strong (n, ∞)-universality of the map $\varphi_n : \mu_n^\infty \cong \widehat{X} \to \widehat{Y} \cong Q^\infty$ is a consequence of condition (3).

We are going to show that the map φ_n is unique up to homeomorphisms. Let $f : \mu_n^\infty \to Q^\infty$ be a strongly (n, ∞)-universal map. Write $\mu_n^\infty = \varinjlim A_i$, $Q^\infty = \varinjlim B_i$, where A_i, B_i are compacta and $f(A_i) \subset B_i$ (we will denote by $f_i : A_i \to B_i$ the restriction of f). Assume that $A_1 = \{x_0\}$, $B_1 = \{y_0\}$.

Claim. *Let $g : \mu_n^\infty \to Q^\infty$ be a strongly (n, ∞)-universal map, $h : A \to B$ be a map of metrizable compacta, where $\dim A \leqslant n$. For any commutative diagram*

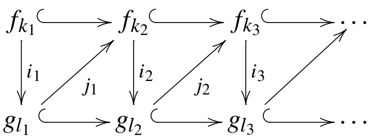


where A', B' are closed subsets in A, B respectively, i', j' are embeddings, there exist embeddings $i : A \to \mu_n^\infty$, $j : B \to Q^\infty$ such that $i|A' = i'$, $j|B' = j'$ and $gi = jh$.

Indeed, there exists an embedding $j : B \to Q^\infty$ that extends j'. By the strong (n, ∞)-universality property, there exists an embedding $i : A \to \mu_n^\infty$ such that $i|A' = i'$ and $gi = jh$.

Suppose now that $g : C \to D$ is a strongly (n, ∞)-universal map, where $C \in \mathcal{MC}(n)^\infty$, $D \in \mathcal{MC}^\infty$. Write $C = \varinjlim C_i$, $D = \varinjlim D_i$, where C_i, D_i are compacta and $g(C_i) \subset D_i$ (we denote by $g_i : C_i \to D_i$ the restriction of g).

Applying the claim, one can easily construct a commutative diagram in the category of maps,



in which $k_1 < k_2 < \cdots$, $l_1 < l_2 < \cdots$, and the morphisms i_p, j_q are embeddings (in the category of maps). Then

$$f \cong \varinjlim f_{k_p} \cong \varinjlim \left\{ f_{k_1} \xrightarrow{i_1} g_{l_1} \xrightarrow{j_1} f_{k_2} \xrightarrow{i_2} g_{l_2} \xrightarrow{j_2} \cdots \right\} \cong \varinjlim g_q = g. \qquad \square$$

The following result is a counterpart of the product theorem of the theory of Q-manifolds (see [5]) in the category $\mathcal{MC}(n)^\infty$.

Theorem 4.4. *Let $\varphi_n : \mu_n^\infty \to Q^\infty$ be a strongly (n,∞)-universal map. Let $X \subset Q^\infty$, $X \in \mathcal{MC}(n)^\infty$ and X be an absolute neighborhood extensor (respectively an absolute extensor) for the class $\mathcal{MC}(n)$. Then $\varphi_n^{-1}(X)$ is a μ_n^∞-manifold (respectively $\varphi_n^{-1}(X) \cong \mu_n^\infty$).*

Proof. We verify the conditions of the characterization theorem 3.1 for μ_n^∞-manifolds. Obviously, $\varphi_n^{-1}(X) \in \mathcal{MC}(n)^\infty$. Given a compact metrizable pair (A,B) with $\dim A \leqslant n$ and an embedding $f : B \to \varphi_n^{-1}(X)$, one can extend the map $\varphi_n f : B \to X$ to a map $g : C \to X$ of a compact neighborhood C of B in A. It follows from the strong (n,∞)-universality of φ_n that there exists an embedding $\bar{f} : C \to \mu_n^\infty$ such that $\varphi_n \bar{f} = g$ and $\bar{f}|B = f$. Then $\bar{f}(C) \subset \varphi_n^{-1}(X)$ and we are done. When X is an absolute extensor for the class $\mathcal{MC}(n)$, we can take $C = A$. □

Theorem 4.5. *There exists a strongly (n,n)-universal map $\psi_n : \mu_n^\infty \to \mu_n^\infty$, which is unique up to homeomorphisms.*

Proof. We suppose that $\mu_n^\infty \subset Q^\infty$. Let $X = \varphi_n^{-1}(\mu_n^\infty)$. We are going to show that X is homeomorphic to μ_n^∞. Obviously, $X \in \mathcal{MC}(n)^\infty$. Let (A,B) be a compact metrizable pair with $\dim A \leqslant n$ and $f : B \to X$ an embedding. Since μ_n^∞ is an absolute extensor for metrizable compacta of dimension $\leqslant n$, there exists an extension $g : A \to \mu_n^\infty$ of the map $\varphi_n f$. It follows from the strong (n,∞)-universality property of φ_n that there exists an embedding $\bar{f} : A \to \mu_n^\infty$ such that $\bar{f}|B = f$ and $\varphi_n \bar{f} = g$. The latter condition means that $\bar{f}(A) \subset X$ and, by the characterization theorem, $X \cong \mu_n^\infty$.

The strong (n,n)-universality of the map ψ_n is an easy consequence of the strong (n,∞)-universality property of the map φ_n.

In turn, the uniqueness of the map ψ_n can be derived from its strong (n,n)-universality similarly as in the proof of Theorem 4.3. □

Theorem 4.6. *There exists a strongly (n,ω)-universal map $\psi_{n,\infty} : \mu_n^\infty \to \mathbb{R}^\infty$, which is unique up to homeomorphisms.*

Proof. We suppose that $\mathbb{R}^\infty \subset Q^\infty$. Let $X = \varphi_n^{-1}(\mathbb{R}^\infty)$. The rest of the proof is completely analogous to that of Theorem 4.5. □

5. Triangulation and classification theorems for μ_n^∞-manifolds

Lemma 5.1. *For every μ_n^∞-manifold X there exists a locally finite polyhedron P of dimension $\leqslant n$ and a map $f : P \to X$ that induces isomorphisms of the homotopy groups of dimensions $\leqslant n-1$.*

Proof. By Theorem 3.3, we can write $X = \varinjlim\{M_i, s_i\}$, where

$$M_1 \xrightarrow{s_1} M_2 \xrightarrow{s_2} M_3 \xrightarrow{s_3} \cdots$$

is a sequence of compact μ_n-manifolds and Z-embeddings. For every i there exist compact μ_n-manifolds M_i' and $M_i'' \subset M_i'$ such that M_i, M_i'' are disjoint Z-sets in M_i' and there exists a polyhedrally n-soft retraction $r_i : M_i' \to M_i$ such that $r_i|M_i'' : M_i'' \to M_i$ is a homeomorphism. This can be easily deduced from the properties of the universal map $F_n : \mu_n \to Q$ (see [9]). Indeed, one can assume that $M_i \times [0,1] \subset Q$ and let $M_i' = F_n^{-1}(M_i \times [0,1])$. It follows from the n-invertibility of F_n that there exist maps $\xi_k : M_i \to M_i'$, $k = 0,1$, such that $F_n \xi_k(x) = (x,k)$ for every $x \in M_i$ and $k \in \{0,1\}$. We identify M_i with its image $\xi_0(M_i)$ and let $M_i'' = \xi_1(M_i)$. The retraction $r_i : M_i' \to M_i$ is given by the formula $r_i(y) = \mathrm{pr}_1 F_n(y)$, where $y \in M_i'$ and $\mathrm{pr}_1 : M_i \times [0,1] \to M_i$ denotes the projection onto the first factor. The required properties of M_i' and M_i'' easily follow from the properties of the universal map F_n.

Define the space X' as the quotient space of the disjoint union $\bigsqcup\{M_i' \mid i \in \mathbb{N}\}$ with respect to the equivalence relation that identifies every point $x \in M_i''$ with the point $s_i \circ r_i(x) \in M_{i+1} \subset M_{i+1}'$. By $q : \bigsqcup\{M_i' \mid i \in \mathbb{N}\} \to X'$ we denote the quotient map.

Define a map $h : X' \to X$ by the condition: if $x \in M'_i$ then $h \circ q(x) = r_i(x) \in M_i \subset X$.

It is not difficult to show that the map h induces isomorphisms of the homotopy groups in dimensions $\leqslant n-1$. Since the space X' is locally compact, metrizable, LC^{n-1}, and $\dim X' = n$, there exists a locally finite polyhedron P of dimension $\leqslant n$ and a map $g : P \to X'$ that induces isomorphisms of the homotopy groups in dimensions $\leqslant n-1$ (see [4, Chapter 6]). The composition $f = h \circ g$ is the required map. □

Lemma 5.2. *Let $f, g : A \to X$ be $(n-1)$-homotopic maps of a metrizable compactum A. Then there exists a compactum $C \subset X$ such that $C \supset f(A) \cup g(A)$ and the maps $f, g : A \to C$ are $(n-1)$-homotopic.*

Proof. There exists an n-invertible map $h : B \to A$, where B is an n-dimensional compactum [9]. Then the maps fh and gh are homotopic; denote by $H : B \times I \to X$ a homotopy connecting them. Let $C = H(B \times I)$.

If $\dim B' \leqslant n$ and a map $h' : B' \to A$ is given, then there exists a map $\alpha : B' \to B$ such that $h\alpha = h'$. Then $H(\alpha \times \mathrm{id}_I)$ is a homotopy between the maps fh' and gh'. Thus, $f, g : A \to C$ are $(n-1)$-homotopic. □

The proof of the following lemma is a direct modification of the proof of Lemma 2.8.7 from [4]. Note that in [4] the notion of μ-homotopy was used where we use the one of $(n-1)$-homotopy.

Lemma 5.3. *Suppose that a map $f : X \to Y$ induces isomorphisms of the homotopy groups of dimension $\leqslant n-1$, Y is an LC^{n-1}-space, (P, L) is a polyhedral pair with $\dim P \leqslant n$ and $\alpha : P \to Y$, $\beta : L \to X$ are maps such that $f\beta = \alpha|L$. Then there exists a map $\hat{\beta} : P \to X$ such that $\hat{\beta}|L = \beta$ and $f\hat{\beta} \sim_{n-1} \alpha$.*

Lemma 5.4. *Let $f : X \to Y$ be a map of μ_n^∞-manifolds which induces isomorphisms of the homotopy groups of dimension $\leqslant n-1$. For every compact metrizable pair (A, B), where $\dim A \leqslant n$, and every pair of maps $\alpha : B \to X$, $\beta : A \to Y$ such that α is an embedding and $f\alpha \simeq_{n-1} \beta|B$ there exists an embedding $\alpha' : A \to X$ such that $\alpha'|B = \alpha$ and $f\alpha' \simeq_{n-1} \beta$.*

Proof. There exists an n-dimensional finite polyhedral pair (P, L) and maps $g : A \to P$, $g' : P \to Y$ such that $g'g \simeq_{n-1} \beta$, $g(B) \subset L$ and there exists a map $h : L \to X$ such that $hg|B \simeq_{n-1} \alpha$ (see [4]).

By Lemma 5.1, there exists a map $g'' : P \to X$ such that $g''|L = h$ and $fg'' \simeq_{n-1} g'$. Then, by Lemma 5.2 and Theorem 3.3, there exists a compact μ_n-manifold $M \subset X$ such that $\alpha(B) \cup g''(P) \subset M$ and the maps $hg|B, \alpha : B \to M$ are $(n-1)$-homotopic. By [6, Proposition 2.2], there exists a map $\tilde{\alpha} : A \to M$ such that $\tilde{\alpha}|B = \alpha$ and $\tilde{\alpha} \simeq_{n-1} g''g$. By Theorem 3.3, there exists a compact μ_n-manifold M' such that $M \subset M' \subset X$ and M is a Z-set in M'. Then, by the Z-set approximation theorem for μ_n-manifolds [4, Theorem 2.3.8], there exists an embedding $\alpha' : A \to M'$ such that $\alpha' \simeq_{n-1} \tilde{\alpha}$ and $\alpha'|B = \alpha$. Then also

$$f\alpha' \simeq_{n-1} f\tilde{\alpha} \simeq_{n-1} fg''g \simeq_{n-1} g'g \simeq_{n-1} \beta. \qquad \square$$

The following result is a classification theorem for μ_n^∞-manifolds.

Theorem 5.5. *Let $f : X \to Y$ be a map of μ_n^∞-manifolds which induces isomorphisms of homotopy groups of dimension $\leqslant n-1$. Then the map f is $(n-1)$-homotopic to a homeomorphism.*

Proof. By Theorem 3.3, the spaces X and Y have representations $X = \varinjlim M_i$, $Y = \varinjlim N_j$, where each M_i and N_j are compact μ_n-manifolds which are Z-sets in M_{i+1} and N_{j+1}, respectively. Set $M_{i_0} = N_{i_0} = \emptyset$ and define by induction sequences $i_0 < i_1 < i_2 < \cdots$ and $j_0 < j_1 < j_2 < \cdots$, maps $f_k : X \to Y$, $\alpha_k : M_{i_k} \to Y$, $\beta_k : N_{j_k} \to X$ such that the following holds:

(1) $f_{k+1} \simeq_{n-1} f_k$;
(2) all α_k, β_k are embeddings, $\alpha_k(M_{i_k}) \subset N_{j_k}$, $\beta_k(N_{j_k}) \subset M_{i_{k+1}}$, and $\beta_k\alpha_k = \mathrm{id}$, $\alpha_{k+1}\beta_k = \mathrm{id}$, $\alpha_{k+1}|M_{i_k} = \alpha_k$, $\beta_{k+1}|N_{j_k} = \beta_k$; and
(3) $f_k|M_{i_k} = \alpha_k$.

Set $f_0 = f$ and suppose that f_l, i_l, j_l, α_l, and β_l are already constructed for $l < k$. Choose $i_k > i_{k-1}$ so that $\beta_{k-1}(N_{j_{k-1}}) \subset M_{i_k}$. It follows from the Z-set approximation theorem that there exists an embedding $\alpha_k : M_{i_k} \to Y$ such that $\alpha_k|\beta_{k-1}(N_{j_{k-1}}) = \beta_{k-1}^{-1}$ and $\alpha_k \simeq_{n-1} f_{k-1}|M_{i_{k-1}}$. By the $(n-1)$-homotopy extension property (see [6]), there exists a map $f_k : X \to Y$ such that $f_k|M_{i_k} = \alpha_k$ and $f_k \simeq_{n-1} f_{k-1}$. By the construction, $f_k \simeq_{n-1} f_0 = f$ and, therefore, the map f_k induces isomorphisms of homotopy groups in dimension $\leqslant n-1$ (see [4]).

Choose $j_k > j_{k-1}$ so that $\alpha_k(M_{i_k}) \subset N_{j_k}$. By Lemma 5.4, for the map f_k and embedding $N_{j_k} \hookrightarrow Y$ there exists an embedding $\beta_k : N_{j_k} \to X$ such that $\beta_k \alpha_k = \mathrm{id}$. By the construction, $\alpha_k \beta_{k-1} = \mathrm{id}$.

Then the map $\alpha = \varinjlim \alpha_k$ is a homeomorphism from $X = \varinjlim M_{i_k}$ into $Y = \varinjlim N_{j_k}$ with $\beta = \varinjlim \beta_k$ as the inverse. It follows from properties (1) and (2) that $\alpha \simeq_{n-1} f$. □

Theorem 5.6. *For every embedding f of a μ_n^∞-manifold X into Q^∞ we have $X \cong \varphi_n^{-1}(f(X))$.*

Proof. It follows from Theorem 4.4 that $\varphi_n^{-1}(f(X))$ is a μ_n^∞-manifold. Note that the map $\varphi_n|\varphi_n^{-1}(f(X)) : \varphi_n^{-1}(f(X)) \to f(X)$ induces isomorphisms of the homotopy groups in dimensions $\leqslant n-1$. The result then follows from Theorem 5.5. □

Theorem 5.7. *For every μ_n^∞-manifold X there exists a locally finite polyhedron P of dimension $\leqslant n$ such that for every embedding $P \subset Q^\infty$ we have $X \cong \varphi_n^{-1}(P)$.*

Proof. By Lemma 5.1, there exists a locally finite polyhedron P of dimension $\leqslant n$ a map $f : P \to X$ that induces isomorphisms of the homotopy groups in dimensions $\leqslant n-1$. We may assume that $P \subset Q^\infty$, then the map $g = f \circ (\varphi|\varphi_n^{-1}(P)) : \varphi_n^{-1}(P) \to X$ is a map of μ_n^∞-manifolds that induces isomorphisms of the homotopy groups in dimensions $\leqslant n-1$. By Theorem 5.5, g is $(n-1)$-homotopic to a homeomorphism. □

6. Open questions

There exist counterparts of the spaces μ_n^∞ which in the class of compact Hausdorff spaces of given weight play the role analogous to that of μ_n^∞ for the class of metrizable compacta. Namely, Dranishnikov constructed n-dimensional spaces D_n^τ that are universal for the class of compact Hausdorff spaces of weight τ and of dimension n. However, these spaces are not absolute extensors in dimension n, because by Dranishnikov's theorem every n-dimensional compact absolute extensor in dimension n is metrizable. This does not allow straightforward extension of our results to the case of spaces of weight τ. As a good starting point we propose the open problem of topological characterization of the countable direct limit of a sequence of spaces D_n^τ and Z-embeddings (see related paper [18]).

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