Homotopy type of the complement of an immersion and classification of embeddings of tori

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This note is devoted to a classification of embeddings of higher dimensional manifolds, a problem actively studied in the 1960s [1], [2] and attracting renewed interest in recent years [3]–[5]. Investigation of the problem started with consideration of knots $S^q \to S^m$, for which an explicit classification in some dimensions was obtained along with a complete *rational* classification in codimension $\geqslant 3$.

Theorem 1 [1]. *Assume that* $q+2 < m < 3q/2+2$. *Then, up to isotopy, the set of smooth embeddings* $S^q \to S^m$ *is infinite if and only if* $q+1$ *is divisible by* 4.

We study the case of embeddings $S^p \times S^q \to S^m$. Such embeddings will be called *knotted tori*. Links are a classical special case of knotted tori (see Fig. 1a). The investigation of knotted tori is a natural next step after knots and links because of the handle decomposition of an arbitrary manifold.

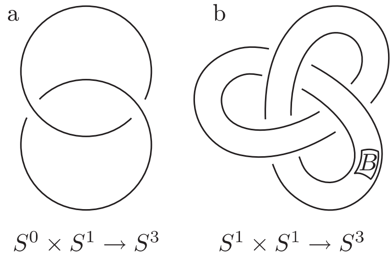

Figure 1

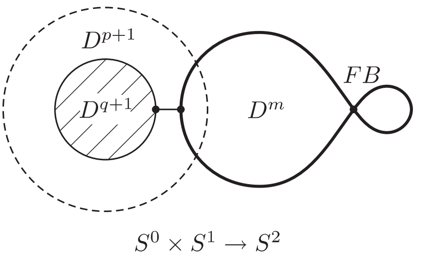

Figure 2

The set of knotted tori in spaces of sufficiently high dimension, namely, in the *metastable* range $m \geqslant p+3q/2+2$, $p \leqslant q$, was explicitly described in [4]. The metastable range is a natural bound for the applicability of classical methods in embedding theory. The aim of this note is to present an approach making it possible to obtain results in lower dimensions.

Theorem 2. *Assume that* $p+4q/3+2 < m < p+3q/2+2$ *and* $m > 2p+q+2$. *Then, up to smooth isotopy, the set of smooth embeddings* $S^p \times S^q \to S^m$ *is infinite if and only if either* $q+1$ *or* $p+q+1$ *is divisible by* 4.

Our approach to the classification is based on investigation of immersions and the homotopy type of their complements.

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We give some necessary definitions. Let us fix points $u \in S^p$ and $v \in S^q$ and a ball $B = B^{p+q} \subset S^p \times S^q$ disjoint from the meridian $S^p \times v$ and the parallel $u \times S^q$ of our torus (see Fig. 1b). A piecewise-smooth map $F\colon S^p \times S^q \to S^m$ is called an *almost embedding* if F is a smooth embedding outside the ball B and $FB \cap F(S^p \times S^q \setminus B) = \varnothing$. An *almost isotopy* is defined similarly, except that B is replaced by $B \times I$. We denote by $\mathscr{M}$ the set of all almost embeddings $F\colon S^p \times S^q \to S^m$ up to almost isotopy. As shown in [5], the set $\mathscr{M}$ has a natural group structure.

The description of the group $\mathscr{M}$ is a much simpler problem than classification of embeddings, and can be realized by classical methods. Thus, the proof of Theorem 2 reduces to the problem of *determining whether a given almost embedding is almost isotopic to an embedding*. Our main lemma asserts that the complete obstruction to this lies in a finite group.

Lemma 2.1. *Assume that $p + 4q/3 + 2 < m < p + 3q/2$ and $m > 2p + q + 2$. Then there is a homomorphism $\beta\colon \mathscr{M} \to \mathscr{G}$ into a certain finite group $\mathscr{G}$, with the property that if $\beta(F) = 0$ then F is almost isotopic to an embedding* (*smooth outside* B).

We sketch our proof of Theorem 2 in the case when $q + 1$ is divisible by 4, assuming that Lemma 2.1 has been proved. It can be shown that the group $\mathscr{M}$ is infinite in this case (and finitely generated). Then the group $|\mathscr{G}| \cdot \mathscr{M}$ is also infinite. For each $F \in |\mathscr{G}| \cdot \mathscr{M}$ we have $\beta(F) = 0$, and hence each such F is almost isotopic to an embedding (smooth outside the ball B). By smoothing these embeddings we get infinitely many distinct smooth embeddings $S^p \times S^q \to S^m$.

Proof of Lemma 2.1. This is based on the following three-step construction (see Fig. 2).

Step 1. *Construction of a web D^{p+1}.* Let $F\colon S^p \times S^q \to S^m$ be an almost embedding. We glue a disc $D^{p+1} \subset S^m$ to the meridian $F(S^p \times v)$ so that the interior of the disc does not intersect the rest of the torus $F(S^p \times S^q)$. This is possible in general position, since $m > 2p + q + 2$. We call such a disc D^{p+1} a *web*.

Step 2. *Construction of a web D^{q+1}.* Using just general position, we cannot construct a second web $D^{q+1} \subset S^m$ glued along the parallel $F(u \times S^q)$. Let us first assume that such a web exists.

Step 3. *Conic construction.* We remove a neighbourhood of the union of our webs from the sphere S^m. Generically, the webs have a unique common point, and hence we obtain an m-dimensional ball D^m as a result. We can assume that $D^m \cap F(S^p \times S^q) = FB$. Replacing FB by a cone over $F\,\partial B$ inside D^m, we transform F into the required embedding.

The outline of the rest of the proof of Lemma 2.1 is as follows. The desired homomorphism $\beta\colon \mathscr{M} \to \mathscr{G}$ is an obstruction to the existence of the web $D^{q+1} \subset S^m$. This obstruction lies in the group $\pi_q(D^m - \operatorname{Im} F, \partial D^m - \operatorname{Im} F)$, where D^m is the complement of a neighbourhood of the web D^{p+1} in S^m. This group can be computed by methods in § 4 of [3], and it is often finite.

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