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Topics in uniform continuity

This paper collects results and open problems concerning several classes of functions that generalize uniform continuity in various ways, including those metric spaces (generalizing Atsuji spaces) where all continuous functions have the property of being close to uniformly continuous.

Keywords: *Closure operator, uniform continuity, Atsuji space, UA space, straight space, hedgehog, magic set, locally connected space.*

Dedicated to the memory of Jan Pelant (1950-2005)

1. INTRODUCTION

The uniform continuity of maps between metric or uniform spaces determines a specific topic in general topology. By the end of the fifties and in the seventies the attention was concentrated on those spaces (called UC spaces, or Atsuji spaces) on which uniform continuity coincides with continuity. Of course, compact spaces are UC spaces, but there also exist non-compact UC spaces (e.g., the uniformly discrete ones).

In this survey we consider several aspects of uniform continuity and its relationship with continuity. We start with a discussion

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of the possibility to capture uniform continuity by means of the so-called closure operators [29]. The most relevant and motivating example of a closure operator is the usual Kuratowski closure K in the category **Top** of topological spaces and continuous maps. It is well known that one can describe the morphisms in **Top** (i.e., the continuous maps) in an equivalent way as the maps “compatible” with the Kuratowski closure (see §2.2). In this setting appear the *uniformly approachable* and the *weakly uniformly approachable* (briefly, UA and WUA , resp.) functions (see Definition 1). Section 3 compares the properties UA and WUA with the property of the u.c. functions which have distant fibers in an appropriate sense. In Section 4 we consider studies those spaces X on which every continuous function $X \rightarrow \mathbb{R}$ is UA . This class contains the well-known *Atsugi* spaces, where every continuous function is u.c. [1, 2, 3, 4].

Section 5 deals with those metric spaces in which the uniform quasi-components of every closed subspace are closed. Every UA space is thin, but there exist complete thin spaces that are not UA . The main result of this section is a separation property of the complete thin spaces.

The last section is dedicated to “additivity”, which turns out to be quite a non-trivial question in the case of uniform continuity. More precisely, we discuss here the *straight* spaces: these are the metric spaces on which a continuous function $X \rightarrow \mathbb{R}$ is uniformly continuous whenever the restrictions $f \upharpoonright_{F_1}$ and $f \upharpoonright_{F_2}$ on each member of an arbitrary closed binary cover $X = F_1 \cup F_2$ are uniformly continuous.

At the end of the paper we collect most of open problems and question (although some of them can be found in the main text).

We did not include in this paper several related issues. One of them, *magic sets*, is a topic that appeared in connection with UA functions, even though it has no apparent connection to uniform continuity. The reader can see [15, 16, 17, 6, 7, 21] for more on this topic.

We are dedicating this survey to the memory of the outstanding topologist and our good friend Jan Pelant, who actively worked on this topic and contributed the most relevant results in this area.

1.1. Notation and terminology. A topological space X is called *hereditarily disconnected* if all connected components of X are trivial, while X is called *totally disconnected*, if all quasi components of X are trivial [32]. The closure of a subset Y of a topological space X will be denoted by $cl(Y)$ or \bar{Y} . All topological spaces considered in this paper are assumed to be Tychonoff. A topological space is said to be *Baire* if it satisfies the Baire Category Theorem, i.e., if every meager subset of X has empty interior. A *Cantor set* is a nonvoid zero-dimensional compact metrizable space with no isolated points, i.e., a homeomorphic copy of the Cantor middle thirds set. If X is a topological space, we write $C(X)$ for the family of all continuous real-valued functions on X , and $C_n(X)$ for the sets of continuous nowhere constant real-valued functions on X . (To say that a continuous function $f: X \rightarrow \mathbb{R}$ is nowhere constant is equivalent to saying that $f^{-1}(y)$ is nowhere dense for each $y \in \mathbb{R}$.)

A metric space is said to be *uniformly locally connected* (shortly, *ULC*) [34, 3-2], if for every $\varepsilon > 0$ there is $\delta > 0$ such that any two points at distance $< \delta$ lie in a connected set of diameter $< \varepsilon$. In other words, close points can be connected by small connected sets. For example, convex subsets of \mathbb{R}^m (or any Banach space) are uniformly locally connected.

2. UNIFORM CONTINUITY VS CONTINUITY

2.1. Global view on closure operators. Closure operators can be introduced in a quite general context [29]. The prominent examples that inspired this general notion were given by Isbell (in the category of semigroups, or more generally, categories of universal algebras) and Salbany (in the category **Top** of topological spaces and continuous maps). We briefly recall here the notion of a closure operator of **Top**, following [29, 30].

A *closure operator* of **Top** is a family $C = (c_X)_{X \in \mathbf{Top}}$ of maps $c_X : 2^X \rightarrow 2^X$ such that for every X in **Top**

- (i) $M \subseteq c_X(M)$ for all $M \in 2^X$;
- (ii) $M \subseteq M' \in 2^X \Rightarrow c_X(M) \subseteq c_X(M')$; and
- (iii) $f(c_X(M)) \subseteq c_Y(f(M))$ for all $f : X \rightarrow Y$ in **Top** and $M \in 2^X$.

A prominent example is the Kuratowski closure operator K . Every continuous function satisfies the “continuity” condition (iii) for every closure operator C . For a closure operator C of **Top** we say that the set map $f : X \rightarrow Y$ is *C-continuous*, if it satisfies (iii). It is easy to see that a map $f : X \rightarrow Y$ is continuous if and only if it is *K-continuous*. In other words, the morphisms in **Top** can be detected by a closure operator (as *K-continuous* maps).

Analogously, a closure operator of **Unif** can be defined as a family $C = (c_X)_{X \in \mathbf{Unif}}$ of maps $c_X : 2^X \rightarrow 2^X$ such that for every X in **Unif** items (i) and (ii) are satisfied, and

- (iii_u) $f(c_X(M)) \subseteq c_Y(f(M))$ for all $f : X \rightarrow Y$ in **Unif** and $M \in 2^X$.

We say that C is *additive (idempotent)* if the equality

$$c_X(M \cup N) = c_X(M) \cup c_X(N)$$

(resp., $c_X(c_X(M)) = c_X(M)$) always holds.

The Kuratowski closure operator K is a closure operator of **Unif**. Analogously, for a closure operator C of **Unif** one can say that the set map $f : X \rightarrow Y$ is *C-continuous*, if it satisfies (iii_u); f is said to be *totally continuous* if it is *C-continuous* for every closure operator C of **Unif**.

In the category **Top** of topological spaces a map is continuous if and only if (iii) is holds for $C = K$. Hence morphisms in **Top** are determined by the closure operator K . Can the same be said of **Unif**? This question was answered in the negative in [28]. We briefly sketch the proof here.

The spaces needed as tools are the uniformly discrete two-point space $D = \{0, 1\}$, the one-point compactification \mathbb{N}_∞ of the naturals \mathbb{N} equipped with its unique uniformity, and two uniformly close sequences, which are not topologically close

$$X_0 := \{(n, 1/n) \mid n \in \mathbb{N}\} \cup \{(n, -1/n) \mid n \in \mathbb{N}\} \subseteq \mathbb{R}^2.$$

Set

$$M_a = \{(n, 1/n) \mid n \in \mathbb{N}\}, \quad M_b = \{(n, -1/n) \mid n \in \mathbb{N}\}$$

and consider the map $\pi : X_0 \rightarrow D$ defined by $\pi(a_n) = 0$ and $\pi(b_n) = 1$ for each n . Clearly π is continuous but not uniformly continuous, since the open disjoint binary cover $X_0 = M_a \cup M_b$ is not uniform.

Lemma 1. ([28]) *In the above notation:*

- (a) *the map π is C -continuous for every additive closure operator C of **Unif** such that*

$$\text{either } c(M_a) \setminus M_a \text{ or } c(M_b) \setminus M_a \text{ is finite;}$$

- (b) *if $c(M_a) \setminus M_a$ is infinite for a closure operator C of **Unif**, then for every metric Baire space $B \in \mathbf{Unif}$ without isolated points there exists a discontinuous map $f_B : \mathbb{N}_\infty \rightarrow B$ which is C -continuous.*

Theorem 1. ([28]) *Let C be an additive closure operator of **Unif**. Then either $\pi : X_0 \rightarrow D$ is C -continuous or for every metric Baire space without isolated points B there exists a discontinuous map $f_B : \mathbb{N}_\infty \rightarrow B$ which is C -continuous.*

This shows that for every additive closure operator C of **Unif** one can find a C -continuous map that is not uniformly continuous. Hence, a single closure operator of **Unif** cannot detect uniform continuity. This theorem also allows us to see which C -continuous maps fail to be u.c.

It is natural to expect that using more than just one closure operator things may change. We shall see now that this is not the case. Even a totally continuous map, which satisfies (iii) for every

closure operator C , is not necessarily uniformly continuous. The tool to achieve this result is the following notion introduced in [28] and [6]:

Definition 1. ([6, 28]) *We say that a map $f \in C(X, Y)$ is **UA (uniformly approachable)**, whenever for every compact set $K \subseteq X$ and every set $M \subseteq X$, there exists a UC function $g \in C(X, Y)$ which coincides with f on K and satisfies $g(M) \subseteq f(M)$.*

*We then say that g is a (K, M) -**approximation of f** . If we require in the definition of UA that K consists of a single point we obtain the weaker notion **WUA (weakly UA)**.*

Clearly, UA implies WUA.

It was shown in [28] that \mathbb{R} with the natural uniformity has the property that every continuous selfmap is uniformly approachable:

Example 10. *Every $f \in C(\mathbb{R})$ is UA. Indeed, let $K = [-n, n]$ and let $M \subseteq \mathbb{R}$ be an arbitrary non-empty set. Pick any $m_1 \in M \cap (-\infty, -n]$ if this set is non-empty, otherwise take $m_1 = -n$. Choose $m_2 \in M$ analogously. Then the function $g : \mathbb{R} \rightarrow \mathbb{R}$ defined by*

$$g(x) = \begin{cases} f(m_1), & \text{if } x \leq m_1 \\ f(x), & \text{if } m_1 \leq x \leq m_2 \\ f(m_2), & \text{if } x \geq m_2 \end{cases}$$

is a (K, M) -approximation of f .

Since “uniformly approachable” implies “totally continuous” and $f(x) = x^2$ is not uniformly continuous it follows that uniform continuity is not detected even by all closure operators in **Unif**. However, Burke noticed [6, Example 3.3] that there are continuous non-WUA functions on \mathbb{R}^2 (in fact, $f : \mathbb{R}^2 \rightarrow \mathbb{R}$, $f(x, y) = xy$, is such a function, see Example 5).

The next theorem easily follows from the definitions:

Theorem 2. ([28]) *Every WUA function is totally continuous.*

2.2. Local view on closure operators. Every additive and idempotent closure operator of **Top** or **Unif** defines a topology on the underlying set of the space. In this sense, the use of topologies that make certain maps (uniformly) continuous in the sequel can be also viewed as a local use of idempotent additive closure operators (i.e., on a single space or on a single pair of spaces, without the axiom (iii)).

2.2.1. Topologies τ on \mathbb{R} that make a given class of functions $\mathcal{F} \subseteq \mathbb{R}^{\mathbb{R}}$ coincide with $C((\mathbb{R}, \tau), (\mathbb{R}, \tau))$. In the sequel X will be a metric space. Following [18] we say that a class \mathcal{F} of functions from X to Y can be topologized if there exist topologies τ_1 on X and τ_2 on Y such that \mathcal{F} coincides with the class of all continuous functions from (X, τ_1) to (Y, τ_2) . The paper [18] gives conditions which under GCH (generalized continuum hypothesis) imply that \mathcal{F} can be topologized. In particular, it is shown that (assuming GCH) there exists a connected Hausdorff topology τ on the real line such that the class of all continuous functions in τ coincides with the class of all linear functions. A similar theorem is valid for the class of all polynomials, all analytic functions, and all harmonic functions. On the other hand, the classes of derivatives, C^∞ , differentiable, or Darboux functions cannot be topologized.

2.2.2. Characterization of uniform continuity as a simple continuity w.r.t. appropriate topologies. We recall here the work of Burke [14] on characterization of uniform continuity as a simple continuity w.r.t. appropriate topologies (or **locally** defined closure operators in the above sense). Within this setting, the problem becomes the question of determining which metric spaces X and Y are such that the uniformly continuous maps $f: X \rightarrow Y$ are precisely the continuous maps between (X, τ_1) and (Y, τ_2) for some new topologies τ_1 and τ_2 on X and Y , respectively.

Theorem 3. *There exist a connected closed subset X of the plane, a homeomorphism $h: X \rightarrow X$, and a connected Polish topology τ on X such that the continuous self-maps of X are precisely the*

maps h^n ($n \in \mathbb{Z}$) and the constant maps, while the continuous self-maps of (X, τ) as well as the uniformly continuous self-maps of X are precisely the maps h^n ($n \leq 0$) and the constant maps.

3. FUNCTIONS WITH DISTANT FIBERS AND UNIFORM CONTINUITY

Definition 2. We say that $f \in C(X)$ has **distant fibers** (briefly, **DF**) if any two distinct fibers $f^{-1}(x)$, $f^{-1}(y)$ of f are at some positive distance.

It is curious to note that this property generalizes two *antipodal* properties of a function:

- f is constant (f has *one* big fiber)
- f has small fibers (e.g., one-to-one functions, or more generally, functions with compact fibers, or briefly, **KF**).

3.1. Uniform continuity coincides with DF for bounded functions $f : \mathbb{R}^m \rightarrow \mathbb{R}$. It is easy to see that *UC* implies *DF* for any function $f : X \rightarrow \mathbb{R}$. Indeed, if $d(f^{-1}(u), f^{-1}(v)) = 0$ for some $u \neq v$ in \mathbb{R} , then any pair of sequences x_n, y_n such that $f(x_n) = u$, $f(y_n) = v$ and $\lim_n d(x_n, y_n) = 0$ witness non-uniform continuity of f .

Let us verify that *DF* implies *UC* for bounded functions $f : \mathbb{R}^m \rightarrow [0, 1]$. Indeed, assume that the sequences x_n, y_n imply non uniform continuity of f with $d(x_n, y_n) \rightarrow 0$ and $|f(x_n) - f(y_n)| \geq \varepsilon$ for every n . Then boundedness of f yields that $f(x_n), f(y_n)$ can without loss of generality be assumed convergent, i.e., $(x_n) \rightarrow a$, $f(y_n) \rightarrow b$ for some $a \neq b$ in $[0, 1]$. Let I_n be the segment in \mathbb{R}^m joining x_n and y_n . Let $a < b$ and take $u, v \in [0, 1]$ with $a < v < u < b$. Then $f(x_n) \in [0, v)$ and $f(y_n) \in (u, 1]$ for sufficiently large n since $f(I_n)$ is an interval (being a connected set) containing $f(x_n)$ and $f(y_n)$. Then $u, v \in f(I_n)$ for sufficiently large n . Since $d(x_n, y_n) \rightarrow 0$, we conclude that $d(f^{-1}(u), f^{-1}(v)) = 0$, a contradiction.

In the argument above \mathbb{R}^m can be replaced by any space that is uniformly locally connected. This condition cannot be omitted, since the argument function on the circle minus a point is DF but not UC (indeed, the circle minus a point is not uniformly locally connected with respect to the metric induced by the plane).

Theorem 4. ([8, Theorem 3.7]) *A bounded function $f \in C(X)$ on a uniformly locally connected space X is u.c. if and only if it is DF.*

Boundedness was essential to prove that DF implies uniform continuity. Indeed, unbounded continuous functions $\mathbb{R} \rightarrow \mathbb{R}$ need not be u.c. even when they are *finite-to-one* (e.g., $x \mapsto x^2$) or even *injective* (e.g., $x \mapsto x^3$).

The next theorem says that uniform continuity of a bounded function $f \in C(X)$ is a property of its fibers. In this form the theorem permits one to remove the hypothesis “uniformly locally connected”.

Theorem 5. ([8, Theorem 3.10]) *Let (X, d) be a connected and locally connected metric space. Suppose that*

$$f, g \in C(X, [0, 1])$$

have the same family of fibers and that f is u.c. Then g is also u.c.

Nevertheless, boundedness cannot be removed, as the pair of functions $x \mapsto x, x \mapsto x^3$ on \mathbb{R} show. It is not clear what is the precise property of the fibres of $f \in C(X, [0, 1])$ which gives uniform continuity. Theorem 4 shows that it is precisely DF when the space X is uniformly locally connected.

How to remove boundedness

In order to remove boundedness we now consider a generalization of UC which coincides with UC for bounded functions.

We start with a notion which is stronger than DF. A function f is said to be **proper** (briefly, P) if the f -preimage of any compact

set is compact. Equivalently, f is a closed map with KF . Even though in general the implication $P \rightarrow KF$ cannot be inverted (e.g., $x \mapsto \arctan x$ in \mathbb{R}), one can prove that $P = KF$ for unbounded functions $\mathbb{R}^m \rightarrow \mathbb{R}$, $m > 1$. In particular, this holds for polynomial functions $\mathbb{R}^m \rightarrow \mathbb{R}$.

The property KF implies DF , but it is much stronger. Indeed, UC need not imply KF . This is why we introduce the auxiliary notion AP that presents a weakening of both the notion of proper function and that of UC function. We plan to show that $DF = AP$ for functions $f \in C(X)$ on a uniformly locally connected space X .

Definition 3. ([8]) $f \in C(X, Y)$ is said to be AP (**almost proper**) if f is u.c. on the f -preimage of every compact set.

Obviously, UC implies AP , whereas bounded AP functions are UC . The same argument given above to prove $UC \rightarrow DF$ also proves that AP implies DF . On the other hand, with the proof of Theorem 4 outlined above one can also show:

Lemma 2. ([8, Lemma 3.5]) $DF \rightarrow AP$ for functions $f \in C(X)$ on a uniformly locally connected space X . Hence, AP coincides with DF on uniformly locally connected spaces.

In this way we have achieved our goal by replacing UC with AP .

Next we discuss an alternative solution, based on a different idea of choosing instead of AP a class of functions close to UC in the sense of approximation, namely UA (and WUA).

Theorem 6. ([8, Theorem 3.15]) DF implies UA in uniformly locally connected spaces.

The proof is based on the notion of **truncation**, which was implicit in Example 10. Now we define a different kind of truncation (for the general definition see Definition 6). For a function $f : X \rightarrow \mathbb{R}$ and real numbers $a \leq b$ define the (a, b) -**truncation** as follows:

$$f_{(a,b)}(x) = \begin{cases} f(x), & \text{if } f(x) \in [a, b], \\ a, & \text{if } f(x) \leq a, \\ b & \text{if } f(x) \geq b. \end{cases}$$

One proves that $f \in DF$ implies $f_{(a,b)} \in DF$, hence $f_{(a,b)} \in UC$ since it is bounded. Now, if K is a compact set and $a, b \in \mathbb{R}$ are chosen such that $f(K) \subseteq [a, b]$, then $f_{(a,b)}$ is a (K, M) -approximation of f if one takes additional care about $g(M) \subseteq f(M)$ as in Example 1.

The above implication cannot be inverted, as the next example shows.

Example 11. *The function $f(x) = \sin x^2$ is not DF, since any two fibers of f are at distance 0. Nevertheless, according to Example 10, $f(x)$ is UA.*

This suggests that the condition DF is too strong. We shall consider an appropriate weaker version below.

Remark 1. *Note that the (a, b) -truncation is different from the truncation g defined in Example 10 in the case $X = \mathbb{R}$. It can easily be shown that if $f([m_1, m_2]) = [a, b]$, then g is a truncation of $f_{(a,b)}$.*

3.2. Distant connected components of fibers. In order to invert the implication in Theorem 6 we need a weaker form of DF. To this end we take a closer look at the *connectedness* structure of the fibers. First of all we recall a notion for continuous maps in **Top**.

Definition 4. *A continuous maps $f : X \rightarrow Y$ between topological spaces is called*

- (a) *monotone if all fibers of f are connected;*
- (b) *light if all fibers of f are totally disconnected.*

The term in item (a) was motivated by the fact that for maps $\mathbb{R} \rightarrow \mathbb{R}$ one obtains the usual monotone maps. It is known that

every continuous map $f : X \rightarrow Y$ between topological spaces can be factorized as $f = l \circ m$, where $m : X \rightarrow Z$ is monotone and $l : Z \rightarrow Y$ is light. (Notice that this factorization for the constant map $f : X \rightarrow Y = \{y\}$ provides as Z exactly the space of connected components of X and $m : X \rightarrow Z$ is the quotient map having as fibers the connected components of X .)

Definition 5. We say that f has “**distant connected components of fibers**” (**DCF**) in the sense that any two components of distinct fibers are at positive distance.

Example 12. Let X be a metric space and $f : X \rightarrow \mathbb{R}$ a continuous map.

- (a) if f is monotone, then f is DCF if and only if f is DF.
- (b) If f is light, then it is DCF.

In particular, the function $f : (0, 1/\pi] \rightarrow \mathbb{R}$ defined by

$$f(x) = \sin 1/x,$$

as well as the function from Example 11, are DCF (being light), but any two non-empty fibers of f are at distance 0.

Theorem 7. ([8, Theorem 4.3]) *UA implies DCF for $f \in C(X)$ and arbitrary metric spaces X .*

In fact, if C_a and C_b are two connected components of fibres of f at distance 0, then for $K = \{a, b\}$ and $M = C_a \cup C_b$ the function f has no (K, M) -approximation.

Along with Example 10 this gives:

Corollary 1. *Every continuous real-valued function $\mathbb{R} \rightarrow \mathbb{R}$ is DCF.*

Actually, we shall see below that even *WUA* implies *DCF* for $f \in C(\mathbb{R}^m)$ (see Theorem 8), therefore,

$$DF \rightarrow UA \rightarrow WUA \rightarrow DCF \text{ for } f \in C(\mathbb{R}^m).$$

Hence all they coincide for polynomial functions (or functions with finitely many connected components of fibres). Let us put all

these implications in the following diagram, where the equivalence (1) for ULS spaces is given by Lemma 2 and the implication (2) for ULS spaces is given by Theorem 6. The implication (3) for \mathbb{R}^n follows from these two implications and the trivial implication $UA \rightarrow WUA$. The implication (4) will be proved in Theorem 8 below which gives a much stronger result.

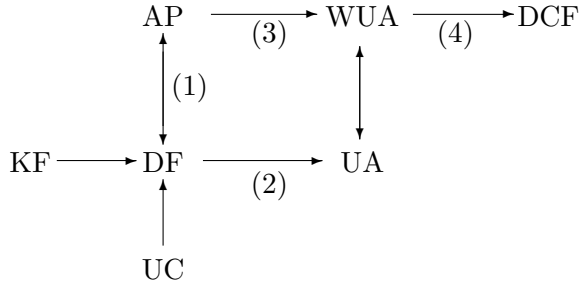


Diagram 1

We shall see below that (4) is not an equivalence. This motivated the introduction of the following weaker version of UA in [20]: a function $f: X \rightarrow \mathbb{R}$ is said to be UA_d (*densely uniformly approachable*) if it admits uniform $\langle K, M \rangle$ -approximations for every dense set M and for every compact set K . Analogously, one can define WUA_d .

Theorem 8. WUA_d coincides with DCF for $f \in C(\mathbb{R}^m)$.

The proof requires a new form of weak UC based on truncations:

Definition 6. $g \in C(X)$ is a **truncation** of $f \in C(X)$ if the space X can be partitioned in two parts $X = A \cup B$ so that $g = f$ on A and g is constant on each connected component of B (that

is, g must be constant on each connected component of $\{x \in X : f(x) \neq g(x)\}$.

This motivates the introduction of the class *TUA* of *truncation-UA* functions, that is, functions $f \in C(X)$ such that for every compact set $K \subseteq X$ there is a u.c. truncation g of f which coincides with f on K .

The following is easy to prove:

Theorem 9. ([8]) *TUA implies DCF on every locally connected space.*

Indeed, if C_a and C_b are two connected components of fibres of f at distance 0, then for $K = \{a, b\}$ the function f has no UC K -truncations.

The proof of Theorem 8 splits in three steps (see in Diagram 2 below)

Step 1:

([8, Cor.7.4]) $DCF \rightarrow TUA$ for $f \in C(\mathbb{R}^m)$;

Step 2:

([20, Th.3.1]) $TUA \rightarrow UA_d$ for $f \in C(\mathbb{R}^m)$;

Step 3:

([8],[20, Cor.4.2]) $WUA_d \rightarrow DCF$ for $f \in C(\mathbb{R}^m)$.

Step 1 together with Theorem 9 ensure the equivalence (4) for $f \in C(\mathbb{R}^m)$ in Diagram 2. Step 2, the trivial implication $UA_d \rightarrow WUA_d$, Step 3 and the equivalence (4) imply (5). This proves all four equivalences for $f \in C(\mathbb{R}^m)$ in the right square of Diagram 2.

The remaining three implications (1), (2) and (3) are trivial.

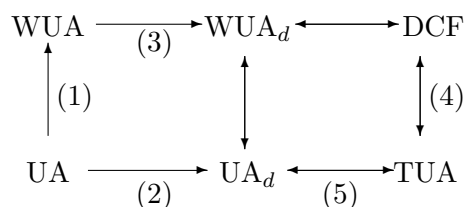


Diagram 2

In view of the four equivalences in the right square of Diagram 2, the next example shows that the implications (2) and (3) cannot be inverted.

Example 13. ([20, §5]) *In $C(\mathbb{R}^2)$, TUA does not imply WUA.*

It remains unclear whether the remaining last implication (1) of Diagram 2 can be inverted for $f \in C(\mathbb{R}^m)$ (see Problem 1).

4. UA SPACES

The main objective of this section are the **UA spaces** – spaces where every continuous function is UA. The first example of this kind is \mathbb{R} (Example 10). The motivation to introduce these spaces are the well known Atsuji spaces.

Here we recall some results from [6] and we anticipate some of the principal results from [9] which give further motivation for studying UA functions.

The next definition will be used in the sequel.

Definition 7. *Two subsets A, B of a topological space X are said to be **separated** if the closure of each of them does not meet the other (this is equivalent to saying that A and B are clopen in $A \cup B$). So X is connected if and only if it cannot be partitioned in two separated sets.*

A subset S of X **separates** the nonempty sets A and B if the complement of S can be partitioned in two separated sets, one of which contains A , the other contains B (see [39, §16, VI]).

4.1. UA spaces. Several criteria for UA-ness are given, and among them the following looks most spectacular:

Theorem 10. ([9]) *Let X be a UA space and let A, B be disjoint closed uniformly connected subsets of X . Then there is a collection $\{H_n \mid n \in \mathbb{N}\}$ of nonempty closed subsets of X such that for every n ,*

- (1) $H_{n+1} \subseteq H_n$;
- (2) H_n separates A and B ; and
- (3) H_n is contained in a finite union of balls of diameter $< 1/n$.

Actually, this property can be proved for a larger class of spaces discussed in §6, where a relevant property is obtained in the case when X is complete (Theorem 21).

The following construction which produces UA spaces from trees and compact sets placed at their vertices was given in [8].

Definition 8. *A metric space X is a tree of compact sets*

$$\{K_n : n \in \omega\}$$

if $X = \bigcup_{n \in \omega} K_n$ where each K_n is compact and

$$|K_{n+1} \cap \bigcup_{i \leq n} K_i| = 1.$$

- (a) *Given a subset $I \subseteq \omega$, we say that the subspace*

$$X_I = \bigcup_{n \in I} K_n$$

of $X = \bigcup_{i \in \omega} K_i$ is a subtree of X if for every n, m with $n < m$, if $n \in I$ and $K_n \cap K_m \neq \emptyset$, then $m \in I$.

- (b) A tree of compact sets $X = \bigcup_{i \in \omega} K_i$ is said to be tame if every K_i has an open neighbourhood which intersects only finitely many K_j 's and every two disjoint subtrees of X are at a distance > 0 .

It is easy to see that the circle minus a point can be represented as a tree of compact sets, but none of these trees is tame. The next theorem shows the reason for that (the circle minus a point is not a UA space).

Theorem 11. [9] *If $X = \bigcup_{i \in \omega} K_i$ is a tame tree of compact sets $\{K_n : n \in \omega\}$, then X is UA.*

Examples of tame trees are given in Figure 1 (see ladders B and C).

4.2. Non-UA spaces: Hedgehogs and some necessary conditions. Let α be a cardinal. In the sequel H_α denotes the hedgehog with α spikes (see [32, Example 4.1.5], note that H_a is separable if and only if $a = \omega$).

Recall the definition of the cardinal \mathfrak{b} as the minimal cardinality of an unbounded family of functions $f : \omega \rightarrow \omega$ with respect to the partial preorder $f \leq^* g$ if $f(n) \leq g(n)$ for all but finite number $n \in \omega$ (see [31]). In ZFC $\omega_1 \leq \mathfrak{b} \leq 2^\omega$, and $\mathfrak{b} = 2^\omega$ consistently (for example under MA or CH, see [31] for more detail).

Surprisingly, one has the following independency result: ZFC cannot decide whether the smallest non-separable hedgehog H_{ω_1} is UA. More precisely, the following holds:

Theorem 12. [9] *Let α be a cardinal. If $\alpha < \mathfrak{b}$ then H_α is UA, whereas if $\alpha \geq \mathfrak{b}$ then H_α is not even WUA.*

In particular, under CH the space H_{ω_1} is not UA, while in models of ZFC where $\neg CH \& MA$ holds, one has $\omega_1 < \mathfrak{b}$, so H_{ω_1} is UA.

We show below that every space H_α is TUA (Corollary 2). Hence $TUA \not\rightarrow WUA$ for nonseparable spaces.

We also consider the following space which is more general than the hedgehog H_α of α spikes:

Definition 9. Let α be an infinite cardinal. A metric space (X, d) is called a hedgehog of compact sets $\{K_\lambda : \lambda \in \alpha\}$ if $X = \bigcup_{\lambda \in \alpha} K_\lambda$ where each K_λ is compact with more than one point and there exists $p \in X$ such that

- (1) $K_\lambda \cap K_{\lambda'} = \{p\}$ for $\lambda \neq \lambda'$; and
- (2) for all $x, y \in X$ we have that

$$d(x, y) < \max\{d(x, p), d(y, p)\} \rightarrow \exists \lambda < \alpha [x, y \in K_\lambda].$$

Sometimes we prefer to say more precisely: a hedgehog of α compact sets.

Definition 10. Let X be a uniform space, let $f \in C(X)$ and let K be a compact subset of the space X . The minimal K -truncation f_K of f is defined as the $(\inf_K f, \sup_K f)$ -truncation of f .

Theorem 13. [9] Let X be a hedgehog of compact sets and K be a compact subset of X containing p . Then for every continuous function $f \in C(X)$ the $(\inf_K f, \sup_K f)$ -truncation of f is u.c.

This gives the following:

Corollary 2. Every hedgehog of compact sets is TUA.

Another source of UA spaces is given by the following:

Theorem 14. [9] Every uniformly zero-dimensional space is UA.

In [10] the Cantor set is characterized as the only compact metrizable space M such that each subspace of M is UA.

Theorem 15. [9] The only manifolds which are WUA are the compact ones and the real line.

Now we see that a metric space having a continuous function which is not uniformly continuous, necessarily also has a bounded uniformly approachable function that is not uniformly continuous. Hence, in some sense, UA is “closer to continuity than to uniform continuity”.

Theorem 16. [9] *A metric space X is UC if and only if every bounded uniformly approachable function is uniformly continuous, so that a space X with $C_u(X) = C_{ua}(X)$ is necessarily a UC space.*

In order to get a necessary condition for being a WUA space, the following notion was proposed in [6]:

Definition 11. *Let X be a uniform space. A family of pseudo-hyperbolas in X is given by a countable family $\{H_n\}$ of disjoint subsets of X such that for every $n \in \mathbb{N}$:*

- (1) H_n is closed and uniformly connected;
- (2) $H_n \cup H_{n+1}$ is uniformly connected;
- (3) $H_n \cap \overline{\bigcup_{m>n} H_m} = \emptyset$; and
- (4) the set $H = \bigcup_n H_n$ is not closed in X .

This notion was inspired by the following example due to Burke.

Example 14. *A family of pseudo-hyperbolas in \mathbb{R}^2 is given by the sets $H_n = \{(x, y) : (xy)^{-1} = n\}$.*

Theorem 17. ([6]) *If a normal uniform space X has a family of pseudo-hyperbolas, then X is not WUA.*

Theorem 18. ([6]) *Let X be a separable uniform space and suppose that there exists $f \in C(X)$ with countable fibers without non-constant uniformly continuous truncations. Then X is not WUA.*

Example 15. *Now we give three non-WUA examples of subsets X_1, X_2 and X_3 of \mathbb{R}^2 that contain no pseudo-hyperbolas. To prove that they are not WUA one can apply Theorem 18. So it is necessary to find, in each case, a continuous function with countable fibers and without non-constant uniformly continuous truncations.*

- (a) *The space X_1 is the unit circle minus a non-empty finite set. So X_1 can be identified with a cofinite subset of the set of complex numbers $e^{i\theta}$ with $0 < \theta < 2\pi$. Define*

$$f: X_1 \rightarrow \mathbb{R}, \quad f(e^{i\theta}) = \theta.$$

Then f is non-WUA. It is easy to see that X_1 contains no pseudo-hyperbolas.

(b) The space X_2 consists of the union of the two hyperbolas

$$H_1 = \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0, xy = 1\},$$

$$H_2 = \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0, xy = 2\}.$$

Obviously, X_2 contains no pseudo-hyperbolas. Define

$$f: X_2 \rightarrow \mathbb{R}$$

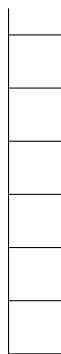
as follows. If $(x, y) \in H_1$, then set $f(x, y) = e^x$.

If $(x, y) \in H_2$, then set $f(x, y) = -e^{-x}$. It is easy to see that this works.

(c) Let X_3 be the space from Diagram 3. It contains no pseudo-hyperbolas. Define $f: X_3 \rightarrow \mathbb{R}$ by identifying X_3 with the subspace of \mathbb{R}^2 consisting of the union of the two vertical axes $x = -1$ and $x = 1$, together with the horizontal segments

$$I_n = \{(x, n) \in \mathbb{R}^2 : -1 \leq x \leq 1\}, \quad n \in \mathbb{N}.$$

Let $f(-1, y) = -y$, $f(1, y) = y$. This defines f on the two axes of the ladder. On each horizontal segment I_n , f is linear. This uniquely defines f since we have already defined f on the extrema of the horizontal segments I_n . f is pseudo-monotone, so each truncation of f is an (a, b) -truncation. Any such non-constant truncation is not uniformly continuous. Since f has countable fibers, Theorem 18 applies and thus X_3 is not WUA.

Diagram 3: A subset of \mathbb{R}^2

5. THIN SPACES

The class of topological spaces X having connected quasi components is closed under homotopy type and it contains all compact Hausdorff spaces (see [32, Theorem 6.1.23]) and every subset of the real line. Some sufficient conditions are given in [33] (in terms of existence of Vietoris continuous selections) and [22] (in terms of the quotient space ΔX in which each quasi-component is identified to a point), but an easily-stated description of this class does not seem to be available (see [22]). The situation is complicated even in the case when all connected components of X are trivial, i.e., when X is hereditarily disconnected. In these terms the question is to distinguish between hereditarily disconnected and totally disconnected spaces (examples to this effect go back to Knaster and Kuratowski [38]).

The connectedness of the quasi component (i.e., the coincidence of the quasi component and the connected component) in topological groups is also a rather hard question. Although a locally compact space does not need to have connected quasi components

[32, Example 6.1.24], all locally compact groups have this property. This is an easy consequence of the well known fact that the connected component of a locally compact group coincides with the intersection of all open subgroups of the group [35, Theorem 7.8]. All countably compact groups were shown to have this property, too ([26], see also [25, 27]). Many examples of pseudocompact group where this property strongly fails in different aspects, as well as further information on quasi components in topological groups, can be found in ([24, 25, 27], see also [45] for a planar group with non-connected quasi components).

Let us recall the definition of the *quasi component* $Q_x(X)$ of a point x in a topological space X . This is the set of all points $y \in X$ such that $f(y) = f(x)$ for every continuous function

$$f : X \rightarrow \{0, 1\},$$

where the doubleton $\{0, 1\}$ is discrete. Analogously, given a uniform space X and a point $x \in X$ the *uniform quasi component* of x consists of all points $y \in X$ such that $f(y) = f(x)$ for every uniformly continuous function $f : X \rightarrow \{0, 1\}$, where the doubleton $D = \{0, 1\}$ has the uniformly discrete structure (i.e., the diagonal of $D \times D$ is an entourage). We denote by $Q_x^u(X)$ the uniform quasi component of x .

In these terms we have the following inclusions

$$C_x(X) \subseteq Q_x(X) \subseteq Q_x^u(X), \quad (*)$$

where $C_x(X)$ denotes the connected component of x . Now we can introduce the relevant notion for this section:

Definition 12. *A uniform space X is said to be **thin** if for every closed subset Y of X and every $y \in Y$, the uniform quasi component of y in Y is connected.*

It is easy to see that all inclusions in (*) become equalities in the case of compact spaces. Hence compact spaces are thin. This also follows from the more general property given in Theorem 20.

Definition 13. For three subsets A, B and S of a topological space X we say that S **cuts** between A and B if S intersects every connected set which meets both A and B . (If S is empty this means that there is no connected set which meets both A and B .)

If a set separates A and B (see Definition 12), then it also cuts between A and B , but the converse is false in general.

Definition 14. We say that a uniform space X has the **compact separation property** (briefly CSP), if for any two disjoint closed connected subspaces A and B there is a compact set K disjoint from A and B such that every neighbourhood of K disjoint from A and B separates A and B (consequently K intersects every closed connected set which meets both A and B , see Definition 7).

It is easy to see that every compact space has CSP since disjoint compact sets are always separated.

It was proved in [9, Lemma 3.2] that if a metric space X contains two disjoint closed sets H and K and a point $a \in H$ such that the uniform quasi component of a in $H \cup K$ intersects K , then X is neither thin nor UA . This yields the following corollary:

Corollary 3. Two disjoint closed uniformly connected subsets A, B of a thin metric space X are at positive distance.

The following notion is relevant to the description of thin spaces.

Definition 15. Given two distinct points a, b of a metric space X such that the uniformly connected component of a contains b , there exists for each n a finite set $L_n \subset X$ whose points form a $1/n$ -chain from a to b . We say that the sets L_n , together with a and b , form a **(discrete) garland**, if there is an open subset V of X which separates a and b and such that $V \cap \bigcup_n L_n$ is closed (and discrete).

One can give a characterization of the thin of metric spaces in terms of existence of garlands in the space.

Proposition 1. [9] For a metric space X the following conditions are equivalent:

- (a) X is thin;
- (b) X contains no garlands;
- (c) X contains no discrete garlands.

The main result of the paper [9] is the following:

Theorem 19. *Every complete thin metric space has CSP.*

A large source of thin spaces is provided by US spaces.

Theorem 20. ([9]) *Every UA metric space is thin.*

Theorem 19 follows from the following more precise result:

Theorem 21. *Let X be a complete thin metric space and let A, B be disjoint closed connected subsets of X . Then:*

- (1) *there is a compact set K such that each neighbourhood of K disjoint from $A \cup B$ separates A and B ;*
- (2) *hence K intersects every closed connected set which meets A and B ;*
- (3) *if X is also locally compact, there is a compact set K' which separates A and B .*

5.1. CSP vs thin and complete. The next examples show that the implications in Theorems 19 and 20 cannot be inverted.

Example 16. *There exist many examples of separable metric space with CSP which are not thin:*

- (i) *the circle minus a point (it has two closed connected subsets at distance zero, so it cannot be thin by Corollary 3);*
- (ii) *the rationals \mathbb{Q} (uniformly connected non-connected, hence not thin).*

None of the above examples is complete. Here is an example of a complete separable metric space with CSP which is not thin.

Example 17. *Let H_1 and H_2 be the branches of hyperbolas $\{(x, y) \in \mathbb{R}^2 : xy = 1\}$ and $\{(x, y) \in \mathbb{R}^2 : xy = 2\}$, respectively, contained in the first quadrant. Then the space $X = H_1 \cup H_2$ with*

the metric induced from \mathbb{R}^2 is a complete separable space. Since H_1 and H_2 are connected and at distance zero, it follows from Corollary 3 that X is not thin. On the other hand, the empty set separates the closed connected sets H_1 and H_2 . So if A and B are closed connected disjoint sets in X , it remains to consider only the case when both A and B are contained in the same component H_i ($i = 1, 2$). Now A and B can be separated by a point.

Theorem 19 can be given the following more general form. A metrizable space X with compatible metrics d_1, d_2 such that (X, d_1) is complete (i.e. X is Čech-complete) and (X, d_2) is thin admits also a compatible metric d such that (X, d) is complete and thin (namely, $d = \max\{d_1, d_2\}$). Hence every Čech-complete metrizable space that admits a compatible thin metric has CSP. This explains why the spaces in (i) above and Example 17 have CSP.

Although completeness was essentially used in the proof of Theorem 19, it is not clear whether it is in fact necessary, in other words:

Question 1. *Are there examples of thin spaces that do not have CSP? What about UA spaces?*

As the following example shows, neither thinness nor UA-ness is preserved by passing to completions, thus an immediate application of Theorem 19 (via passage to completions) cannot help attempts to answer Question 1.

Example 18. *There is a UA metric space whose completion is not thin (hence not UA). Let $X = \bigcup_{n \in \mathbb{N}} \{1/n\} \times I$, where I is the unit interval $[0, 1] \subset \mathbf{R}$, let $a = (0, 0)$, $b = (0, 1)$ and $Y = X \cup \{a, b\}$. We put on Y the following metric. The distance between two points (x_1, y_1) and (x_2, y_2) is $|y_1 - y_2|$ if $x_1 = x_2$. Otherwise the distance is the minimum between*

$$y_1 + y_2 + |x_1 - x_2|$$

and

$$(1 - y_1) + (1 - y_2) + |x_1 - x_2|.$$

With this metric Y is the completion of X and the two points a, b are the limits for $n \rightarrow \infty$ of $(1/n, 0)$ and $(1/n, 1)$, respectively. The space Y is not thin since there is a garland consisting of a, b and $\langle L_n \mid n \in \mathbf{N} \rangle$ where L_n is a $1/n$ -chain between a and b in $\{1/n\} \times I$. The space X is UA since X is a union of a chain of compact sets, each attached to the next by at most one point (see [6, Theorem 11.4] and the introduction).

5.2. Thin does not imply UA for complete metric spaces.

We give an example of a complete connected thin metric space that is not UA .

Example 19. For any cardinal α the hedgehog $J(\alpha)$ is thin. Indeed, if $J(\alpha)$ were not thin, then by Proposition 1, it would contain a discrete garland $a, b, \langle L_n \mid n \in \mathbf{N} \rangle$. Let V be an open set separating a, b such that $V \cap \bigcup_n L_n$ is closed and discrete. The minimal connected set C containing a, b must non-trivially intersect V , so it contains an open interval I on one of the spikes. Now, whenever $1/n$ is less than the diameter of I , L_n must intersect I , so $V \cap \bigcup_n L_n$ has an accumulation point, which is a contradiction.

This gives the following immediate corollary of Theorem 12

Corollary 4. For every $\alpha \geq \mathfrak{b}$ the hedgehog $J(\alpha)$ is thin (so has the property CSP), but not UA .

The space $J(\alpha)$ is not separable for $\alpha > \omega$. On the other hand, $\mathfrak{b} > \omega$ ([31]), hence the above examples are not separable. According to Theorem 12 the hedgehogs $J(\alpha)$ are UA for all $\alpha < \mathfrak{b}$, so one cannot get in this way an example of a separable space with the above properties (see Question 4).

6. GLUING UNIFORMLY CONTINUOUS FUNCTIONS

It is well-known fact that a map $f : X \rightarrow Y$ between topological spaces is continuous whenever its restriction to each member of a locally finite closed cover of X is continuous. This section is dedicated to the analogue of this property for uniform continuity.

6.1. Straight spaces. In order to characterize the spaces where uniformly continuous functions can be glued as the continuous ones, the following definition was introduced in [10]:

Definition 16. *A space X is called **straight** if whenever X is the union of two closed sets, then $f \in C(X)$ is u.c. if and only if its restriction to each of the closed sets is u.c.*

Apparently, it would be more natural to ask about the possibility to glue together *finite* number of u.c. functions instead of just two. The following geometric criterion obtained in [10] justifies this choice.

Two subsets A and B of a uniform space X are called *U -distant* (or simply, *distant*) if there exists an entourage U such that

$$A[U] \cap B = \emptyset$$

(or equivalently, there exists an entourage U such that $A \cap B[U] = \emptyset$).

Definition 17. *Let (X, \mathcal{U}) be a uniform space. A pair C^+, C^- of closed sets of X is said to be u-placed if C_U^+ and C_U^- are distant for every entourage U , where*

$$C_U^+ = \{x \in C^+ \mid x \notin (C^+ \cap C^-)[U]\},$$

$$C_U^- = \{x \in C^- \mid x \notin (C^+ \cap C^-)[U]\}.$$

Remark 2. (a) *In the case of a metric space (X, d) we always consider the metric uniformity of X , so that in such a case a pair C^+, C^- of closed sets of X is u-placed if $d(C_\varepsilon^+, C_\varepsilon^-) > 0$ holds for every $\varepsilon > 0$, where*

$$C_\varepsilon^+ = \{x \in C^+ : d(x, C^+ \cap C^-) \geq \varepsilon\},$$

$$C_\varepsilon^- = \{x \in C^- : d(x, C^+ \cap C^-) \geq \varepsilon\}.$$

(b) *Note that $C_\varepsilon^+ = C^+$ and $C_\varepsilon^- = C^-$ when $C^+ \cap C^- = \emptyset$ in Definition 17. Hence a partition $X = C^+ \cup C^-$ of X*

into clopen sets is u -placed if and only if C^+, C^- are uniformly clopen (a subset U of a uniform space X is uniformly clopen if the characteristic function $X \rightarrow \{0, 1\}$ of U is uniformly continuous where $\{0, 1\}$ is discrete).

Theorem 22. For a uniform space (X, \mathcal{U}) and a pair C^+, C^- of closed sets the following statements are equivalent:

- (1) the pair C^+, C^- is u -placed;
- (2) a continuous function $f : C^+ \cup C^- \rightarrow \mathbb{R}$ is $u.c.$ whenever $f|_{C^+}$ and $f|_{C^-}$ are $u.c.$
- (3) same as (2) with \mathbb{R} replaced by a general uniform space (M, \mathcal{V})

The next theorem extends the defining property of straight spaces to arbitrary finite products.

Theorem 23. [10] If a metric space X is straight and X can be written as a union of finitely many closed sets

$$C_1, \dots, C_n$$

it follows that $f \in C(X)$ is $u.c.$ if and only if each restriction $f|_{C_k}$ ($k = 1, 2, \dots, n$) of f is $u.c.$

Definition 18. Let X be a metric space. We say that X is *WULC*, if for every pair of sequences x_n, y_n in X with

$$d(x_n, y_n) \rightarrow 0$$

and such that the set $\{x_n\}$ is closed and discrete there exist a $n_0 \in \mathbb{N}$ and connected sets $I(x_n, y_n)$ containing x_n and y_n for every $n \geq n_0$ in such a way that the $\text{diam} I(x_n, y_n) \rightarrow 0$.

Proposition 2. Every *WULC* space is straight.

Proof. Assume X is the union of finitely many closed sets F_1, \dots, F_m and the restriction of a function $f \in C(X)$ to each of the closed

so that also the set $\{y_n\}$ is closed and discrete.

sets F_k is u.c. We have to check that f is u.c. Pick $\varepsilon > 0$ and assume that

$$|f(x_n) - f(y_n)| \geq 2\varepsilon \quad (*)$$

for some x_n, y_n such that $d(x_n, y_n) \rightarrow 0$. It is clear, that the sequence x_n cannot have an accumulation point x in X , since then some subsequence $x_{n_k} \rightarrow x$ and also $y_{n_k} \rightarrow x$. Now the continuity of f would imply $|f(x_{n_k}) - f(x)| \rightarrow 0$ and $|f(y_{n_k}) - f(x)| \rightarrow 0$. Consequently, $|f(x_{n_k}) - f(y_{n_k})| \rightarrow 0$ contrary to (*). Therefore, the double sequence x_n, y_n satisfies the condition (a) of Definition 18. Therefore, for large enough n we have a connected set I_n containing x_n, y_n such that $\text{diam} I_n < \delta$. We can choose $\delta > 0$ such that also $|f(x) - f(y)| < \varepsilon/m$ whenever x, y belong to the same closed set F_k and $d(x, y) < \delta$. Note that $f(I_n)$ is an interval with length $\geq 2 \cdot \varepsilon$ covered by m subsets $f(I_n \cap F_k)$, $k = 1, \dots, m$ each with diameter $\leq \varepsilon/m$. This leads to a contradiction since an interval of length $\geq 2 \cdot \varepsilon$ cannot be covered by m sets of diameter $\leq \varepsilon/m$. \square

As a corollary we obtain that ULC spaces are straight.

Theorem 24. *Let (X, d) be locally connected. Then (X, d) is straight if and only if it is uniformly locally connected.*

Theorem 25. *Let (X, d) be a totally disconnected metric space. Then X is straight if and only if X is UC.*

6.2. Stability properties of straight spaces. The next theorem shows that straightness spectacularly fails to be preserved under taking closed spaces.

Theorem 26. *For every metric space X with $\text{Ind}X = 0$ the following are equivalent:*

- (1) X is UC;
- (2) every closed subspace of X is straight;
- (3) whenever X can be written as a union of a locally finite family $\{C_i\}_{i \in I}$ of closed sets we have that $f \in C(X)$ is u.c. if and only if each restriction $f|_{C_i}$ of f , $i \in I$, is u.c.

The following notion is relevant for the description of the dense straight subspaces.

Definition 19. ([12]) *An extension $X \subseteq Y$ of topological spaces is called **tight** if for every closed binary cover*

$$X = F^+ \cup F^-$$

one has

$$\overline{F^+}^Y \cap \overline{F^-}^Y = \overline{F^+ \cap F^-}^Y. \quad (6.1)$$

With this notion one can characterize straightness of extensions.

Theorem 27. ([12]) *Let X, Y be metric spaces, $X \subseteq Y$ and let X be dense in Y . Then X is straight if and only if Y is straight and the extension $X \subseteq Y$ is tight.*

6.3. Products of straight spaces. Here we discuss preservation of straightness under products.

Nishijima and Yamada [40] proved the following

Theorem 28. ([40]) *Let X be a straight space. Then $X \times K$ is straight for each compact space K if and only if $X \times (\omega + 1)$ is straight.*

The next lemma easily follows from the definitions.

Lemma 3. ([13]) *A product $X \times Y$ is ULC if and only if both X and Y are ULC.*

The next proposition, proved in [13], plays a crucial role in the proof of Theorem 29:

Proposition 3. *If $X \times Y$ is straight, then X is ULC or Y is precompact.*

Theorem 29. ([13]) *The product $X \times Y$ of two metric spaces is straight if and only if both X and Y are straight and one of the following conditions holds:*

- (a) both X and Y are precompact;
- (b) both X and Y are ULC;

(c) one of the spaces is both precompact and ULC.

It turns out that the straightness of an infinite product of ULC spaces is related to connectedness:

Theorem 30. ([13]) *Let X_n be a ULC space for each $n \in \mathbb{N}$ and $X = \Pi_n X_n$.*

- (a) *X is ULC if and only if all but finitely many X_n are connected.*
- (b) *The following are equivalent:*
 - (b₁) *X is straight.*
 - (b₂) *either X is ULC or each X_n is precompact.*

This theorem completely settles the case of infinite powers of ULC space:

Corollary 5. *Let X be ULC. Then*

- (a) *X^ω is ULC if and only if X is connected;*
- (b) *X^ω straight if and only if X is either connected or precompact.*

The above results leave open the question about when infinite products of precompact straight spaces are still straight (see Questions 7 and 6).

7. QUESTIONS

Our first open problem is about the implication (1) in Diagram 2:

Problem 1. ([20, Problem 1.4]) *Does WUA imply UA in $C(\mathbb{R}^n)$? What about $C(\mathbb{R}^2)$?*

7.1. Questions on UA functions and UA spaces. A general question is to characterize the UA and WUA spaces and functions. We list below more specific questions ([6]).

- (1) Characterize the UA functions $f: \mathbb{R}^2 \rightarrow \mathbb{R}$.
- (2) Characterize the UA subsets of \mathbb{R} .

- (3) Characterize the topological spaces which admit a UA uniformity, and those which are UA under every uniformity compatible with their topology. Does the latter class of spaces also include the UC spaces?
- (4) Do WUA and UA coincide for connected spaces?
- (5) Suppose that a uniform space X has a dense UA subspace. Does it follow that X is UA ? (This fails for WUA according to Example 7.7 from [6].)
- (6) Let X be the pushout of two WUA spaces over a single point. Is X WUA ? (This holds for UA by Theorem 11.1 from [6].)
- (7) Suppose that every pseudo-monotone function $f \in C(X)$ is UA . Is then X a UA space?
- (8) Define $2-UA$ similarly as UA but with the set K of cardinality at most 2. Is then $2-UA$ equivalent to UA ?

7.2. Questions on thin spaces. The next question is related to Theorem 19:

Question 2. *Is it true that a complete thin uniform space has CSP? What about a complete UA uniform space?*

Our next question is about how much one needs the fact that **uniform** quasi components are connected.

Question 3. *Is it true that every complete metric space X such that every closed subspace of X has connected quasi components necessarily has CSP?*

Question 4. *Is it true that every (complete) metric thin separable space is UA ?*

7.3. Questions on straight spaces. Theorem 27 gives a criterion for straightness of a dense subspace Y of a straight space X in terms of properties of the embedding $Y \hookrightarrow X$ (namely, when X is a tight extension of Y). The analogue of this question for *closed* subspaces is somewhat unsatisfactory. We saw that uniform retracts, clopen subspaces, as well as direct summands, of straight

spaces are always straight, [9]. On the other hand, closed subspaces even of ULC spaces may fail to be straight (see [BDP2]). Another instance when a closed subspace of a straight space fails to be straight is given by the following fact proved in [10]: the spaces X in which *every* closed subspace is straight are precisely the UC spaces [10]. Hence every straight space that is not UC has closed non-straight subspaces. This motivates the following general

Problem 2. Find a sufficient condition ensuring that a closed subspace Y of a straight space X is straight.

Question 5. *Generalize the results on straight spaces from the category of metric spaces to the category of uniform spaces.*

The results from §6.3 describe when infinite products of ULC spaces are again ULC or straight. The case of precompact spaces is still open, so we start with the following still unsolved

Question 6. *Let X be a precompact straight space. Is the infinite power X^ω necessarily straight?*

More generally:

Question 7. *Let X_n be a precompact straight space for every $n \in \mathbb{N}$. Is the infinite product $\prod_n X_n$ necessarily straight?*

It is easy to see that a positive answer to this question is equivalent to a positive answer to item (b) of the following general question: (i.e., the version of Theorem 30 for products of *precompact* spaces):

Question 8. *Let the metric space Y_i be a tight extension of X_i for each $i \in \mathbb{N}$.*

- (a) *Is $\prod_i Y_i$ a tight extension of $\prod_i X_i$.*
- (b) *What about precompact metric spaces Y_i ?*

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