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L-fuzzy strongest postcondition predicate transformers as L-idempotent linear or affine operators between semimodules of monotonic predicates

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Abstract

For a completely distributive quantale L, L-fuzzy strongest postcondition predicate transformers are introduced, and it is shown that, under reasonable assumptions, they are linear or affine continuous mappings between continuous L-idempotent semimodules of L-fuzzy monotonic predicates.

Keywords: Monotonic predicate; Strongest postcondition; Linear operator; Idempotent semimodule

0. Introduction

Predicate transformers, which were introduced in the pioneering work of Dijkstra [6], are powerful tools for analyzing the total or partial correctness of computer programs. The main idea is that a final state after execution of a program depends on its initial state; hence there is an interdependency between validity of statements (predicates) about the initial and the final states. One can ask, e.g., what are minimal requirements on an initial state that ensure that the final state satisfies a certain condition. Then these requirements form the weakest precondition for the given condition. On the other hand, the most precise knowledge about an output of a program for an input, that satisfies some predicate, is the strongest postcondition for this predicate. Such "forward" and "backward" dependencies are called predicate transformers.

Things become more complicated because of randomness or/and non-determinism, which can arise from unpredictable influence, "angelic" or "demonic" (with the obvious connotations). For simplicity, assume first that only randomness is present, and a set S of possible states is finite. We mostly follow [16], but notation will partially vary. A *subprobabilistic distribution* $D : S \to [0, 1]$ guarantees that the probability of each state $s \in S$ is at least $D(s)$. Obviously it is required that $\sum_{s\in S} D(s) \le 1$, and $1 - \sum_{s\in S} D(s)$ "goes to" unspecified state of the system. We say that a subprobabilistic distribution D is *refined* by another subprobabilistic distribution D' on S (written $D \sqsubseteq D'$)

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if $D(s) \leq D'(s)$ for all $s \in S$; this means that D' offers more precise knowledge than D. This partial order makes the set $\bar{S}$ of all subprobabilistic distributions on S a complete lower semilattice, with the bottom element 0="no information".

A random variable $\alpha : S \to \mathbb{R}_+$ is called a *probabilistic predicate*, and $\alpha(s)$ can be treated as a degree of appropriateness of $s \in S$ for some purpose (the more, the better). In particular, if $\alpha(S) \subset \{0, 1\}$, then all elements of S are divided into "bad" and "good". For a subprobabilistic distribution D, the *expectation* $\int_D \alpha = \sum_{s\in S} D(s) \cdot \alpha(s)$ is a maximal expected degree guaranteed by D.

A *deterministic probabilistic program* $p : S \to \bar{S}$ sends each initial state $s \in S$ to a subprobabilistic distribution $p(s)$ of possible finite states, where the probability $1 - \sum_{s'\in S} p(s)(s')$ is related to unknown behavior of the program, in particular, to the cases when the program does not terminate. Similarly, a program $p' : S \to \bar{S}$ refines a program $p : S \to \bar{S}$ (written $p \sqsubseteq p'$) if $p(s) \sqsubseteq p'(s)$ for each initial state $s \in S$. If an initial probability distribution is partially described (estimated from below) by a subprobabilistic distribution $D \in \bar{S}$, then a probability of a final state $s' \in S$ is greater or equal than $D'(s') = \sum_{s\in S} D(s) \cdot p(s)(s')$. Therefore, for a probabilistic predicate $\beta : S \to \mathbb{R}_+$, the expectation after execution of the program has the best estimate from below

$$\int_{D'} \beta = \sum_{s,s'\in S} D(s) \cdot p(s)(s') \cdot \beta(s').$$

A predicate $\alpha : S \to \mathbb{R}_+$ is called a (*probabilistic*) *precondition* for β, and β then is a (*probabilistic*) *postcondition* for α, if for *each* initial subprobabilistic distribution $D \in \bar{S}$ and the respective final subprobabilistic distribution $D' \in \bar{S}$, we have $\int_D \alpha \leq \int_{D'} \beta$, i.e., the expected value $\varepsilon \geq 0$ of α guarantees that the expectation of β is also equal or greater than ε. It is easy to see that the *strongest* (i.e., the least) *postcondition* $sp(p)(\alpha)$ of α is determined with the formula

$$sp(p)(\alpha)(s') = \sum_{s\in S} \alpha(s) \cdot p(s)(s'), \quad s' \in S.$$

Observe that all probabilistic predicates on S form a cone, and the mapping $sp(p)$ is additive and positively uniform, i.e., preserves multiplication by non-negative numbers. In this paper we shall construct and investigate an analogue of this mapping. Similarly, for a given predicate $\beta \in \bar{S}$, a *weakest* (greatest) *precondition* $wp(p)(\beta)$ can be found. See [16] on how nondeterminism can be incorporated into this model by mapping each initial state not to a single distribution, but to a set of distributions.

This is also closely related to the notion of *approximate correctness* of a computer program [15]. Although a number that expresses "approximateness" can be also treated as degree of belief, the entire theory by Mingsheng Ying is based on probabilistic logic and well suited to study probabilistic programs. It is also focused more on uncertainty of assumptions and conclusions than on imprecision in description of input and output data, as one could expect based on the term "approximate". For example, the refinement index of two probabilistic predicates is defined as the belief probability to which one probabilistic predicate is refined by another. There are several parallels between this theory and what we are doing in the sequel.

This approach, however, has intrinsic restrictions: we assume that a system is sufficiently described with knowledge which states or random events (sets of states) are realized, or what are the probabilities of their realization. For a simple program, like the examples in [16], this assumption is realistic, but if, e.g., our program removes artifacts from a sufficiently large color image, then the state space S is too huge to apply the above apparatus. To reduce S, one can divide all possible images into a reasonable number of classes. Boundaries between these classes cannot be clear; therefore the predicates will not be tolerant to small changes in images. Next, careful study of probability distributions of the class of a possible output for a given class of an input image is a nontrivial task. Even if this goal is achieved, the respective predicate transformers describe *average* results, and say nothing about rare extreme cases, which may make the program unusable.

For such "huge-dimensional" cases we suggest to resign from the purely probabilistic approach and to decrease the "dimensionality" by allowing fuzzy predicates. The idea is to have less predicates, which may be "more or fewer" true, and their values for each possible portion of information about a system present the greatest known degrees of truth, certainty, precision, quality, etc., which we can reliably count for. For example, such a predicate can assign to each square part, with integer coordinates of the vertices, of a given image a numerical measure of its quality. Then an image is incompletely but efficiently described with a finite collection of numbers, which is considered to be the value of the predicate. Observe that two such collections can be incomparable, e.g., if two images are damaged in different places.

Hence, the considered predicates can attain values in sets which are only partially ordered, although fuzziness is most often expressed on a numeric scale, e.g., [0, 1].

From now on we shall talk about "truth values" of fuzzy predicates, but this term is used for the sake of convenience and does not restricts possible interpretations to fuzzy logic only, although it is also possible. We expect that all known semantics of fuzziness [2,8] can be applied; see the examples in the next section.

Fuzzy predicate transformers also have been studied mostly in [0, 1]-settings [3,4]. This paper is devoted to constructing and investigating L-fuzzy (where L is a suitable lattice) strongest postcondition predicate transformers that are determined by state transformers, i.e., by L-fuzzy knowledge about what we can expect (more precisely, what is guaranteed in the worst case) for each initial state of a system. We are interested in order and topological properties of predicate transformers. It will be shown that spaces of predicates are idempotent semimodules, which are analogues of vector spaces, and under certain (not very restrictive) conditions the strongest postcondition predicate transformers are linear or affine continuous mappings between these semimodules.

1. Semimodules of monotonic predicates

Throughout this paper, if f, g are functions with a common domain, α is a constant, and $*$ is a binary operation, then we denote by $f*g$, $\alpha*f$ and $f*\alpha$ the functions with the same domain obtained by pointwise application of the operation $*$ (provided it is defined for the corresponding values). In the sequel $\sup_p$ and $\inf_p$ for a family of functions with a common domain to a poset will denote the pointwise suprema and infima, respectively.

See [11] for basic definitions and facts on partially ordered sets, including continuous semilattices and lattices. Here we shall recall only notation and a few definitions. For a poset X, the same set, but with the reversed order, is denoted by X^{op}. An element *a approximates b* or is *way below b*, in a poset X, which is written as $a \ll b$, if, for each directed subset $C \subset X$ such that $b \le \sup C$, there is $c \in C$ such that $a \le c$. A poset X is called *continuous* if, for each $b \in X$, the set of all $a \ll b$ is directed and has b as its lowest upper bound. A poset is *directed complete* if each its non-empty *directed* subset has a least upper bound. A continuous directed complete poset is called a *domain*. A domain which is additionally a meet semilattice (a complete lattice) is called a *continuous semilattice* (respectively a *continuous lattice*).

The *Scott topology* on a poset X is the least topology such that all lower sets C that are closed under directed suprema are closed. The *lower topology* on X is the least topology such that the sets $\{a \in X | b \le a\}$ are closed for all $b \in X$. The join, i.e., the least topology that contains the Scott and the lower topologies, is called the *Lawson topology*.

In the sequel L will be a completely distributive lattice, i.e., a compact Hausdorff distributive Lawson lattice with its Lawson topology. A topological lattice (semilattice) is said to be Lawson if for each point it possesses a local base that consists of sublattices (respectively of subsemilattices). Note that the same is true for L^{op}. We denote by 0, 1, $\oplus$, and $\otimes$ the bottom element, the top element, the join, and the meet in L, respectively. The elements of this (arbitrary, but fixed throughout the paper) lattice will be used to express truth values. The operation $\oplus$ is the disjunction, but the conjunction does not necessarily coincide with $\otimes$. Although complete distributivity is a very strong requirement, a lot of important lattices fall into this class, e.g., all complete linearly ordered sets, including $I = [0, 1]$ or any other segment in $\mathbb{R}$, all finite distributive lattices, all products of completely distributive lattices, in particular, I^τ for all cardinals τ. In fact, a lattice is completely distributive if and only if it is order isomorphic to a complete sublattice of some I^τ.

We shall also use basic notions of denotational semantics of programming languages. Consider a state of a computational process or a system. All possible (probably incomplete) portions of information we can have about this state form a *domain of computation D* [9]. This set carries a partial order $\le$ which represents a hierarchy of information or knowledge: the more information an element contains (i.e., the more specific/restrictive it is), the higher it is. See [9] for more details, in particular, for an explanation why it is natural to demand that D is a domain, i.e., a continuous directed complete poset. In addition to this, it is also often required that there is a least element $0 \in D$ (no information at all), and that for all a and b in D there is a meet $a \wedge b$, which, e.g., can be (but not necessarily is) treated as "a or b is true".

Following [13], for a domain D we call elements of the set $[D \to L^{op}]^{op}$ *L-fuzzy monotonic predicates* on D (here $[A \to B]$ stands for the set of mappings from A to B that are Scott continuous, i.e., they preserve directed suprema). For $m \in [D \to L^{op}]^{op}$ and $a \in D$, we regard $m(a)$ as the truth value of a; hence it is required that $m(b) \le m(a)$ for all $a \le b$. The second^{op} means that we order fuzzy predicates pointwise, i.e., $m_1 \le m_2$ iff $m_1(a) \le m_2(a)$ in L (not in L^{op}!) for all $a \in D$. We denote $\underline{M}_{[L]}D = [D \to L^{op}]^{op}$, and, for a domain D with a bottom element, consider also the subset $M_{[L]}D \subset \underline{M}_{[L]}D$ of all *normalized* predicates that take $0 \in D$ (no information) to $1 \in L$ (complete truth). Observe that $M_{[L]}D$ is a complete sublattice of $\underline{M}_{[L]}D$.

Example 1.1. Let a system have a finite or countable state space S. Each subset $A \subset S$ is identified with it characteristic mapping $\chi_A : S \to \{0, 1\}$, which is a Boolean predicate "current state s is in A". A smaller subset A corresponds to more information; therefore the set D of all subsets of S is partially ordered by reverse inclusion. Then D is a continuous lattice, and the $\{0, 1\}$-fuzzy monotonic predicates on D are precisely χ_A for all $A \subset S$.

If the system changes its state randomly, then different schemes are possible. Generally, an incomplete probabilistic knowledge is a mapping $m : D \to [0, 1]$ such that for all $A \subset S$ the probability $P(A)$ is at least $m(A)$. Of course, $A \leq B$, i.e., $A \supset B$, implies $m(A) \geq m(B)$, and σ-additivity of probability requires that m sends the directed unions of subsets of S to the corresponding suprema in [0, 1]. Thus m is a [0, 1]-fuzzy monotonic predicate.

Observe that m may not necessarily be reduced to a collection of estimates for the probabilities of individual states $s \in S$. For example, if all that we know is $P(\{s_1, s_2\}) \geq 0, 5$, then the only subprobabilistic distribution that is surely less or equal than the actual distribution is trivial, i.e., zero for all states.

Of course, m can be determined by (sub)probabilistic distributions. Let an exact probability distribution be unknown, but one of n possible, which are bounded from below respectively by subprobabilistic distributions $P_1, P_2, \ldots, P_n \in \bar{S}$. The greatest guaranteed probability of a random event $A \in D$ is equal to $m(A) = \inf_{1 \leq i \leq n} \sum_{s \in A} P_i(s)$. Then m is a [0, 1]-valued fuzzy monotonic predicate, which "aggregates" all possible probability distributions in the assumption of "demonic" non-determinism.

Thus numeric fuzzy predicates can arise in purely probabilistic settings, with the semantics "truth value = guaranteed probability". Observe that the probability of S is always 1, hence the mentioned predicates may be considered normalized.

Example 1.2. Let an image be divided into n parts, and the quality of each of them can be rated in the scale $L = \{0, 1, \ldots, m\}$, e.g., 0="awful", 1="bad", ..., m="perfect". Then the state space is equal to $S = L^n$. The domain of computation D can also be put equal to L^n, and $d = (d_1, d_2, \ldots, d_n)$ will mean "the actual quality s_i of ith part is not worse than d_i for all $1 \leq i \leq n$". This implies that $(d_1, d_2, \ldots, d_n) \leq (d'_1, d'_2, \ldots, d'_n)$ in D if and only if $d_1 \leq d'_1$, $d_2 \leq d'_2, \ldots, d_n \leq d'_n$.

For each $q = (q_1, q_2, \ldots, q_n) \in L^n$, let the predicates $m_q, m'_q, m''_q : D \to L$ be defined by the formulae

$$m_q((d_1, d_2, \ldots, d_n)) = \max\{k \in L | d_i \geq q_i - (m - k) \text{ for all } i = 1, 2, \ldots, n\},$$

$$m'_q((d_1, d_2, \ldots, d_n)) = \max\{k \in L | d_i \geq \min\{k, q_i\} \text{ for all } i = 1, 2, \ldots, n\},$$

$$m''_q((d_1, d_2, \ldots, d_n)) = \max\{k \in L | \max\{d_i, m - k\} \geq q_i \text{ for all } i = 1, 2, \ldots, n\},$$

for all $(d_1, d_2, \ldots, d_n) \in S$. Then $m_q((d_1, d_2, \ldots, d_n))$ shows the worse relative loss of quality w.r.t. $(q_1, q_2, \ldots, q_n)$, $m'_q((d_1, d_2, \ldots, d_n))$ shows "below what degree" the quality of $(d_1, d_2, \ldots, d_n)$ is not worse than $(q_1, q_2, \ldots, q_n)$, and $m''_q((d_1, d_2, \ldots, d_n))$ shows "above what degree" the quality of $(d_1, d_2, \ldots, d_n)$ is not worse than $(q_1, q_2, \ldots, q_n)$. In all these cases the predicates compare the guaranteed quality of an input with a desired one. Thus we can construct a predicate like "the image is perfect at the center and at least good at the angles".

Moreover, we can rate parts of an image in several aspects, with separate scales $L_1, L_2, \ldots, L_r$ for each, then the resulting $L = L_1 \times L_2 \times \cdots \times L_r$ will be a finite distributive lattice, which is not linearly ordered.

It follows from [10, Theorem 4] (classified as "folklore knowledge" in [13]) that, for a domain D and a completely distributive lattice L, the set $[D \to L^{op}]$ is a completely distributive lattice as well. Hence, this is also valid for $\underline{M}_{[L]}D$ and (if D contains a least element) $M_{[L]}D$.

For an element $d_0 \in D$, we denote by $\eta_{[L]}D(d_0)$ the function $D \to L$ that sends each $d \in D$ to 1 if $d \leq d_0$ and to 0 otherwise. It is easy to see that $\eta_{[L]}D(d_0) \in M_{[L]}D \subset \underline{M}_{[L]}D$, and $\delta_L^D = \eta_{[L]}D(0)$ is a least element of $M_{[L]}D$.

Lemma 1.3. *For a domain D, the mapping $\eta_{[L]}D : D \to \underline{M}_{[L]}D$ is continuous w.r.t. the Scott topologies and w.r.t. the lower topologies. If D is a complete continuous semilattice, then $\eta_{[L]}D$ is an embedding w.r.t. the Scott topologies, the lower topologies, and the Lawson topologies.*

Proof. Obviously, $\eta_{[L]}D(d_1) \le \eta_{[L]}D(d_2)$ if and only if $d_1 \le d_2$. Observe also that $\eta_{[L]}D(d_0)$ is a least $m \in \underline{M}_{[L]}D$ such that $m(d_0) = 1$. If $\mathcal{D} \subset D$ is directed and $\sup \mathcal{D} = d_0$, then $\sup\{\eta_{[L]}D(d) | d \in \mathcal{D}\}$ is a least $m \in \underline{M}_{[L]}D$ such that $m \ge \eta_{[L]}D(d)$ for all $d \in \mathcal{D}$, which is equivalent to $m(d) = 1$ for all $d \in \mathcal{D}$. Since $m : D \to L^{op}$ is Scott continuous, i.e., it preserves directed suprema, which is, in turn, equivalent to $m(\sup \mathcal{D}) = m(d_0) = 1$. By the above such m is equal to $\eta_{[L]}D(d_0)$. Hence, $\eta_{[L]}D$ preserves directed suprema as well.

To show that $\eta_{[L]}D$ is lower continuous, it suffices to show that, for all $m \in \underline{M}_{[L]}D$, the set

$$\eta_{[L]}D^{-1}(\{m\}\uparrow) = \{d_0 \in D | \eta_{[L]}D(d_0) \ge m\}$$

is closed in the lower topology on D. The inequality $\eta_{[L]}D(d_0) \ge m$ means that $\eta_{[L]}D(d_0)(d) = 1$ for all $d \in D$ such that $m(d) \ne 0$; in other words, d_0 is an upper bound of the set $\{d \in D | m(d) \ne 0\}$. This implies that

$$\eta_{[L]}D^{-1}(\{m\}\uparrow) = \bigcap\{\{d\}\uparrow \subset D | m(d) \ne 0, d \in D\},$$

which is closed in the lower topology on D.

If D is a complete continuous semilattice, then it is compact Hausdorff in its Lawson topology; therefore, a continuous injective mapping from it to a compactum $\underline{M}_{[L]}D$ is an embedding. Due to the completeness of D, this implies that the isotone mapping $\eta_{[L]}D$ is also an embedding w.r.t. the Scott topologies and w.r.t. the lower topologies. □

Therefore, we consider D as a sub*dcpo* of $\underline{M}_{[L]}D$, and a complete continuous semilattice D is additionally a subspace of $\underline{M}_{[L]}D$ w.r.t. the Scott, the lower, and the Lawson topologies on the both sets.

Infima and finite suprema in the complete lattices $\underline{M}_{[L]}D$ and $M_{[L]}D$ of functions are taken pointwise, whereas arbitrary suprema are described by the following easy, but useful statement. For a function $f : D \to L$, let

$$f^u(b) = \inf\{f(a) | a \in D, a \ll b\} \quad \text{for all } b \in D.$$

Observe that f^u is always a monotonic predicate. Moreover [21, Lemma I.4]:

Lemma 1.4. *For an antitone function $f : D \to L$, the function f^u is the least monotonic predicate f' such that $f \le f'$ pointwise.*

Hence, for a family $\mathcal{F} \subset \underline{M}_{[L]}D$ (or $\mathcal{F} \subset M_{[L]}D$), we have $\inf \mathcal{F} = \inf_p \mathcal{F}$, $\sup \mathcal{F} = (\sup_p \mathcal{F})^u$. For finite $\mathcal{F}$, the latter^u can be dropped.

Lemma 1.5. *Let a set $\mathcal{F} \subset \underline{M}_{[L]}D$ (or $\mathcal{F} \subset M_{[L]}D$) be compact in the relative lower topology. Then $\sup_p \mathcal{F} \in \underline{M}_{[L]}D$ (resp. $\sup_p \mathcal{F} \in M_{[L]}D$); therefore $\sup \mathcal{F} = \sup_p \mathcal{F}$.*

Proof. Assume to the contrary, that there exists $a_0 \in D$ such that

$$\sup\{f(a) | f \in \mathcal{F}\} \ge \alpha \not\le \alpha_0 = \sup\{f(a_0) | f \in \mathcal{F}\}$$

for all $a \in D$, $a \ll a_0$. The complete distributivity of L implies that there is $\beta \in L$ such that $\beta \le \alpha$, $\beta \not\le \alpha_0$, and if $\Gamma \subset L$ satisfies $\sup \Gamma \ge \alpha$, then there is $\gamma \in \Gamma$, $\gamma \ge \beta$ (such β is said to be *way-way below* α, cf. [11]). The set

$$\mathcal{F}_a = \{f \in \mathcal{F} | f(a) \ge \beta\} = \{f \in \mathcal{F} | f \ge m\},$$

where

$$m(a') = \begin{cases} \beta, & a' \le a \\ 0, & a' \not\le a \end{cases} \quad \text{for } a' \in D$$

is closed in $\mathcal{F}$. The family $\{\mathcal{F}_a | a \ll a_0\}$ of nonempty sets is directed; therefore by compactness it has a common element $f_0 \in \mathcal{F}$, i.e., $f_0(a) \ge \beta$ for all $a \ll a_0$. Then by the Scott continuity of $f_0 : D \to L^{op}$ we obtain

$$\alpha_0 = \sup\{f(a_0) | f \in \mathcal{F}\} \ge f_0(a_0) \ge \beta,$$

which is a contradiction. □

We use notation $\bar{\oplus}$ and $\bar{\otimes}$ for, respectively, joins and meets both in $\underline{M}_{[L]}D$ and $M_{[L]}D$.

In the sequel we shall additionally require that L be a *unital quantale* [20], i.e., there exists an associative binary operation $*: L \times L \to L$ such that 1 is a two-sided unit and $*$ is infinitely distributive w.r.t. supremum in both variables, which is equivalent to being continuous w.r.t. the Scott topology on L. Observe that, for such $*$, its infinite distributivity w.r.t. infima also means continuity w.r.t. the Lawson topology on L. Recall that we treat $\oplus$ as a disjunction, and $*$ will be a (possibly noncommutative) conjunction in an L-valued fuzzy logic [12]. The Boolean case is obtained for $L = \{0, 1\}$, $\oplus = \vee$ and $* = \wedge$. On the other hand, let the finite linearly ordered set $L = \{0, 1, \ldots, m\}$ be used to express absolute and relative quality of input, certainty, precision, etc., cf. Example 1.2. Then the operations $i*j \equiv \min\{i, j\}$ and $i*j \equiv \max\{i + j - m, 0\}$ can be reasonable choices, which reflect the natural assumption that combination of two distorted, imprecise, or uncertain inputs produces an equally or more distorted, imprecise, or uncertain output.

Lemma 1.6. *For $\alpha \in L$, a predicate $m \in \underline{M}_{[L]}D$, and an antitone function $f: D \to L$, we have $m(b) \geq \alpha * f(b)$ (resp. $m(b) \geq f(b)*\alpha$) for all $b \in D$ if and only if $m(b) \geq \alpha * f^u(b)$ (resp. $m(b) \geq f^u(b)*\alpha$) for all $b \in D$.*

Proof. Since $f \leq f^u$, "if" is trivial. Assume that $m(b) \geq \alpha * f(b)$ for all $b \in D$. Then for all $a \in D$, $a \ll b$ the inequality $f^u(a) \geq f(b)$ implies $m(a) \geq \alpha * f^u(b)$. Putting $a \to b$, we obtain $m(b) \geq \alpha * f^u(b)$. $\square$

Remark 1.7. The latter statement can be expressed by the formulae

$$(\alpha * f)^u = (\alpha * f^u)^u, \quad (f * \alpha)^u = (f^u * \alpha)^u,$$

for each antitone function $f: D \to L$ and $\alpha \in L$. It is also easy to see that, for a family $\{f_i | i \in \mathcal{I}\}$ of antitone functions $D \to L$, the equality

$$\left(\sup_{i \in \mathcal{I}}{}_p\, f_i\right)^u = \left(\sup_{i \in \mathcal{I}}{}_p (f_i)^u\right)^u$$

is valid.

The operation $*$ induces binary operations $\bar{\odot}$ and $\bar{\circledast}$ on the posets $\underline{M}_{[L]}D$ and $M_{[L]}D$, which make them *L-idempotent compact Lawson semimodules* [19]. Recall that a (left idempotent) $(L, \oplus, *)$-semimodule [1] is a set X with operations $\bar{\oplus}: X \times X \to X$ and $\bar{*}: L \times X \to X$ such that for all $x, y, z \in X$, $\alpha, \beta \in L$:

(1) $x \bar{\oplus} y = y \bar{\oplus} x$;
(2) $(x \bar{\oplus} y) \bar{\oplus} z = x \bar{\oplus} (y \bar{\oplus} z)$;
(3) there is an (obviously unique) element $\bar{0} \in X$ such that $x \bar{\oplus} \bar{0} = x$ for all x;
(4) $\alpha \bar{*} (x \bar{\oplus} y) = (\alpha \bar{*} x) \bar{\oplus} (\alpha \bar{*} y)$, $(\alpha \oplus \beta) \bar{*} x = (\alpha \bar{*} x) \bar{\oplus} (\beta \bar{*} x)$;
(5) $(\alpha * \beta) \bar{*} x = \alpha \bar{*} (\beta \bar{*} x)$;
(6) $1 \bar{*} x = x$; and
(7) $0 \bar{*} x = \bar{0}$.

Observe that these axioms imply that $(X, \bar{\oplus})$ is an upper semilattice with a bottom element $\bar{0}$, and $\alpha \bar{*} \bar{0} = \bar{0}$ for all $\alpha \in L$. The operation $\bar{*}$ is isotone in both variables.

Hence, an $(L, \oplus, *)$-semimodule is an analogue of a vector space. Similarly, analogues exist for linear and affine mappings. A mapping $f: X \to Y$ between $(L, \oplus, *)$-semimodules is called *linear* if, for all $x_1, \ldots, x_n \in X$ and $\alpha_1, \ldots, \alpha_n \in L$, the equality

$$f(\alpha_1 \bar{*} x_1 \bar{\oplus} \ldots \bar{\oplus} \alpha_n \bar{*} x_n) = \alpha_1 \bar{*} f(x_1) \bar{\oplus} \ldots \bar{\oplus} \alpha_n \bar{*} f(x_n)$$

is valid. If the latter equality is ensured only whenever $\alpha_1 \oplus \cdots \oplus \alpha_n = 1$, then f is called *affine*. Observe that an affine mapping f preserves joins, i.e., $f(x_1 \bar{\oplus} x_2) = f(x_1) \bar{\oplus} f(x_2)$ for all $x_1, x_2 \in X$. An affine mapping is linear if and only if it preserves the least element.

We call a triple $(X, \bar{\oplus}, \bar{*})$ a *continuous* $(L, \oplus, *)$-semimodule [19] if $(X, \bar{\oplus}, \bar{*})$ is an $(L, \oplus, *)$-semimodule, X is a continuous (hence complete) lattice, and $\bar{*}: L \times X \to X$ is infinitely distributive w.r.t. all suprema in both variables.

Then X with its Lawson topology is a compact Hausdorff Lawson lower semilattice with a top element, and $\bar{*}$ is jointly continuous w.r.t. the Scott topologies on L and X.

For $m \in \underline{M}_{[L]}D$, we define $\alpha\bar{\odot}m$ to be a least predicate $m' : D \to L$ such that $\alpha * m(b) \le m'(b)$ for all $b \in D$, i.e., $\alpha\bar{\odot}m = (\alpha * m)^u$. Then

$$(\alpha\bar{\odot}m)(b) = \inf\{\alpha * m(a) | a \in D, a \ll b\}.$$

For $m \in M_{[L]}D$, we need to "adjust" the result

$$(\alpha\bar{\circledast}m)(b) = (\alpha\bar{\odot}m)(d)\bar{\oplus}\delta_L^D = \begin{cases} (\alpha\bar{\odot}m)(b), & b \neq 0; \\ 1, & b = 0. \end{cases}$$

Lemma 1.8. *For* $\alpha, \beta \in L$, $m \in \underline{M}_{[L]}D$

$$\alpha\bar{\odot}(\beta\bar{\odot}m) = (\alpha * \beta)\bar{\odot}m.$$

Proof. By Remark 1.7

$$\alpha\bar{\odot}(\beta\bar{\odot}m) = (\alpha * (\beta * m)^u)^u = (\alpha * (\beta * m))^u = (\alpha * \beta)\bar{\odot}m. \quad \square$$

Now the equality

$$\alpha\bar{\circledast}(\beta\bar{\circledast}m) = (\alpha * \beta)\bar{\circledast}m,$$

for all $\alpha, \beta \in L$, $m \in M_{[L]}D$ is immediate. Both operations $\bar{\odot}$ and $\bar{\circledast}$ are infinitely distributive w.r.t. supremum in the both arguments (because $*$ is such an operation); hence, both are lower semicontinuous. Using routine, but straightforward calculations ([19]; the same but in terms of hyperspaces in [18]) we obtain:

Proposition 1.9. *The triples* $(\underline{M}_{[L]}D, \bar{\oplus}, \bar{\odot})$ *and* $(M_{[L]}D, \bar{\oplus}, \bar{\circledast})$ *are continuous* $(L, \oplus, *)$-semimodules.

Remark 1.10. It is easy to see that, if $*$ is also infinitely distributive w.r.t. infimum, then $\alpha * m \in [D \to L^{op}]^{op}$ for all $\alpha \in L$, $m \in [D \to L^{op}]^{op}$. Therefore, in this case $\alpha\bar{\odot}m$ coincides with $\alpha * m$.

For two predicates $m_1, m_2 : D \to L$, their join (i.e., the argumentwise supremum) $m_1\bar{\oplus}m_2$ can be interpreted as disjunction: "m_1 or m_2". Multiplication of a predicate $m : D \to L$ by $\alpha \in L$ either does not change this predicate or makes it more "pessimistic", or, equivalently, more "demanding". Since the sets of L-fuzzy monotonic predicates are "vector-like" spaces, we can apply to them the tools of idempotent linear algebra and idempotent functional analysis, although these theories are rather limited and poor comparing to the "conventional" classical analogues. In particular, results of [19] allow:

- to approximate L-fuzzy monotonic predicates from below and from above with predicates that attain only finite sets of values;
- to study and approximate predicates with special properties, e.g., meet- and join-preserving; and
- to construct the predicate that is dual to a given one, if the latter expresses an undesirable property which have to be avoided, etc.

2. Strongest postcondition predicate transformers

We treat each mapping $m : D \to L$ as "it is known that, for each $d \in D$, its truth value is at least $m(d)$". Similarly, an arbitrary mapping $\varphi : D \to \underline{M}_{[L]}D'$ is interpreted as "if $a \in D$ is true, then the truth value of each $b \in D'$ is at least $\varphi(a)(b)$". Note that $\varphi(a)(b)$ is implicitly considered as a "conditional" truth value, i.e., if a is "partially true" at a degree $\ge \alpha$, then b is true at least at a degree $\alpha * \varphi(a)(b)$.

Hence, such a φ is an L-fuzzy *state transformer*. For a given φ, we say that $m : D \to L$ is a *precondition* and $m' : D' \to L$ is a *postcondition* for each other w.r.t. φ, if, for all $a \in D$ and $b \in D'$, the "guaranteed" truth value $m'(b)$ is greater or equal to $m(a) * \varphi(a)(b)$, i.e., to the result of *modus ponens*.

Obviously, for an antitone function $m : D \to L$, its *strongest* (*least*) postcondition $\underline{sp}(\varphi)(m)$ in $\underline{M}_{[L]}D'$ is determined by the equality

$$\underline{sp}(\varphi)(m)(b) = \inf\{\sup\{m(a)*\varphi(a)(b')|a \in D\}|b' \in D', b' \ll b\}, \quad b \in D'.$$

Again, if we restrict ourselves to normalized predicates, the strongest postcondition must be corrected

$$sp(\varphi)(m)(b) = \underline{sp}(\varphi)(m)(b)\bar{\oplus}\delta_L^D = \begin{cases} \underline{sp}(\varphi)(m)(b), & b \neq 0; \\ 1, & b = 0. \end{cases}$$

It is easy to see that, for all $d \in D$ and isotone $\varphi : D \to \underline{M}_{[L]}D'$, we have $\underline{sp}(\varphi)(\eta_{[L]}D(d)) = \varphi(d)$, hence $\underline{sp}(\varphi)$ is an isotone extension of φ. Similarly, for an isotone mapping $\varphi : D \to M_{[L]}D'$, the mapping $sp(\varphi)(\eta_{[L]}D(d))$ is an isotone extension as well. The mapping $\underline{sp}(\varphi)$ and $sp(\varphi)$ are called (L-fuzzy) *strongest postcondition predicate transformers* induced by the state transformer φ, and are analogues of crisp (i.e., Boolean) predicate transformers, which were introduced by Dijkstra [6]. Compare also with the *weakest precondition predicate transformers*, cf. [3,4]. Their L-valued "angelic" and "demonic" analogues were introduced and investigated in [5] by means of topology. The latter reference contains also an example of a security system, which analyzes security threats of different severities and nature and imposes security measures of the corresponding level. This is naturally expressed with elements of lattices; therefore, the authors propose to "consider possible definitions for lattice-valued predicate transformers". Here is another example.

Example 2.1. Assume that a program processes a sequence of n frames. The quality s_i of ith frame is rated in the scale $L = \{0, 1, \ldots, m\}$. The domain of computation is equal to $D = L^n$, and the meaning of $d = (d_1, d_2, \ldots, d_m)$ is "$s_1 \geq d_1, s_2 \geq d_2, \ldots, s_n \geq d_n$". The multiplication $i*j = \max\{i + j - m, 0\}$ is considered on L, making it a finite quantale. The truth value of $d = (d_1, d_2, \ldots, d_n)$ is defined as

$$\max\{k \in L|s_i \geq d_i*k \quad \text{for all } i = 1, 2, \ldots, n\}$$

(observe that it is $m_s(d)$ for $s = (s_1, s_2, \ldots, s_n)$, cf. Example 1.2). Assume that it is known that, if the quality of ith frame, $0 < i < n$, is $\geq k - 1$, and the quality of the two neighboring frames is $\geq k$, then, after the program execution, the quality of ith frame will be $\geq k$, for all $1 \leq k \leq m$. This information can be expressed via the state transformer $\varphi : D \to \underline{M}_{[L]}D$ that sends

$$s = (0, \ldots, \underset{i-1}{m}, \underset{i}{m-1}, \underset{i+1}{m}, \ldots, 0) \text{ for } 0 < i < n,$$

to m_q, where

$$q = (0, \ldots, \underset{i-1}{0}, \underset{i}{m}, \underset{i+1}{0}, \ldots, 0),$$

and all other $s \in D$ to the constant zero predicate. Similarly we can add the fact that the quality of each frame will not be worse than before, etc. The resulting predicate transformer $\underline{sp}(\varphi) : \underline{M}_{[L]}D \to \underline{M}_{[L]}D$ sends a known quality of the frames *before* the program run to the most guaranteed quality *after* its execution.

To simplify our exposition, we consider in this section not necessarily normalized monotonic predicates.

Lemma 2.2. *With respect to a Scott continuous mapping $\varphi : D \to \underline{M}_{[L]}D'$, a monotonic predicate $m' : D' \to L$ is a postcondition for an antitone function $m : D \to L$ if and only if m' is a postcondition for $m^u : D \to L$.*

Proof. Since $m \leq m^u$, "if" is immediate. Let $m'(b) \geq m(a)*\varphi(a)(b)$ for all $a \in D$, $b \in D'$. Then $m'(b) \geq m(a')*\varphi(a')(b) \geq m^u(a)*\varphi(a')(b)$ for all $a' \ll a$. This implies $m' \geq m^u(a)*\sup_{p\, a' \ll a}\varphi(a')$, therefore, by Lemma 1.6

$$m' \geq m^u(a)*\left(\sup_{p \atop a' \ll a} \varphi(a')\right)^u = m^u(a)*\sup_{a' \ll a} \varphi(a') = m^u(a)*\varphi(a'),$$

the last equality is due to the Scott continuity of φ. $\square$

Proposition 2.3. *Let φ be a mapping $D \to \underline{M}_{[L]}D'$. Then $\underline{sp}(\varphi) : \underline{M}_{[L]}D \to \underline{M}_{[L]}D'$ preserves joins (hence finite suprema). For an isotone φ, the mapping $\underline{sp}(\varphi)$ preserves all suprema if and only if φ is Scott continuous, i.e., preserves directed suprema.*

Proof. Let $m = m_1 \bar{\oplus} m_2$, for $m, m_1, m_2 \in \underline{M}_{[L]}D$. Then, for $m' \in \underline{M}_{[L]}D'$, $a \in D$, $b \in D'$, the inequality $m'(b) \geq (m_1 \bar{\oplus} m_2)(a) * \varphi(a)(b)$ is valid if and only if both $m'(b) \geq m_1(a) * \varphi(a)(b)$ and $m'(b) \geq m_2(a) * \varphi(a)(b)$ are satisfied. Therefore

$$\begin{aligned} &\min\{m' \in \underline{M}_{[L]}D' | m'(b) \geq (m_1 \bar{\oplus} m_2)(a) * \varphi(a)(b) \quad \text{for all } a \in D, b \in D'\} \\ &\quad = \min\{m' \in \underline{M}_{[L]}D' | m'(b) \geq m_1(a) * \varphi(a)(b) \quad \text{for all } a \in D, b \in D'\} \\ &\quad \bar{\oplus} \min\{m' \in \underline{M}_{[L]}D' | m'(b) \geq m_2(a) * \varphi(a)(b) \quad \text{for all } a \in D, b \in D'\}, \end{aligned}$$

i.e.,

$$\underline{sp}(\varphi)(m_1 \bar{\oplus} m_2) = \underline{sp}(\varphi)(m_1) \bar{\oplus} \underline{sp}(\varphi)(m_2).$$

Now let φ be isotone. If $\underline{sp}(\varphi)$ preserves all suprema, than it is Scott continuous, as well as $\varphi = \underline{sp}(\varphi) \circ \eta_{[L]}D'$. If φ is Scott continuous and $\{m_i | i \in \mathcal{I}\} \subset \underline{M}_{[L]}D$, then due to monotonicity

$$\underline{sp}(\varphi)\left(\sup_{i \in \mathcal{I}} m_i\right) \geq \sup_{i \in \mathcal{I}} \underline{sp}(\varphi)(m_i).$$

On the other hand, $\sup_{i \in \mathcal{I}} \underline{sp}(\varphi)(m_i)$ is a postcondition for all m_i; hence, by Lemma 2.2 for $(\sup_{p_{i \in \mathcal{I}}} m_i)^u = \sup_{i \in \mathcal{I}} m_i$. Therefore

$$\sup_{i \in \mathcal{I}} \underline{sp}(\varphi)(m_i) \geq \underline{sp}(\varphi)\left(\sup_{i \in \mathcal{I}} m_i\right),$$

and $\underline{sp}(\varphi)$ preserves all suprema. □

Unfortunately, an analogue of Proposition 2.3 for lower topologies is not valid, even if $*$ is infinitely distributive w.r.t. both suprema and infima.

Example 2.4. Let $D = \{0, 1, 1'\} \cup \{1 + (1/n) | n = 1, 2, 3, \ldots\}$ with the usual numeric order, except that $1'$ is an extra copy of 1, and 1 and $1'$ are incomparable. Each directed set in D has a greatest element; hence, D is a directed complete continuous poset. Thus, D is an incomplete continuous semilattice with a least element 0. All upper sets in D are lower closed and Scott open; therefore, all isotone mappings from D to any poset are continuous w.r.t. both the lower and the Scott topologies.

Also, let $L = D' = \{0, 1\}$; $* = \wedge$; and $\varphi : D \to \underline{M}_{[L]}D'$ be an isotone mapping defined as follows:

$$\varphi(d) = \begin{cases} \bar{0}, d \in \{0, 1, 1'\}, \\ \delta_L^{D'}, d \notin \{0, 1, 1'\}, \end{cases} \quad d \in D.$$

Then

$$\underline{sp}(\varphi)(m)(0) = \begin{cases} 1 \ \text{ if there is } d \in \left\{1 + \dfrac{1}{n} | n = 1, 2, 3, \ldots\right\}, \quad m(d) = 1, \\[2ex] 0 \ \text{ otherwise.} \end{cases}$$

Therefore, there is a greatest element m_1 in the complement of the preimage $\underline{sp}(\varphi)^{-1}(\{\delta_L^{D'}\}\uparrow)$ in $\underline{M}_{[L]}D$

$$m_1(d) = \begin{cases} 1, d \in \{0, 1, 1'\}, \\ 0, d \notin \{0, 1, 1'\}, \end{cases} \quad d \in D.$$

However, there are no minimal elements in the preimage itself; hence, it is not lower closed.

Thus $\underline{sp}(\varphi)(m)(0)$ is not lower continuous.

To obtain the required analogue, we must apply additional requirements.

Proposition 2.5. *Let D and D' be complete continuous semilattices, $\varphi : D \to \underline{M}_{[L]}D'$ an isotone mapping, and $* : L \times L \to L$ infinitely distributive also w.r.t. infimum in both variables. Then $\underline{sp}(\varphi)$ is lower continuous if and only if φ is lower continuous, and in this case $\underline{sp}(\varphi)$ is defined by a simpler formula*

$$\underline{sp}(\varphi)(m)(b) = \sup\{m(a)*\varphi(a)(b)|a \in D\}, b \in D'.$$

Proof. Recall that such an operation $* : L \times L \to L$ is continuous w.r.t. the lower and the Lawson topologies on L, while the previously required infinite distributivity w.r.t. supremum implies only the Scott continuity of φ. The semilattices D and D' with the Lawson topologies are compact Hausdorff topological semilattices.

Necessity is due to Lemma 1.3, because $\varphi = \underline{sp}(\varphi) \circ \eta_{[L]}D$, and $\eta_{[L]}D$ is lower continuous.

Sufficiency: The mapping that sends each $a \in D$ to $m(a)*\varphi(a) \in \underline{M}_{[L]}D$ is continuous w.r.t. the Lawson topology on D and the lower topology on $\underline{M}_{[L]}D$. Hence, the set $\{m(a)*\varphi(a)|a \in D\}$ is compact in the lower topology on $\underline{M}_{[L]}D$. By Lemma 1.5 its pointwise limit is in $\underline{M}_{[L]}D$; therefore, it coincides with $\underline{sp}(\varphi)(m)$.

Let $m \in \underline{M}_{[L]}D \setminus \underline{sp}(\varphi)^{-1}(\{m'\}\uparrow)$, $m' \in \underline{M}_{[L]}D'$, then $\underline{sp}(\varphi)(m)(b) = \sup\{m(a)*\varphi(a)(b)|a \in D\} = \gamma \not\geq m'(b)$ for some $b \in D'$.

The set $\{(m(a), \varphi(a)(b))|a \in D\}$ is contained in the closed, therefore compact, lower set $\{(\alpha, \beta) \in L \times L|\alpha*\beta \leq \gamma\}$. The operation $*$ is isotone and Lawson continuous. Hence, there are $\alpha_1, \beta_1, \ldots, \alpha_n, \beta_n \in L$ such that the open set

$$U = (L \times L) \setminus (\{\alpha_1\}\uparrow \times \{\beta_1\}\uparrow \cup \ldots \cup \{\alpha_n\}\uparrow \times \{\beta_n\}\uparrow)$$

contains

$$\{(\alpha, \beta) \in L \times L|\alpha*\beta \leq \gamma\},$$

and $\sup\{\alpha*\beta|(\alpha, \beta) \in U\} = \gamma' \not\geq m'(b)$. By the above, for neither of $a \in D$ and $i = 1, \ldots, n$, the inequalities $m(a) \geq \alpha_i$ and $\varphi(a)(b) \geq \beta_i$ are valid simultaneously. The set

$$B_i = \{a \in D|\varphi(a)(b) \geq \beta_i\} = \{a \in D|\varphi(a) \geq \beta_i * \eta_{[L]}D'(b)\}$$

is closed w.r.t. the lower topology due to the continuity of φ. It has an empty intersection with the Scott closed set

$$A_i = \{a \in D|m(a) \geq \alpha_i\}.$$

By compactness, there is a finite collection $a_{i1}, \ldots, a_{ik_i} \in D$ such that the set

$$\{a \in D|a_{ij} \leq a \text{ for some } 1 \leq j \leq k_i\}$$

contains B_i and has an empty intersection with A_i. Then the set

$$V = \{c \in \underline{M}_{[L]}D|c \not\geq \alpha_i * \eta_{[L]}D(a_{ij}) \quad \text{for all } 1 \leq i \leq n, 1 \leq j \leq k_i\}$$

is an open neighborhood of m in the lower topology, and, if $c \in V$, then $c(a) \not\geq \alpha_i$ whenever $\varphi(a)(b) \geq \beta_i$, $1 \leq i \leq n$.

Therefore, if $c \in V$, then

$$\sup\{c(a)*\varphi(a)(b)|a \in D\} \leq \gamma' \not\geq m'(b),$$

hence, $\underline{sp}(\varphi)(c)(b) \not\geq m'(b)$, and all preimages $\underline{sp}(\varphi)^{-1}(\{m'\}\uparrow)$ are closed, which implies the required continuity of $\underline{sp}(\varphi)$. □

Proposition 2.6. *Let φ be a mapping $D \to \underline{M}_{[L]}D'$. If (a) φ is Scott continuous, or (b) $*$ is infinitely distributive w.r.t. infimum, then the mapping $\underline{sp}(\varphi) : \underline{M}_{[L]}D \to \underline{M}_{[L]}D'$ is linear.*

Proof. Join preservation is due to Proposition 2.3.

Let a mapping $\varphi : D \to \underline{M}_{[L]}D'$ be Scott continuous (a). Then

$$\underline{sp}(\varphi)(\alpha\bar{\odot}m) = \underline{sp}(\varphi)((\alpha*m)^u) \overset{\text{Lemma 2.2}}{=\!=\!=} \underline{sp}(\varphi)(\alpha*m)$$
$$= \left(\sup_{p\, a\in D} \alpha*m(a)*\varphi(a)\right)^u \overset{\text{Lemma 1.6}}{=\!=\!=} \left(\alpha*\left(\sup_{p\, a\in D} m(a)*\varphi(a)\right)^u\right)^u = \alpha\bar{\odot}\underline{sp}(\varphi)(m).$$

Assume (b). Then

$$\underline{sp}(\varphi)(\alpha\bar{\odot}m)(b) = \underline{sp}(\varphi)(\alpha*m)(b)$$
$$= \inf\{\sup\{\alpha*m(a)*\varphi(a)(b')|a \in D\}|b' \in D', b' \ll b\}$$
$$= \inf\{\alpha* \sup\{m(a)*\varphi(a)(b')|a \in D\}|b' \in D', b' \ll b\}$$
$$= \alpha* \inf\{\sup\{m(a)*\varphi(a)(b')|a \in D\}|b' \in D', b' \ll b\}$$
$$= \alpha\bar{\odot}\underline{sp}(\varphi)(m)(b), \quad \text{for all } m \in \underline{M}_{[L]}D, b \in D'. \qquad \square$$

Remark 2.7. In the presence of (a) or (b), the mapping $\underline{sp}(\varphi)$ can be characterized as the least linear mapping $\Phi : \underline{M}_{[L]}D \to \underline{M}_{[L]}D'$ such that $\Phi(\eta_{[L]}D(d)) = \varphi(d)$ for all $d \in D$.

Remark 2.8. *All* statements in this section have straightforward analogues for normalized predicates. The only significant distinction is that, if a mapping $\varphi : D \to M_{[L]}D'$ satisfies the conditions that are analogous to ones of Proposition 2.6, then the mapping $sp(\varphi) : M_{[L]}D \to M_{[L]}D'$ is affine, instead of linear. Proofs can be obtained *mutatis mutandis*, without any major changes.

3. Epilogue

We have shown that L-fuzzy strongest postcondition predicate transformers are related to L-idempotent linear or affine operators between continuous L-semimodules. Now it is possible to study linear and affine approximations of predicate transformers from above and from below. These approximations are related to attempts to describe a program behavior in a more economical way, dropping less important details.

It has been observed, e.g., by Doberkat [7] that monads and Kleisli composition [14] arise in description of combining several programs into a pipe and composing the respective predicate transformers. While, for probabilistic programs, these monads are based on (sub)probability measures, for non-probabilistic fuzzy semantics we propose to use monads of lattice-valued non-additive measures [17].

Treatment of L-fuzzy weakest precondition predicates transformers, similar to a proposed one for strongest precondition predicate transformers, as well as a demonstration that relations between these classes can be properly expressed in terms of category theory, will be the topic of our future publications. In particular, Galois connections [19] will be used to investigate compatibility of L-fuzzy knowledge and of nondeterministic programs.

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