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CONTINUITY-LIKE PROPERTIES AND CONTINUOUS SELECTIONS

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1. Introduction

Recall the definition of continuity of a map f between metric spaces (X,d) and (Y,ρ):

$$(\forall x \in X)(\forall \varepsilon > 0)(\exists \delta > 0)\,(\forall x' \in X)\Big(d(x,x') < \varepsilon \Rightarrow \rho\big(f(x),f(x')\big) < \varepsilon\Big).$$

The question arises whether it is possible to choose $\delta > 0$, which continuously depends on the triple $(x,\varepsilon,f) \in X \times \mathbf{R}^+ \times \mathcal{C}(X,Y)$, where $\mathcal{C}(X,Y)$ denotes the set of all continuous maps from X into Y, endowed with the metric of *uniform convergence*:

$$\operatorname{dist}(f,g) = \sup\Big\{\min\big\{1,\rho\big(f(x),g(x)\big)\big\} \mid x \in X\Big\}.$$

In [6] the following result was proved:

THEOREM 1.1. *Let (X,d) and (Y,ρ) be metric spaces and suppose that X is locally compact. Then there exists a continuous function $\hat{\delta}: X \times \mathbf{R}^+ \times C(X,Y) \to \mathbf{R}^+$ such that for every $(x,\varepsilon,f) \in X \times \mathbf{R}^+ \times C(X,Y)$ and for every $x' \in X$ the following implication holds:*

$$d(x,x') < \hat{\delta}(x,\varepsilon,f) \Rightarrow \rho\big(f(x),f(x')\big) < \varepsilon. \quad \square$$

The purpose of the present paper is: (a) to prove an analogue of Theorem 1.1 where the continuous choice depends on *five* variables: three of them are as in Theorem 1.1 and the remaining two are the metrics on the spaces X and Y, compatible with the given metrizable topologies on X and Y; (b) to avoid the local compactness restriction in Theorem 1.1; and (c) to examine some similar problems for noncontinuous maps, e.g. for the lower or upper semicontinuous real-valued functions.

We shall answer (a), (b) and (c) from a rather formal point of view. Namely, we shall substitute the inequality $\rho\big(f(x),f(x')\big) < \varepsilon$ in the standard definition of continuity by some suitable predicate P in the variables

$x, x', \varepsilon, f, \rho$. We shall call such a predicate a *continuity-like predicate*. A positive answer to (b) was suggested by [1] and [2] and in the present paper we actually exploit an idea of G. de Marco (as explained in [1]).

For metrizable spaces X and Y we denote by $F = F(X,Y)$ the set of all single-valued maps: $X \to Y$. We endow the set F with the topology of the uniform convergence. If ρ is a metric on a space Y, compatible with the topology on Y, then the following metric on a space F is compatible with the topology of the uniform convergence:

$$\tilde{\rho}(f,g) = \sup_{x \in X} \min \{ 1, \rho(f(x), g(x)) \}, \qquad f, g \in F. \tag{1}$$

For a metrizable space X we denote by M_X the set of all metrics which are compatible with the topology on X.

Since each metric $d : X \times X \to \mathbf{R}$ is a single-valued function we endow the set M_X with the relative topology, induced by the inclusion $M_X \subset F(X \times X, \mathbf{R})$. Hence the metric on the space M_X is defined as follows:

$$\operatorname{dist}(d, d') = \sup_{x, x' \in X} \min \{ 1, |d(x,x') - d'(x,x')| \}. \tag{2}$$

We shall represent different types of continuity of maps from X into Y as predicates, defined on the domain $X \times X \times \mathbf{R}^+ \times F \times M_X \times M_Y$. Let $P(x, x', \varepsilon, f, d, \rho)$ be a predicate (i.e. a logical function) of the variables

$$(x, x', \varepsilon, f, d, \rho) \in X \times X \times \mathbf{R}^+ \times F \times M_X \times M_Y.$$

Denote by P^+ the subset of $X \times X \times \mathbf{R}^+ \times F \times M_X \times M_Y$ consisting of all 6-tuples $(x, x', \varepsilon, f, d, \rho)$ such that the proposition $P(x, x', \varepsilon, f, d, \rho)$ is valid.

DEFINITION 1.2. A map $f : X \to Y$ is said to be *P-continuous* if for each $x \in X$, $\varepsilon \in \mathbf{R}^+$, $d \in M_X$, $\rho \in M_Y$, there exists a neighborhood $\mathcal{U} = \mathcal{U}_{x,\varepsilon,f,d,\rho} \subset X$ of the point x such that $\{x\} \times \mathcal{U} \times \{\varepsilon\} \times \{f\} \times \{d\} \times \{\rho\} \subset P^+$. Denote by F_P the set of all P-continuous maps from X into Y. A predicate P is said to be *continuity-like* if the set F_P is nonempty.

As special cases of continuity-like predicates one can consider the usual properties of continuity, lower (upper) semicontinuity of real-valued functions, α-continuity, locally uniform continuity, etc. (See also Section 3.)

DEFINITION 1.3. Let X and Y be metrizable spaces and let P be a continuity-like predicate on $X \times X \times \mathbf{R}^+ \times F \times M_X \times M_Y$. The multivalued map $\Delta_P : X \times \mathbf{R}^+ \times F_P \times M_X \times M_Y \to \mathbf{R}^+$, defined by the relation

$$\Delta_P(x, \varepsilon, f, d, \rho) = \{ \delta > 0 \mid \forall x' \in X \; d(x,x') < \delta \Rightarrow (x, x', \varepsilon, f, d, \rho) \in P^+ \}$$

is said to be the *modulus* of a predicate P.

REMARK. From the definition of P-continuous functions it follows immediately that the set $\Delta_P(x,\varepsilon,f,d,\rho)$ is nonempty for a continuity-like predicate P. We also recall that a single-valued map $\phi : A \to B$ is called a *selection* of a given multivalued map $\Phi : A \to B$ if $\phi(x) \in \Phi(x)$, for all $x \in A$.

The main result in this note is a criterion for the existence of a continuous selection $\hat{\delta}$ of the modulus Δ_P of a continuity-like predicate P, formulated in Theorem 1.4 below. Of course, one can consider Δ_P as a multivalued map from $X \times \mathbf{R}^+ \times F_P \times M_X \times M_Y$ into $\mathbf{R}^+$ with nonempty convex values. An attempt of a direct application of E. Michael's theory of continuous selections [5] leads to some restrictions for the domain X, because of the restriction of lower semicontinuity for Δ_P, see [6]. Here we avoid E. Michael's selection theorems altogether.

Denote by $\operatorname{diag} X$ the diagonal subset $\{(x \times x) \mid x \in X\}$ in $X \times X$, and let

$$P_0^+ = P^+ \cap (X \times X \times \mathbf{R}^+ \times F_P \times M_X \times M_Y).$$

The definition of a P-continuous map implies that

$$(\operatorname{diag} X) \times \mathbf{R}^+ \times F_P \times M_X \times M_Y \subset P_0^+.$$

THEOREM 1.4 (criterion for the existence of a continuous selection). *Let P be a continuous-like predicate and let Δ_P be its modulus. Then the following two assertions are equivalent:*

(i) *There exists a continuous single-valued selection $\hat{\delta}$ of modulus Δ_P; and*

(ii) *The set* $(\operatorname{diag} X) \times \mathbf{R}^+ \times F_P \times M_X \times M_Y$ *lies in the interior of the set P_0^+.*

2. Proof of Theorem 1.4

Proof of (i) $\Rightarrow$ (ii). Let $(x_0, x_0, \varepsilon_0, f_0, d_0, \rho_0) \in (\operatorname{diag} X) \times \mathbf{R}^+ \times F_P \times M_X \times M_Y$ be an arbitrary point and let $\hat{\delta}$ be a continuous selection of Δ_P. Denote $\hat{\delta}_0 = \hat{\delta}(x_0, \varepsilon_0, f_0, d_0, \rho_0)$. Since $\hat{\delta}$ is continuous, the preimage $G = \hat{\delta}^{-1}(\hat{\delta}_0/2, +\infty)$ of the interval $\left(\frac{\hat{\delta}_0}{2}, +\infty\right)$ is an open set in the space $X \times \mathbf{R}^+ \times F_P \times M_X \times M_Y$. The set G contains the point $(x_0, \varepsilon_0, f_0, d_0, \rho_0)$, so this point is an interior point of G. By the definition of the product topology there exists an open neighborhood $\mathcal{U}$ of the point $(\varepsilon_0, f_0, d_0, \rho_0)$ in the space $\mathbf{R}^+ \times F_P \times M_X \times M_Y$ and there exists an open ball $B(x_0; r)$ with radius r such that $B(x_0; r) \times \mathcal{U} \subset G$. We can assume that $r < \frac{\hat{\delta}_0}{4}$. The set $\mathcal{V} = B\left(x_0; \frac{\hat{\delta}_0}{4}\right)$

$\times B(x_0;r)\times\mathcal{U}$ is open and contains the point $(x_0,x_0,\varepsilon_0,f_0,d_0,\rho_0)$. Moreover, $\mathcal{V}$ is a subset of P_0^+. To see this, let $x\in B\left(x_0;\frac{\hat\delta_0}{4}\right)$ and $(x',\varepsilon,f,d,\rho)\in B(x_0;r)\times\mathcal{U}$ be arbitrary points.

Since $d(x,x_0)<\frac{\hat\delta_0}{4}$ and $d(x_0,x')<\frac{\hat\delta_0}{4}$ it follows $d(x,x')<\frac{\hat\delta_0}{2}$. But $\hat\delta(x',\varepsilon,f,d,\rho)>\frac{\hat\delta_0}{2}$, therefore $d(x,x')<\hat\delta(x',\varepsilon,f,d,\rho)$. This implies that $(x,x',\varepsilon,f,d,\rho)\in P_0^+$. We have proved that the point $(x_0,x_0,\varepsilon_0,f_0,d_0,\rho_0)$ is an interior point of P_0^+.

To prove the inverse implication (ii) $\Rightarrow$ (i) we need some lemmas on the spaces of metrics and on maps between them. Let d and ρ be metrics on the spaces X and Y, respectively. It is known that the metric $\tau_{d\rho}$ which is defined by the equality

$$\tau_{d\rho}\big((x,y),(x',y')\big)=d(x,x')+\rho(y,y')$$

where (x,y) and (x',y') are points in $X\times Y$, induces the product topology on the space $X\times Y$.

LEMMA 2.1 (on transfer of metrics onto the product space). *The map $\tau: M_X\times M_Y\to M_{X\times Y}$ which assigns to the pair $(d,\rho)\in M_X\times M_Y$ of metrics the metric $\tau_{d\rho}\in M_{X\times Y}$, is continuous.*

PROOF. Let $d,d'\in M_X$ and $\rho,\rho'\in M_Y$. It suffices to prove the inequality

$$\operatorname{dist}(\tau_{d\rho},\tau_{d'\rho'})\leqq\operatorname{dist}(d,d')+\operatorname{dist}(\rho,\rho'). \tag{3}$$

Let $(x,y)\in X\times Y$ and $(x',y')\in X\times Y$. Then

$$\begin{aligned}&\left|\tau_{d\rho}\big((x,y),(x',y')\big)-\tau_{d'\rho'}\big((x,y),(x',y')\big)\right|\\&=\left|d(x,x')+\rho(y,y')-d'(x,x')-\rho'(y,y')\right|\\&\leqq\left|d(x,x')-d'(x,x')\right|+\left|\rho(y,y')-\rho'(y,y')\right|.\end{aligned}$$

Taking a minimum between 1 and the value of the expression on the left, and between 1 and the value of the expression on the right, respectively, and then taking the supremum over all four variables x,x',y and y', we obtain the inequality (3). □

In formula (1) we assigned to the metric ρ, acting on the space Y, the metric $\tilde\rho$, acting on the space $F=F(X,Y)$.

LEMMA 2.2 (on the transfer of a metric onto the space of functions). *The map $\tau: M_Y\to M_F$ which assigns to each metric $\rho\in M_Y$ the metric $\tilde\rho\in M_F$, is continuous.*

PROOF. Let $\rho,\rho'\in M_Y$. It suffices to prove the inequality

$$\operatorname{dist}(\tilde\rho,\tilde\rho')\leqq\operatorname{dist}(\rho,\rho').$$

Let $f,g \in F(X,Y)$ be arbitrary functions. By the definition of the metric in the space M_F it suffices to prove that

$$\min\left\{1, \left|\tilde\rho(f,g) - \tilde\rho'(f,g)\right|\right\} \leqq \operatorname{dist}(\rho,\rho').$$

Moreover, by the definition of metrics $\tilde\rho$ and $\tilde\rho'$ in F it suffices to prove that for each $x \in X$, the following inequality holds:

(4)

$$\min\Big\{1, \big|\min\{1,\rho(f(x),g(x))\} - \min\{1,\rho'(f(x),g(x))\}\big|\Big\} \leqq \operatorname{dist}(\rho,\rho').$$

It is easy to show that for arbitrary $a,b \in \mathbf{R}$ we have

$$\big|\min\{1,a\} - \min\{1,b\}\big| \leqq \min\{1,|a-b|\}.$$

Therefore, instead of (4) it suffices to prove the following inequality:

$$\min\Big\{1, \big|\rho(f(x),g(x)) - \rho'(f(x),g(x))\big|\Big\} \leqq \operatorname{dist}(\rho,\rho').$$

But since $f(x)$ and $g(x)$ are points in Y, the last inequality holds because of Definition 1.3 of the metric dist $\in M_Y$. □

Let $G = G(X,Y)$ be some subspace of the space of functions $F = F(X,Y)$. Also formula (1), used with G instead of F, defines a map $\tau_G : M_Y \to M_G$.

LEMMA 2.3. *The map* $\tau_G : M_Y \to M_G$ *is continuous.*

PROOF. Let $r : M_F \to M_G$ be the restriction map. If $\tilde\rho \in M_F$ is a metric, then $\tilde\rho$ is a function $F \times F \to \mathbf{R}$ and $r(\tilde\rho)$ is simply its restriction $\tilde\rho|_{G\times G}$. Since r is continuous and τ_G factorizes as

$$\tau_G : M_Y \xrightarrow{\tau} M_F \xrightarrow{r} M_G$$

τ_G is also continuous. □

As usual, let $\mathcal{C} = \mathcal{C}(X,Y)$ denote the subspace of all continuous functions in the space $F(X,Y)$, i.e. $\mathcal{C}(X,Y)$ is endowed with the topology of uniform convergence. It is well-known (see [4]) that the *evaluation* map $e : X \times \mathcal{C}(X,Y) \to Y$ which maps every pair (x,f) to the point $f(x) \in Y$, is jointly continuous, when $\mathcal{C}(X,Y)$ is endowed with the topology of uniform convergence.

Let Z be a metrizable space. Since the space M_Z is a subspace of the space $\mathcal{C}(Z \times Z, \mathbf{R})$, the following lemma holds.

LEMMA 2.4. *The map* $e : Z \times Z \times M_Z \to \mathbf{R}$ *defined by* $e(z, z', d) = d(z, z')$ *is continuous.* □

Let S be a fixed subset of a metrizable space Z. If $d \in M_Z$ is a metric, then for each $z \in Z$ denote as usual

$$d(z, S) = \inf_{s \in S} d(z, s)$$

and call the number $d(z, S)$ the distance of the point z from the set S. The following lemma is a modification of Lemma 2.4.

LEMMA 2.5. *The map* $e_S : Z \times M_Z \to \mathbf{R}$ *for a fixed subset* S, *defined by* $e_S(z) = d(z, S)$ *is continuous.* □

It is well-known that the function $f_{S,d} : Z \to \mathbf{R}$ defined by $f_{S,d}(z) = d(z, S)$ is continuous [3].

Introduce the map $f_S : M_Z \to C(Z, \mathbf{R})$ by setting $f_S(d) = f_{S,d}$. The function $e_S : Z \times M_Z \to \mathbf{R}$ can be factorized as follows:

$$e_S : Z \times M_Z \xrightarrow{\mathrm{id}_Z \times f_S} Z \times C(Z, \mathbf{R}) \xrightarrow{e} \mathbf{R}.$$

Here e is a jointly continuous map. Therefore only the continuity of f_S is to be proved. It suffices to prove that the following inequality holds for each $z \in Z$:

$$\min\left\{1, \left|d(z, S) - d'(z, S)\right|\right\} \leqq \operatorname{dist}(d, d'). \tag{5}$$

Inequality (5) can easily be obtained from the fact that for each $\varepsilon > 0$, there exists a point $s \in S$ such that

$$\left|d(z, S) - d'(z, S)\right| < \left|d(z, s) - d'(z, s)\right| + \varepsilon. \tag{6}$$

To prove the inequality (6) it is necessary to consider two different possibilities:

$$d(z, S) > d'(z, S) \quad \text{or} \quad d(z, S) < d'(z, S).$$

In the case when $d(z, S) > d'(z, S)$ we choose an $s \in S$ such that

$$d'(z, s) \geqq d'(z, S) - \varepsilon. \tag{7}$$

Combining (7) with $d(z, S) \leqq d(z, s)$ we obtain

$$d(z, S) - d'(z, S) \leqq d(z, s) - d'(z, s) + \varepsilon$$

hence also (6). In the case when $d(z, S) < d'(z, S)$, the proof is analogous. □

Proof of (ii) $\Rightarrow$ (i). Assume that each point in the set $\operatorname{diag} X \times \mathbf{R}^+ \times F_P \times M_X \times M_Y$ is an interior point in P_0^+ and construct a continuous selection

$$\hat{\delta} : X \times \mathbf{R}^+ \times F_P \times M_X \times M_Y \to \mathbf{R}^+$$

for the modulus Δ_P.

Let "dist" be the product metric in the space $X \times X \times \mathbf{R}^+ \times F_P \times M_X \times M_Y$ and let P^- be the complement of the set P^+ in this space. If $P^- = \emptyset$ then $\Delta_P \equiv \mathbf{R}^+$ and we can put $\hat{\delta} \equiv 1$, for example. In the case $P^- \neq \emptyset$ for arbitrary point $(x, \varepsilon, f, d, \rho) \in X \times \mathbf{R}^+ \times F_P \times M_X \times M_Y$ let

$$\hat{\delta}(x, \varepsilon, f, d, \rho) = \operatorname{dist}\left((x, x, \varepsilon, f, d, \rho), P^-\right).$$

Since the point $(x, x, \varepsilon, f, d, \rho)$ is an interior point of P_0^+, $\hat{\delta}$ is strictly positive.

Let $x' \in X$ be a point such that $d(x, x') < \hat{\delta}(x, \varepsilon, f, d, \rho)$. By definition of the product metric we have that

$$\operatorname{dist}\left((x, x, \varepsilon, f, d, \rho), (x, x', \varepsilon, f, d, g)\right) \leqq d(x, x')$$

hence

$$\operatorname{dist}\left((x, x, \varepsilon, f, d, \rho), (x, x', \varepsilon, f, d, \rho)\right) \leqq d(x, x') < \hat{\delta}(x, \varepsilon, f, d, \rho)$$
$$= \operatorname{dist}\left((x, x, \varepsilon, f, d, \rho), P^-\right).$$

It follows that $(x, x', \varepsilon, f, d, \rho) \in P^+$. We have proved that $\hat{\delta}$ is a selection for the modulus Δ_P. It remains to prove that the function $\hat{\delta}(x, \varepsilon, f, d, \rho)$ is a continuous function of all of its variables.

Denote by $Z = X \times X \times \mathbf{R}^+ \times F_P \times M_X \times M_Y$. The construction of the function $\hat{\delta} : X \times \mathbf{R}^+ \times F_P \times M_X \times M_Y \to \mathbf{R}^+$ implies that $\hat{\delta}$ can be composed from the following sequence of maps:

(1) $X \times \mathbf{R}^+ \times F_P \times M_X \times M_Y \to X^2 \times \mathbf{R}^+ \times F_P \times M_X^4 \times M_Y^2$, given by the diagonal embeddings $X \to X^2$, $M_X \to M_X^4$, $M_Y \to M_Y^2$ and identities on the remaining factors;

(2) $X^2 \times \mathbf{R}^+ \times F_P \times M_X^4 \times M_Y^2 \to Z \times M_{X^2} \times M_{F_P} \times M_{M_X} \times M_{M_Y}$, given by the transfers of metrics $M_X^2 \to M_{X^2}$, $M_Y \to M_{F_P}$, $M_X \to M_{M_X}$, $M_Y \to M_{M_Y}$ and identities on the remaining factors;

(3) $Z \times M_{X^2} \times M_{F_P} \times M_{M_X} \times M_{M_Y} \to Z \times M_Z$, given by the transfer of metrics into the product space; and

(4) $Z \times M_Z \to \mathbf{R}^+$, given by the evaluation map e_S as in Lemma 2.5, for $S = P^-$.

All these maps are continuous because of Lemmas 2.1–2.5. Theorem 1.4 is thus finally proved. $\square$

3. Applications

(a) *Continuity.* The predicate C is defined on the domain of variables $X \times X \times \mathbf{R}^+ \times F \times M_X \times M_Y$ as follows:

$$C(x, x', \varepsilon, f, d, \rho) = \big(\rho\big(f(x), f(x')\big) < \varepsilon\big).$$

We assert that C is a continuity-like predicate. Indeed, F_C coincides with $\mathcal{C}(X, Y) \neq \emptyset$. Hence the predicate C is the predicate of the *ordinary continuity.*

PROPOSITION 3.1. *The predicate C of the ordinary continuity satisfies the criterion for existence of continuous selections of the modulus Δ_C.*

PROOF. By Theorem 1.4 it suffices to prove that the set C_0^+ is an open subset in the space

$$Z = X \times X \times \mathbf{R}^+ \times F_C \times M_X \times M_Y.$$

Take the function $c : Z \to \mathbf{R}$ defined as follows:

$$c(x, x', \varepsilon, f, d, \rho) = \varepsilon - \rho\big(f(x), f(x')\big).$$

Obviously, $c^{-1}(\mathbf{R}^+) = C_0^+$. Hence it remains to prove that c is continuous. It suffices to prove that the function $b : X \times X \times F_C \times M_Y \to \mathbf{R}$, given by $b : (x, x', f, \rho) \mapsto \rho\big(f(x), f(x')\big)$, is continuous.

The map b can be expressed as the composition of the following maps:

(1) $X \times X \times F_C \times M_Y \to X \times X \times F_C \times F_C \times M_Y$, given by the diagonal embedding $F_C \to F_C \times F_C$ and the identity maps on the remaining factors;

(2) $X \times X \times F_C \times F_C \times M_Y \to Y \times Y \times M_Y$, given by the jointly continuous maps $X \times F_C \to Y$ and the identity map on M_Y; and

(3) $Y \times Y \times M_Y \to \mathbf{R}$, given by the jointly continuous map for the metric.

All these maps are continuous because of Lemmas 2.1–2.5. □

COROLLARY 3.2. *Let X and Y be metrizable spaces. Then there exists a continuous function*

$$\hat{\delta} : X \times \mathbf{R}^+ \times \mathcal{C}(X, Y) \times M_X \times M_Y \to \mathbf{R}^+$$

such that for any $(x, \varepsilon, f, d, \rho) \in X \times \mathbf{R}^+ \times \mathcal{C}(X, Y) \times M_X \times M_Y$ and for any $x' \in X$ the following implication holds:

$$d(x, x') < \hat{\delta}(x, \varepsilon, f, d, \rho) \Rightarrow \rho\big(f(x), f(x')\big) < \varepsilon.$$

Corollary 3.2 is a generalization of Theorem 1.1: our continuous choice depends on five variables $x, \varepsilon, f, d, \rho$ and the local compactness restriction of the space X has been deleted.

(b) *Semicontinuity.* Let $Y = \mathbf{R}$. The function $f : X \to \mathbf{R}$ is said to be *upper semicontinuous or lower semicontinuous* at the point $x \in X$ respectively, if for each $\varepsilon > 0$ there exists a neighborhood $\mathcal{U}$ of the point $x \in X$ such that for any $x' \in \mathcal{U}$ $f(x') < f(x) + \varepsilon$ or $f(x') > f(x) - \varepsilon$, respectively.

Therefore, the predicates USC and LSC such that USC-continuous functions are upper semicontinuous functions and LSC-continuous functions are lower semicontinuous functions, are defined as follows:

$$\mathrm{USC}(x, x', \varepsilon, f, d, \rho) = \big(f(x') < f(x) + \varepsilon\big)$$

and

$$\mathrm{LSC}(x, x', \varepsilon, f, d, \rho) = \big(f(x') > f(x) - \varepsilon\big)$$

Obviously, USC and LSC are continuty-like predicates.

Now we use the predicate USC to explain an important detail in Definition 1.2 and Theorem 1.4. It might seem that P-continuity of maps from X into Y in Definition 1.2 implies the assertion (ii) in Theorem 1.4, hence that each continuity-like predicate P satisfies the criterion for existence of a continuous selection of the modulus Δ_P.

However, this conjecture is not valid. As a counterexample consider the following: $X = Y = \mathbf{R}$, $P = \mathrm{USC}$, $\varepsilon = \frac{1}{2}$, $d = \rho =$ the usual metric on $\mathbf{R}$ and $f : \mathbf{R} \to \mathbf{R}$ defined by

$$f(x) = \begin{cases} 1, & x \leqq 0 \\ 0, & x > 0. \end{cases}$$

Obviously, f is upper semicontinuous, i.e. $f \in F_{\mathrm{USC}}$. Projection of the set P^+ onto the (x, x') plane is the set of points which satisfies the inequality

$$f(x') < f(x) + \frac{1}{2}$$

i.e. the union of the first three quadrants.

Obviously, the point $\big(0, 0, \frac{1}{2}, f, d, \rho\big)$ is not an interior point in the set P_0^+. Hence the predicate USC does not satisfy the criterion for the existence of a continuous selection of the modulus Δ_{USC}, in the special case when $X = \mathbf{R}$. This fact is valid in general:

PROPOSITION 3.3. *Let X be a nondiscrete metric space. Then there are no continuous selections for the moduli Δ_{USC} an Δ_{LSC} of the predicates* USC *and* LSC.

PROOF. Let $x \in X$ be an accumulation point, let $\varepsilon = \frac{1}{2}$ and let $f : X \to \mathbf{R}$ be defined as follows:

$$f(x) = 1, \qquad f(y) = 0 \text{ for } y \neq x.$$

The function f is upper semicontinuous. It is obvious that

$$\Delta_{\mathrm{USC}}\left(x, \frac{1}{2}, f, d\right) = (0, \infty)$$

and

$$\Delta_{\mathrm{USC}}\left(y, \frac{1}{2}, f, d\right) \subset \left(0, d(x,y)\right) \quad \text{for} \quad y \neq x.$$

Hence, for any selection δ of the modulus Δ_{USC} the following has to hold:

$$\delta\left(y, \frac{1}{2}, f, d\right) < d(x,y) \quad \text{for} \quad y \neq x$$

or

$$\lim_{y \to x} \delta\left(y, \frac{1}{2}, f, d\right) = 0.$$

Since $\delta\left(x, \frac{1}{2}, f, d\right) > 0$, the selection δ is discontinuous at the point $\left(x, \frac{1}{2}, f, d\right)$. □

Although there is no continuous selection

$$\delta : X \times \mathbf{R}^+ \times F_{\mathrm{USC}} \times M_X \times M_{\mathbf{R}} \to \mathbf{R}^+$$

of the modulus Δ_{USC} with respect to all variables $(x, \varepsilon, f, d, \rho) \in X \times \mathbf{R}^+ \times F_{\mathrm{USC}} \times M_X \times M_Y$, there exists a selection $\hat{\delta}$ which is continuous with respect to the variable ε, only.

PROPOSITION 3.4. *For each quadruple of the variables* (x, f, d, ρ) *there exists a continuous function*

$$\hat{\delta}_{x,f,d,\rho} : \mathbf{R}^+ \to \mathbf{R}^+$$

such that the function

$$\hat{\delta}(x, \varepsilon, f, d, \rho) = \hat{\delta}_{x,f,d,\rho}(\varepsilon)$$

is a selection of the modulus Δ_{USC}.

PROOF. It is obvious that for each quintuple $(x, \varepsilon, f, d, \rho)$ of variables the set $\Delta_{\mathrm{USC}}(x, \varepsilon, f, d, \rho)$ is an interval with the number 0 as the left endpoint. Also, Δ_{USC} is a nondecreasing multivalued map of ε, i.e. if $\varepsilon' > \varepsilon$, then

$$\Delta_{\mathrm{USC}}(x, \varepsilon, f, d, \rho) \subseteqq \Delta_{\mathrm{USC}}(x, \varepsilon', f, d, \rho).$$

Now the problem is elementary. Namely,

$$\Delta(\varepsilon) = \Delta_{\mathrm{USC}}(x, \varepsilon, f, d, \rho)$$

is a multivalued map $\Delta : \mathbf{R}^+ \to \mathbf{R}^+$ such that for each $\varepsilon \in \mathbf{R}^+$ the set $\Delta(\varepsilon) \subset \mathbf{R}^+$ is an interval with zero as the left endpoint. Moreover, $\Delta(\varepsilon)$ is a nondecreasing map of the variable ε, i.e.

$$\forall \varepsilon' \quad (\varepsilon' > \varepsilon) \Rightarrow \big(\Delta(\varepsilon) \subseteqq \Delta(\varepsilon')\big).$$

The problem is to construct a continuous single-valued selection $\hat{\delta} : \mathbf{R}^+ \to \mathbf{R}^+$ for the multivalued map Δ.

First we construct a step function $\delta : \mathbf{R}^+ \to \mathbf{R}^+$ which is a selection for Δ. For each natural number n, construct the set

$$\Delta^{-1}\left(\frac{1}{n}\right) = \left\{\varepsilon \,\middle|\, \frac{1}{n} \in \Delta(\varepsilon)\right\}.$$

Since $\Delta(\varepsilon)$ is a nondecreasing function, each set $\Delta^{-1}\left(\frac{1}{n}\right)$ is either empty set or an interval of the form (ε_n, ∞) or $[\varepsilon_n, \infty)$. Since Δ is a strictly positive function, there are sets $\Delta^{-1}\left(\frac{1}{n}\right)$ which are not empty.

Moreover, for each $n \in \mathbf{N}$ the following holds:

$$\Delta^{-1}\left(\frac{1}{n}\right) \subset \Delta^{-1}\left(\frac{1}{n+1}\right)$$

hence $\varepsilon_n > \varepsilon_{n+1}$. Set

$$\delta|_{(\varepsilon_{n+1}, \varepsilon_n]} = \frac{1}{n+1}.$$

By construction, δ is a nondecreasing step-function and for each $\varepsilon > 0$, $\delta(\varepsilon) \in \Delta(\varepsilon)$, i.e. δ is a selection for Δ.

Now, since the nondecreasing step function δ is constructed, it is easy to construct a continuous "lower" selection $\hat{\delta}$. Fig. 1 illustrates the idea of the construction.

It is clear that $\hat{\delta}$ is a piecewise linear function which attains the value $\frac{1}{n+1}$ at the point ε_n.

(c) *α-continuity.* Let X, Y be metric spaces with metrics d and ρ, respectively, and let $\alpha : X \to [0, +\infty]$ be a function. A map $f : X \to Y$ is said to be *α-continuous* if

$$\forall \varepsilon > 0 \quad \forall x \in X\ \exists \delta > 0 \text{ such that } (\forall x' \in X)\Big(d(x, x') < \delta$$

$$\Rightarrow \rho\big(f(x'), f(x)\big) < \alpha(x) + \varepsilon\Big).$$

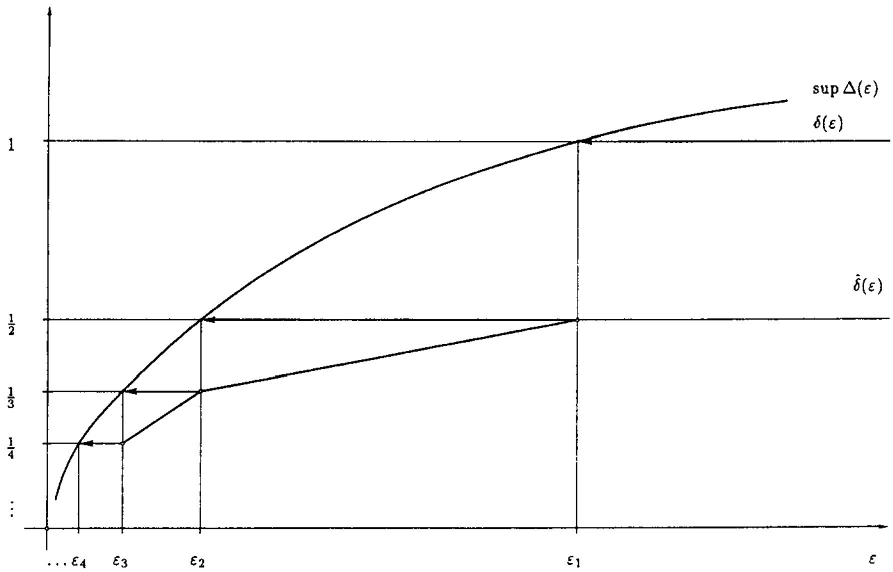

Fig. 1

The function α is called the *degree of discontinuity*. Denote by 0 and ∞ functions on X, identically equal to 0 and ∞, respectively. Then 0-continuous maps are exactly ordinary continuous maps and ∞-continuous maps are all maps. If α, β are degrees of discontinuity and if $\alpha(x) \leqq \beta(x)$ for all $x \in X$ then each α-continuous map is also a β-continuous map.

In particular, each ordinarily continuous map is α-continuous for an arbitrary degree α of discontinuity. But the converse does not hold. For example, let $x_0 \in X$, $y_0, y_1 \in Y$ be points such that $\rho(y_0, y_1) = \alpha(x_0) > 0$. Then the map

$$f(x) = \begin{cases} y_1, & x \neq x_0 \\ y_0, & x = x_0 \end{cases}$$

is α-continuous but not ordinarily continuous.

For a given degree α of discontinuity, let us introduce the predicate P_α of α-continuity, by the following formula:

$$P_\alpha(x, x', \varepsilon, f, d, \rho) = \Big(\rho\big(f(x'), f(x)\big) < \alpha(x) + \varepsilon\Big).$$

Since each (ordinarily) continuous map is also α-continuous, the predicate P_α is continuity-like (cf. Definition 1.2).

The following result is an immediate consequence of Theorem 1.4.

PROPOSITION 3.5. *If (Y, ρ) is a connected metric space with infinite diameter and if the degree α of discontinuity is not a lower semicontinuous* *function, then the modulus of α-continuity*

$$\Delta_{P_\alpha} : X \times \mathbf{R}^+ \times F_{P_\alpha} \times M_X \times M_Y \to \mathbf{R}^+$$

does not admit a continuous selection $\hat{\delta}$.

PROOF. Let α be not lower semicontinuous at a point $x_0 \in X$. Then, there exists a positive number ε_0 such that for each neighborhood $\mathcal{U}$ of the point x_0 there is a point $x \in \mathcal{U}$ such that $\alpha(x) < \alpha(x_0) - \varepsilon_0$. Since Y is connected and has infinite diameter, it is possible to choose points $y_0, y_1 \in Y$ such that $\rho(y_0, y_1) = \alpha(x_0)$. Let us introduce the map

$$f_0(x) = \begin{cases} y_1, & x \neq x_0 \\ y_0, & x = x_0. \end{cases}$$

Obviously, the map f_0 is α-continuous. We assert that for arbitrary metrics d_0, ρ_0 the point

$$(x_0, x_0, \varepsilon_0, f_0, d_0, \rho_0) \in \operatorname{diag} X \times X \times \mathbf{R}^+ \times F_{P_\alpha} \times M_X \times M_Y$$

is not an interior point in the set $P^+_{\alpha,0}$. The assertion holds since there is a point x in each neighborhood $\mathcal{U}$ of the point x_0 such that

$$\rho\big(f_0(x_0), f_0(x)\big) = \alpha(x_0) > \alpha(x) + \varepsilon_0$$

and therefore, the point $(x, x_0, \varepsilon_0, f_0, d_0, \rho_0)$ does not belong to the set P^+_α. By Theorem 1.4, the modulus Δ_{P_α} does not admit a continuous selection $\hat{\delta}$. □

CONJECTURE 3.6. *If the degree α of discontinuity is a lower semicontinuous function then the modulus Δ_{P_α} admits a continuous selection $\hat{\delta}$.*

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