Hawaiian groups of topological spaces

U. H. Karimov and D. Repovš

The n-dimensional *Hawaiian earring* ($n = 0, 1, 2, \dots$) is defined to be the following subspace of the $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$:

$$\mathscr{H}^n = \left\{ \bar{x} = (x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1} \;\middle|\; (x_0 - 1/k)^2 + \sum_{i=1}^{n} x_i^2 = (1/k)^2,\ k \in \mathbb{N} \right\}.$$

The point $\theta = (0, 0, \dots)$ will be regarded as a base point of $\mathscr{H}^n$.

The n-dimensional *Hawaiian set* of a space X with base point x_0 is defined as the set of homotopy classes $[f]$ of maps $f\colon (\mathscr{H}^n, \theta) \to (X, x_0)$. We denote this set by $\mathscr{H}_n(X, x_0)$. For $n \geqslant 1$ a group operation in $\mathscr{H}_n(X, x_0)$ comes naturally from the groups $\pi_n(X, x_0)$. The groups $\mathscr{H}_n(X, x_0)$ (and the sets $\mathscr{H}_0(X, x_0)$) are homotopy invariants in the category of all topological spaces with base points.

A space X is said to be *locally n-connected* if for every $x \in X$ and every neighbourhood $U \subset X$ of x there is a neighbourhood $V \subset U$ of x such that the homomorphism $\pi_n(V, x) \to \pi_n(U, x)$ induced by inclusion is zero.

Theorem 1. *If the space X is locally n-connected at the point x_0 and satisfies the first countability axiom, then the group $\mathscr{H}_n(X, x_0)$ is isomorphic to the weak direct product $\prod_{i=0}^{\infty} G_i$ with each factor G_i equal to $\pi_n(X, x_0)$.*

Proof. Let $f\colon (\mathscr{H}^n, \theta) \to (X, x_0)$ be an arbitrary map. Since X is locally n-connected at x_0, there exists a neighbourhood V_{x_0} such that the embedding $V_{x_0} \subset X$ is n-trivial. By the continuity of f, there exists a positive integer K such that $S_k^n \subset f^{-1}(V_{x_0})$ for $k > K$, where S_k^n is the kth n-sphere in $\mathscr{H}^n$. Therefore, all the maps $f|_{S_k^n}$ are n-trivial for $k > K$. We define the map $\varphi\colon \mathscr{H}_n(X, x_0) \to \prod_{i=1}^{\infty} G_i$ as follows: $\varphi([f]) = ([f|_{S_1^n}], [f|_{S_2^n}], [f|_{S_3^n}], \dots, [f|_{S_K^n}], e, e, \dots) \in \prod_{i=0}^{\infty} G_i$. Clearly, φ is surjective. Let us show that φ is injective. To this end, we consider two maps f and g such that $\varphi(f) = \varphi(g)$. Since the space X is locally n-connected and satisfies the first countability axiom, there exists a countable nested system of neighbourhoods U_i of x_0 such that all the embeddings $U_{i+1} \subset U_i$ are homotopically n-trivial. There exists an increasing sequence $\{K_i\}_{i \in \mathbb{N}}$ of positive integers such that $\operatorname{Im}(f|_{S_k^n}) \cup \operatorname{Im}(g|_{S_k^n}) \subset U_{m+1}$ for all $k > K_m$. For $k \leqslant K_1$ we take an arbitrary homotopy with respect to the point θ connecting $f|_{S_k^n}$ with $g|_{S_k^n}$ (this can be done since $\varphi(f) = \varphi(g)$). For k in the interval $K_1 < k \leqslant K_2$ we take an arbitrary homotopy in U_1 connecting $f|_{S_k^n}$ with $g|_{S_k^n}$. In general, for k in the interval $K_m < k \leqslant K_{m+1}$ we take an arbitrary homotopy in U_m connecting $f|_{S_k^n}$ with $g|_{S_k^n}$. As a result we obtain a homotopy with respect to the point θ connecting f with g, and hence φ is injective.

Theorem 2. *If the space X has a countable system of neighbourhoods at the point x_0 and the groups $\mathscr{H}_n(X, x_0)$ (and the sets $\mathscr{H}_0(X, x_0)$) are countable, then X is locally n-connected at x_0.*

Proof. Suppose that X is not locally n-connected at the point x_0. Then there exists a nested system of open neighbourhoods V_i of x_0 such that the embeddings $V_i \subset V_1$

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are essential in dimension n (that is, the embeddings are not n-trivial) and $\bigcap_{i=1}^{\infty} V_i = x_0$. With each index i we associate a map $f_i\colon S^n \to V_i$ whose composition with the embedding $V_i \subset V_1$ is homotopically essential. Furthermore, to each sequence $\sigma = (\sigma_1, \sigma_2, \sigma_3, \dots)$ of zeros and ones ($\sigma_i = 0$ or 1) there obviously corresponds a map $f_\sigma\colon (\mathscr{H}^n, \theta) \to (X, x_0)$. Let us take two such sequences σ and σ' with the property that $\sigma_i \neq \sigma'_i$ for an infinite set of indices. The map f_σ is not homotopy equivalent to $f_{\sigma'}$. Indeed, assuming their homotopy equivalence, let the homotopy $H\colon (\mathscr{H}^n, \theta) \times I \to (X, x_0)$ connect f_σ with $f_{\sigma'}$. Since $H(\theta \times I) = x_0 \in V_1$, there exists an integer K such that $H(S^n_k \times I) \subset V_1$ for $k > K$. And since $\sigma_i \neq \sigma'_i$ for an infinite number of indices, there exists a $k_0 > K$ such that $\sigma_{k_0} \neq \sigma'_{k_0}$. Then one of the two maps $f_\sigma|_{S^n_{k_0}}\colon S^n_{k_0} \to V_{k_0} \to V_1$ and $f_{\sigma'}|_{S^n_{k_0}}\colon S^n_{k_0} \to V_{k_0} \to V_1$ is homotopically essential, while the other is homotopically constant. This contradicts the embedding $H(S^n_{k_0} \times I) \subset V_1$, thus showing that f_σ and $f_{\sigma'}$ are not homotopy equivalent.

Since the set of all sequences σ differing from each other on an infinite set of indices is uncountable, the set $\mathscr{H}_n(X, x_0)$ is uncountable. This contradicts the hypothesis of Theorem 2.

Corollary 1. *A compact connected metrizable space X is a Peano continuum if and only if the set $\mathscr{H}_0(X, x_0)$ is countable for every point x_0 of X.*

Corollary 2. *A finite-dimensional compact metrizable space X is an ANR if and only if the groups $\mathscr{H}_n(X, x_0)$ are countable for all n and all points $x_0 \in X$.*

Corollary 3. *A finite-dimensional compact metrizable space X is an AR if and only if $\mathscr{H}_n(X, x_0) = e$ for all n and all points x_0 in X.*

Remark 1. There exists a contractible compact space X such that $\mathscr{H}_1(X, *) \neq e$ for some point $*$.

The cone $C(\mathscr{H}^1, \theta)$ over the 1-dimensional Hawaiian earring is such a contractible space (here $*$ is an arbitrary interior point of the segment $C(\theta)$). This cone is not locally 1-connected at $*$, and hence $\mathscr{H}_1(C(\mathscr{H}^1, \theta), *) \neq e$.

Remark 2 (K. Eda). There exists a compact space X that is locally 1-connected at all points and such that the group $\mathscr{H}_1(X, *)$ is uncountable for any interior point $*$ of X. The suspension ΣC of a Cantor compactum C is an example of such a space X.

Remark 3. There exists a locally 2-connected Peano continuum X such that the groups $\mathscr{H}_2(X, *)$ are uncountable for all points $*$.

The bouquet of a 2-dimensional sphere and the 1-dimensional Hawaiian earring provides an example of such a space.

Theorem 3. *There exists a non-contractible cell-like compact space X such that the group $\mathscr{H}_n(X, x_0)$ is trivial for all n and all points $x_0 \in X$.*

Proof. We consider a countable compact bouquet $\bigvee_{i=1}^{\infty} S^i$ of spheres of increasing dimension with base point θ. Let $C(\bigvee_{i=1}^{\infty} S^i)$ be the cone over the bouquet $\bigvee_{i=1}^{\infty} S^i$ with vertex a and with base identified with $\bigvee_{i=1}^{\infty} S^i$. Let $\theta \in \bigvee_{i=1}^{\infty} S^i \subset C(\bigvee_{i=1}^{\infty} S^i)$ be a base point of the cone, and let X_1 and X_2 be two copies of this cone with vertices a_1, a_2 and base points θ_1, θ_2, respectively. Define the space X as the one-point union of the spaces X_1 and X_2 with respect to the points θ_1 and θ_2. Obviously, X satisfies the conditions of the theorem.

Question. Let P and P^* be the one-point compactifications of countable polyhedra by points θ and θ^*, respectively, and let $f\colon (P, \theta) \to (P^*, \theta^*)$ be a continuous map such that $\mathscr{H}_n(f)\colon \mathscr{H}_n(P, \theta) \to \mathscr{H}_n(P^*, \theta^*)$ is an isomorphism for any n. Is it true that f is a homotopy equivalence?

U. H. Karimov
Institute of Mathematics, Academy of Sciences of Tajikistan, Dushanbe, Tajikistan
E-mail: umed-karimov@mail.ru

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D. Repovš
Institute of Mathematics, Physics, and Mechanics, University of Ljubljana, Ljubljana, Slovenia
E-mail: dusan.repovs@fmf.uni-lj.si