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ON BASIC EMBEDDINGS INTO THE PLANE

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ABSTRACT. A subset $K \subset \mathbf{R}^2$ is said to be *basic* if for each function $f: K \to \mathbf{R}$ there exist functions $g, h: \mathbf{R} \to \mathbf{R}$ such that $f(x,y) = g(x) + h(y)$ for each point $(x,y) \in K$. If all the three functions in this definition are assumed to be *continuous* (*differentiable*), then the embedding is C^0*-basic* (C^1*-basic*). This notion appeared in studies of Hilbert's 13th problem on superpositions. We prove that *if a finite graph is* C^0*-basically embeddable in the plane, then it is* C^1*-basically embeddable in the plane.* In our proof we construct an explicit C^1-basic embedding and use the Skopenkov characterization of graphs C^0-basically embeddable in the plane. Our result is nontrivial because the plane contains graphs which are C^0-basic but not C^1-basic and graphs which are C^1-basic but not C^0-basic (Baran-Skopenkov). We also prove that *given any integer* $k \geq 0$*, there is a subset of the plane which is* C^r*-basic for each* $0 \leq r \leq k$ *but not* C^r*-basic for each* $k < r \leq \omega$*.*

1. Introduction. The notion of a basic embedding appeared implicitly in the Kolmogorov-Arnold solution of Hilbert's 13th problem [**1, 5, 6**]. A compactum $K \subset \mathbf{R}^2$ is said to be *basic* if, for each continuous function $f: K \to \mathbf{R}$ there exist continuous functions $g, h: \mathbf{R} \to \mathbf{R}$ such that $f(x,y) = g(x) + h(y)$ for each point $(x,y) \in K$. One can replace in the definition of a basic embedding *continuous* functions by *smooth* functions (by Lipschitz, Hölder, analytic, etc., functions) and obtain a notion of basic embeddability in a smooth, Lipschitz, Hölder, analytic, etc. sense.

This note is motivated by the following problems.

Problem 1. *Find conditions on a compactum* $K \subset \mathbf{R}^2$*, under which* K *is basically embeddable into the plane in the smooth sense.*

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Problem 2. *Find conditions on a finite graph K, under which K is basically embeddable into the plane in the smooth sense.*

Problem 3. *Find conditions on an arbitrary compactum K, under which K is basically embeddable into the plane in the smooth sense.*

The answer to Problem 2 is given in the paper; the other two problems remain open.

For a subset K of the plane, not necessarily open, a function $f\colon K \to \mathbf{R}$ is said to be *r-analytic*, $0 \leq r < \infty$, if for each point $(x_0, y_0) \in K$ there exists

$$\{a_{ij}\}_{i,j=0}^{r} \subset \mathbf{R} \quad \text{such that} \quad a_{00} = f(x_0, y_0)$$

and

$$f(x_0 + x, y_0 + y) = \sum_{i,j=0}^{r} a_{ij} x^i y^j + o((|x| + |y|)^r),$$

where $(x_0 + x, y_0 + y) \in K$ and $(x, y) \to (0, 0)$. Since $\mathbf{R} \subset \mathbf{R}^2$, this definition applies to functions $\mathbf{R} \to \mathbf{R}$ as well. Note that 0-analytic is the same as continuous, 1-analytic for functions $\mathbf{R} \to \mathbf{R}$ is the same as differentiable and r-analytic for functions $\mathbf{R} \to \mathbf{R}$ is approximately (but not precisely) the same as C^r.

For a subset K of the plane (not necessarily open) a function $f\colon K \to \mathbf{R}$ is said to be *analytic* (or ω-analytic), if for each point $(x_0, y_0) \in K$ there exists

$$\{a_{ij}\}_{i,j=0}^{\infty} \subset \mathbf{R} \quad \text{such that} \quad f(x_0 + x, y_0 + y) = \sum_{i,j=0}^{\infty} a_{ij} x^i y^j$$

for $(x_0 + x, y_0 + y)$ belonging to some neighborhood of (x_0, y_0) in K.

A compactum $K \subset \mathbf{R}^2$ is said to be *C^r-basic*, $1 \leq r \leq \omega$, if for each r-analytic function $f\colon K \to \mathbf{R}$ there exist r-analytic functions $g, h\colon \mathbf{R} \to \mathbf{R}$ such that $f(x, y) = g(x) + h(y)$ for each point $(x, y) \in K$.

Theorem 1.1. *Given any integer $k \geq 0$, there is a subset of the plane which is C^r-basic for each $0 \leq r \leq k$ but not C^r-basic for each $k < r \leq \omega$.*

In Theorem 1.1 we can take *the graph* V_k *of the function* $y = |x|^k$, $x \in [-1, 1]$ for k odd, and $W_{k+1} = (V_{k+1} - (2, 0)) \sqcup (V_{k+1} + (2, 0))$ for k even.

The main result of this paper is the following.

Theorem 1.2. *If a finite graph* K *is* C^0*-basically embeddable into the plane, then* K *is* C^1*-basically embeddable into the plane.*

Theorem 1.2 is nontrivial because the plane contains graphs which are C^1-basic but not C^0-basic and graphs which are C^1-basic but not C^0-basic [**3**].

In the proof of Theorem 1.2 we use the following result, answering the Sternfeld problem [**13**].

Theorem 1.3 [**11**], cf. [**7, 8**], [**10**, Section 5]. *For any finite graph* K *the following conditions are equivalent*:

(C) K *is* C^0*-basically embeddable in* $\mathbf{R}^2$;

(G) K *does not contain any of the following three graphs: a circle* S*, a pentod* P *or a cross* C *with branched ends*;

(R) K *can be embedded in* R_n *for some* n.

Definition of the graphs R_n is given in Section 2. Our proof of Theorem 1.2 is based on a construction of a C^1-basic embedding $R_n \subset \mathbf{R}^2$ (Section 2). We prove elementary that this embedding is also C^0-basic, which yields an elementary proof of Theorem 1.3 as explained in Section 3.

2. Proofs.

Proof of Theorem 1.1 *for k odd.* First we prove that $V = V_1$ is C^1-basic. Take a 1-analytic function $f: V \to \mathbf{R}$. Since f is 1-analytic at $(0, 0)$, it follows that there exist $a, b \in \mathbf{R}$ such that

$$f(x, |x|) = f(0, 0) + ax + b|x| + o(|x| + |x|), \quad \text{where} \quad x \to 0.$$

Take $h(y) = by$ and $g(x) = f(x, |x|) - h(|x|)$. Clearly, h is 1-analytic, i.e. differentiable, and g is 1-analytic outside 0. Since $g(x) = f(0,0) + ax + o(x)$ when $x \to 0$, it follows that g is 1-analytic also at 0.

Now we prove that V_k is C^r-basic for each $0 \leq r \leq k$. Take an r-analytic function $f\colon V_k \to \mathbf{R}$. Since f is r-analytic at $(0,0)$, it follows that there exists $\{a_{ij}\}_{i,j=0}^{r} \subset \mathbf{R}$ such that

$$a_{00} = f(0,0) \quad \text{and} \quad f(x, |x|^k) = \sum_{i,j=0}^{r} a_{ij} x^i |x|^{kj} + o((|x| + |x|^r)^r),$$

where $x \to 0$. Since

$$o((|x| + |x|^r)^r) = o_1(x^r),$$

we have

$$f(x, |x|^k) = a_{00} + a_{01}|x|^k + a_{10}x + \cdots + a_{r0}x^r + o_2(x^r).$$

Take $h(y) = a_{01}y$ and $g(x) = f(x, |x|^k) - h(|x|^k)$. Clearly, h is r-analytic and g is r-analytic outside 0. We also have $g(x) = a_{00} + a_{10}x + \cdots + a_{r0}x^r + o_2(x^r)$ when $x \to 0$. So g is r-analytic also at 0.

Next we prove that $V = V_1$ is not C^r-basic for each $1 < r \leq \omega$. Define an analytic function $f\colon V \to \mathbf{R}$ by $f(x,y) = xy$, where $y = |x|$. If V is C^r-basic for some $r \geq 2$, then there are r-analytic functions

$$g, h\colon \mathbf{R} \to \mathbf{R} \quad \text{such that} \quad f(x, |x|) = x|x| = g(x) + h(|x|)$$

for each $x \in [0,1]$. Hence $g(x) - g(-x) = 2x^2$. But this is impossible because g is 2-analytic, hence

$$g(x) = g(0) + ax + bx^2 + o(x^2) \quad \text{and so} \quad g(-x) = g(0) - ax + bx^2 + o(x^2)$$

for $x \to +0$.

At last we prove that V_k is not C^r-basic for k odd and each $k < r \leq \omega$. Define an analytic function $f\colon V_k \to \mathbf{R}$ by $f(x,y) = xy$, where $y = |x|^k$. If V is C^r-basic for some $r > k$, then there are r-analytic functions

$$g, h\colon \mathbf{R} \to \mathbf{R} \quad \text{such that} \quad f(x, |x|^k) = x|x|^k = g(x) + h(|x|^k)$$

for each $x \in [0,1]$. Hence $g(x)-g(-x) = 2x|x|^k$. But this is impossible for k odd because g is $(k+1)$-analytic, hence

$$g(x) = g_0 + g_1 x + \cdots + g_{k+1}x^{k+1} + o(x^{k+1})$$

and so

$$g(-x) = g_0 - g_1 x + \cdots + g_{k+1}x^{k+1} + o(x^{k+1})$$

for $x \to +0$. □

Note that a function $f(x,y)$ on the graph V is 1-analytic if and only if $p(t) = f(t,|t|)$ is differentiable on $[-1,0]$ and on $[0,1]$.

Proof of Theorem 1.1 *for k even.* Let us prove that W_{k+1} is C^r-basic for each $0 \le r \le k$. Given an r-analytic function $f\colon W_{k+1} \to \mathbf{R}$, take functions $h(y) = 0$ and $g(x) = f(x, |x - 2\operatorname{sign} x|^{k+1})$. Clearly, h is r-analytic and $f(x,y) = g(x) + h(y)$ for each $(x,y) \in W_{k+1}$. Since the function $p(t) = |t|^{k+1}$ is k-analytic and $r \le k$, it follows that g is r-analytic.

Let us prove that W_{k+1} is not C^r-basic for k even and each $k < r \le \infty$. Define an analytic function $f\colon W_{k+1} \to \mathbf{R}$ by $f(x,y) = y\operatorname{sign} x$. If W_{k+1} is C^r-basic, then there are r-analytic functions g and h such that $f(x,y) = g(x) + h(y)$.

For $x \in [-1,1]$ we have

$$g(x-2) + h(|x|^{k+1}) = f(x-2, |x|^{k+1}) = -|x|^{k+1}$$

and

$$g(x+2) + h(|x|^{k+1}) = f(x+2, |x|^{k+1}) = |x|^{k+1}.$$

Hence $g(2-x) = g(2+x)$ and $g(-x-2) = g(x-2)$ for $x \in [-1,1]$. Now $d^{k+1}g/dx^{k+1}|_{x=2} = d^{k+1}g/dx^{k+1}|_{x=-2} = 0$. This leads to a contradiction since g is $(k+1)$-analytic, $k+1$ is odd, and $g(x+2) - g(x-2) = 2|x|^{k+1}$. □

Let us define inductively the graphs R_n together with an embedding $R_n \to \mathbf{R}^2$. We embed R_1 into $[-10,10]\times[-10,10]$ as shown in Figure 1.

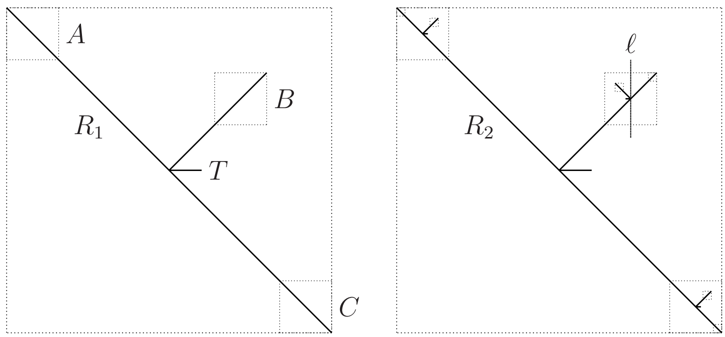

FIGURE 1.

Then we repeat the procedure by embedding copies of R_1 into squares A, B and C shown in Figure 1 to get R_2. Note that the embedded R_1 into B was mirrored over ℓ to get a connected R_2.

In general, the graph R_n is constructed by embedding R_{n-1} into appropriate small squares A, B, C attached to R_1. The squares A, B and C have to be chosen carefully. Let $p_1: \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ and $p_2: \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ denote projections onto x and y axes. We require that $p_1(A)$, $p_1(B)$, $p_1(C)$, $p_1(T)$ are disjoint and $p_2(A)$, $p_2(B)$, $p_2(C)$, $p_2(T)$ are disjoint.

Proof of Theorem 1.2. The boundary in R_n of any subgraph $K \subset R_n$ consists of a finite number of points. Hence any 1-analytic mapping $K \to \mathbf{R}$ can be extended to a 1-analytic mapping $R_n \to \mathbf{R}$. So it suffices to prove that R_n is C^1-basic. We prove this by induction. Given a mapping $f: R_n \to \mathbf{R}$ we shall find functions $g, h: \mathbf{R} \to \mathbf{R}$ such that $f(x, y) = g(x) + h(y)$. Then we shall show that we can obtain g and h to be 1-analytic, i.e. differentiable, when f is 1-analytic.

Put $h(0) = 0$ and define $g(x) = f(x, 0)$ for every $x \in [0, 2]$. Extend g to a function $g: [0, 10] \to \mathbf{R}$.

Note that for every $y \in [-10, 6]$ there exists an unique $x_y = |y| \in [0, 10]$ such that $(x_y, y) \in R_1$. (See Figure 2 for details.) Therefore,

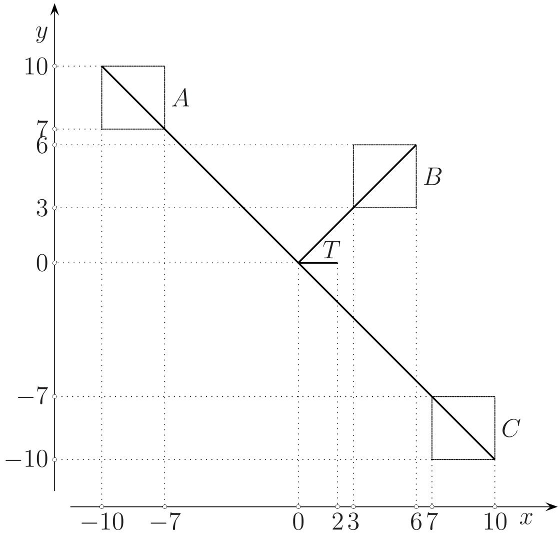

FIGURE 2.

using g and f for $x \in [0,10]$ we can define $h\colon[-10,6] \to \mathbf{R}$ as $h(y) = f(|y|, y) - g(|y|)$. Extend h to $h\colon[-10,10] \to \mathbf{R}$.

Note that for every $x \in [-10,0]$ there exists a unique $y_x = -x$ such that $(x, y_x) \in R_1$. Therefore using h we can define $g\colon[-10,0] \to \mathbf{R}$ as $g(x) = f(x,-x) - h(-x)$. Finally, we extend g and h to $g, h\colon \mathbf{R} \to \mathbf{R}$.

Now let $f\colon R_n \to \mathbf{R}$, $n > 1$, be given. We put $h(0) = 0$ and define $g(x) = f(x,0)$ for every $x \in [0,2]$. As R_n is constructed by embedding R_{n-1} into appropriate small squares A, B, C attached to R_1, by inductive hypothesis there exist functions $g', h'\colon \mathbf{R} \to \mathbf{R}$ such that $f(x,y) = g'(x) + h'(y)$ on $(x,y) \in (A \cup B \cup C) \cap R_n$. Hence we can extend g smoothly onto $[0,10]$ so that $g = g'$ on $p_1(B \cup C)$. Using functions g and f for $x \in [0,10]$ we can define $h\colon[-10,6] \to \mathbf{R}$ as $h(y) = f(|y|, y) - g(|y|)$. Then we extend h onto $[-10,10]$ so that $h = h'$ on $[7,10]$. Using h we finally define $g\colon[-10,0] \to \mathbf{R}$ as $g(x) = f(x,-x) - h(-x)$.

For $n = 1$, if f is 1-analytic, then it is clear that at each step the constructed functions g and h are differentiable except maybe at 0. So all the extensions can be chosen to be differentiable. Since f is

1-analytic at $(0,0)$, it follows that there exist $a, b \in \mathbf{R}$ such that

$$\begin{gathered} f(x,y) = f(0,0) + ax + by + o(|x| + |y|), \\ \text{where} \quad (x,y) \in R_1 \quad \text{and} \quad (x,y) \to (0,0). \end{gathered}$$

We may assume that $f(0,0) = g(0) = h(0) = 0$. Then according to the structure of R_1 one can write

$$\begin{cases} f(x,x) = g(x) + h(x) \\ f(x,-x) = g(x) + h(-x) \\ f(x,0) = g(x) \\ f(-x,x) = g(-x) + h(x), \end{cases}$$

so

$$\begin{cases} g(x) = f(x,0) \\ h(x) = f(x,x) - f(x,0) \\ h(-x) = f(x,-x) - f(x,0) \\ g(-x) = f(-x,x) - f(x,x) + f(x,0) \end{cases}$$

for small $x \geq 0$. Hence

$$g(x) = ax + o(x)$$

and

$$g(-x) = -ax + bx - ax - bx + ax + o(x) = -ax + o(x)$$

when $x \to +0$. So g is differentiable at 0. Also,

$$h(x) = ax + bx - ax + o(x) = bx + o(x)$$

and

$$h(-x) = ax - bx - ax + o(x) = -bx + o(x)$$

when $x \to +0$. So h is differentiable at 0.

Hence, for $n > 1$, if f is 1-analytic, then it is clear that at each step the constructed functions g and h are differentiable everywhere. So all the extensions can be chosen to be differentiable and thus the resulting functions are differentiable. □

An elementary proof of $(R) \Rightarrow (C)$ *in Theorem* 1.3. Analogously to the proof of Theorem 1.2 above. The reduction from K to R_n follows also by the Tietze-Uryhson extension theorem. We construct g and h from f as above. From the construction it is clear that at each step the constructed functions g and h are continuous. So all the extensions can be chosen to be continuous and thus the resulting functions are continuous. $\square$

Note that for each function $f\colon R_1 \to \mathbf{R}$ the functions $g, h\colon \mathbf{R} \to \mathbf{R}$ such that $f(x,y) = g(x) + h(y)$ are *uniquely* defined by f in a neighborhood of 0. Hence *any* such functions g and h are 0- or 1-analytic in a neighborhood of 0, if f is 0- or 1-analytic. Surprisingly, this is false for r-analytic functions with $1 < r \leq \omega$: the subset $R_1 \subset \mathbf{R}^2$ is C^1-basic but not C^r-basic for each $1 < r \leq \omega$. This is proved analogously to Theorem 1.1 for k odd.

3. The Sternfeld criterion. The proof of Theorem 1.3 in [**11**] was based on the solution of the Arnold problem [**2**]: find conditions on a compactum $K \subset \mathbf{R}^2$, under which K is C-basic. This problem was solved by Sternfeld [**12, 13**] (who was apparently unaware of [**2**]). In order to formulate the Sternfeld criterion, let us introduce some definitions. Let p_1 and p_2 be projections onto the coordinate axes in $\mathbf{R}^2$. For $Z \subset \mathbf{R}^2$, let

$$E(Z) = \{z \in Z : \ |Z \cap p_1^{-1}(p_1(z))| \geq 2 \quad \text{and} \quad |Z \cap p_2^{-1}(p_2(z))| \geq 2\}.$$

Set $E^2(Z) = E(E(Z))$, $E^3(Z) = E(E(E(Z)))$, etc. An ordered sequence $\{a_1, \dots, a_n\} \subset \mathbf{R}^2$ is called an *array* if, for each i, we have $p_1(a_i) = p_1(a_{i+1})$ for i even and $p_2(a_i) = p_2(a_{i+1})$ for i odd ($a_i \neq a_{i+1}$, but it is not required that all the points of an array should be distinct).

Theorem 3.1 [**12, 13**]. *For any compactum* $K \subset \mathbf{R}^2$ *the following conditions are equivalent*:

(B) *the embedding* $K \subset \mathbf{R}^2$ *is basic*;

(E) $E^n(K) = \varnothing$ *for some* n;

(A) K *does not contain any array of* n *points for some* n.

In this paper we prove Theorem 3.1 following [**13**] (we believe our exposition is clearer). One can see that the proof of Theorem 3.1 is non-elementary in a sense that it used the Banach inverse operator theorem.

The proof of $(R) \Leftrightarrow (G)$ in Theorem 1.3 is elementary, cf. [**4**]. The proof of $(C) \Rightarrow (G)$ in Theorem 1.3 is elementary modulo the implication $(B) \Rightarrow (A)$ of Theorem 3.1 [**11**]. The latter implication has an elementary proof by [**9**]. The proof of $(R) \Rightarrow (C)$ in Theorem 1.3 used the non-elementary implication $(E) \Rightarrow (B)$ of Theorem 3.1 [**11**]. In this paper we give an elementary proof of $(R) \Rightarrow (C)$ in Theorem 1.3, which yields an elementary proof of the whole Theorem 1.3.

The Sternfeld proof of Theorem 3.1. First we prove the easy assertion $(A) \Rightarrow (E)$. Suppose to the contrary that $E^n(K) \neq \varnothing$. Take a point $a_0 \in E^n(K)$. Then there exist points $a_{-1}, a_1 \in E^{n-1}(K)$ such that $p_1(a_{-1}) = p_1(a_0)$ and $p_2(a_1) = p_2(a_0)$. Analogously, there exist points $a_{-2}, a_2 \in E^{n-2}(K)$ such that $\{a_{-2}, a_{-1}, a_0, a_1, a_2\}$ is an array. Analogously we construct an array of $2n+1$ points in K.

The proof of $(E) \Rightarrow (\Phi) \Rightarrow (A)$ is based on a reformulation of (B) terms of linear operators in functional spaces. Denote by $C(X)$ the space of continuous functions on X with the norm $|f| = \sup\{|f(x)| : x \in X\}$. For a subset $K \subset I^2$ define the *linear superposition operator*

$$\phi\colon C(I) \oplus C(I) \to C(K) \quad \text{by} \quad \phi(g,h)(x,y) = g(x) + h(y).$$

Clearly, the embedding $K \subset I^2$ is basic if and only if $\phi = \phi_K$ is epimorphic. Denote by $C^*(X)$ the space of bounded linear functionals on $C(X)$ with the norm $|\mu| = \sup\{|\mu(f)| : f \in C(X),\ |f| = 1\}$. For a subset $K \subset I^2$ define the *dual linear superposition operator*

$$\phi^*\colon C^*(K) \to C^*(I) \oplus C^*(I) \quad \text{by} \quad \phi^*\mu(g,h) = (\mu(g \circ p_1), \mu(h \circ p_2)).$$

Since $|\phi^*\mu| \leq 2|\mu|$, it follows that ϕ^* is bounded. By duality, ϕ_K is epimorphic if and only if $\phi^* = \phi_K^*$ is monomorphic. By the Banach inverse operator theorem, ϕ^* is monomorphic if and only if

(Φ) *there exists* $\varepsilon > 0$ *such that* $|\phi^*\mu| > \varepsilon|\mu|$ *for each* $\mu \in C^*(K)$

(because this condition ensures that $\operatorname{im}\phi^*$ is closed). Thus $(B) \Leftrightarrow (\Phi)$. So it remains to prove $(E) \Rightarrow (\Phi) \Rightarrow (A)$.

First we prove $(\Phi) \Rightarrow (A)$. If (A) is false, then for each n there exists an array $\{a_1, \dots, a_n\} \subset K$. Define a linear functional $\mu \in C^*(K)$ by $\mu(f) = \sum_{i=1}^n (-1)^i f(a_i)$. Then $|\mu| = n$ and $|\phi^*\mu| \le 4$. Hence (Φ) is false.

Now we prove $(E) \Rightarrow (\Phi)$. We use the fact that $C^*(X)$ is the space of σ-additive regular real valued Borel measures (in the sequel – simply 'measures') on X. We have

$$\phi^*\mu = (\mu_x, \mu_y), \quad \text{where} \quad \mu_x(U) = \mu(p_1^{-1}U) \quad \text{and} \quad \mu_y(U) = \mu(p_2^{-1}U).$$

If $\mu = \mu^+ - \mu^-$ is the decomposition of a measure μ to its positive and negative parts, then $|\mu| = \bar{\mu}(X)$, where $\bar{\mu} = \mu^+ + \mu^-$ is the absolute value of μ. Let D_x (D_y) be the set of points of K which are not shadowed by some other point of K in x- (y-) direction. Take any measure μ on K of the norm 1.

If

$$E(K) = \varnothing, \quad \text{then} \quad D_x \cup D_y = K, \quad \text{so} \quad 1 = \bar{\mu}(K) \le \bar{\mu}(D_x) + \bar{\mu}(D_y).$$

Therefore without loss of generality, $\bar{\mu}(D_x) \ge 1/2$. Since p_1 is injective over D_x, it follows that $|\mu_x| \ge 1/2$, thus (Φ) holds.

If

$$E(E(K)) = \varnothing, \quad \text{then} \quad D_x \cup D_y = K - E(K), \quad \text{so} \quad E(D_x \cup D_y) = \varnothing.$$

Therefore in the case when $\bar{\mu}(E(K)) < 3/4$ we have $\bar{\mu}(D_x \cup D_y) > 1/4$ and without loss of generality $\bar{\mu}(D_x) > 1/8$. Then as above $|\mu_x| > 1/8$, thus (Φ) holds. In the case when $\bar{\mu}(E(K)) \ge 3/4$ we have $\bar{\mu}(K - E(K)) \le 1/4$. By the case $E(K) = \varnothing$ above without loss of generality $\bar{\mu}_x(p_1(E(K))) \ge \bar{\mu}(E(K))/2$. Hence $|\mu_x| \ge 1/2 \cdot 3/4 - 1/4 = 1/8$, thus (Φ) holds. The case of arbitrary n is proved analogously. □

We remark that not only some linear relation on $\operatorname{im}\phi_K$ can force it to be strictly less than $C(K)$. Or, in other words, φ_K^* can be injective but not monomorphic. If an embedding $K \subset \mathbf{R}^2$ is basic, then we can prove that ϕ^* is monomorphic without use of ϕ as follows. Define a linear operator

$$\Psi\colon C^*(I) \oplus C^*(I) \to C^*(K) \quad \text{by} \quad \Psi(\mu_x, \mu_y)(f) = \mu_x(g) + \mu_y(h),$$

where $g, h \in C(I)$ are such that $g(0) = 0$ and $f(x, y) = g(x) + h(y)$ for $(x, y) \in K$. Clearly, $\Psi\Phi = \mathrm{id}$ and Ψ is bounded, hence Φ is monomorphic.

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