Cantor set problems

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Introduction

A Cantor set is characterized as a topological space that is totally disconnected, perfect, compact and metric. Any two such spaces C_1 and C_2 are homeomorphic, but if C_1 and C_2 are subspaces of $\mathbb{R}^n$, $n \geq 3$, there may not be a homeomorphism of $\mathbb{R}^n$ to itself taking C_1 to C_2. In this case, C_1 and C_2 are said to be *inequivalent embeddings* of the Cantor set. There has been recent renewed attention to properties of embeddings of Cantor sets since these sets arise in the settings of dynamical systems, ergodic theory and group actions. The bibliography, while not complete, gives a sampling of the various mathematical areas where Cantor sets naturally arise.

A Cantor set C in $\mathbb{R}^n$ is *tame* if it is equivalent to the standard middle thirds Cantor set. If it is not tame, it is *wild*. A Cantor set C is *strongly homogeneously embedded* in $\mathbb{R}^n$ if every self homeomorphism of C extends to a self homeomorphism of $\mathbb{R}^n$. At the opposite extreme, a Cantor set C in $\mathbb{R}^n$ is *rigidly embedded* if the identity homeomorphism is the only self homeomorphism of C that extends to a homeomorphism of $\mathbb{R}^n$. A Cantor set C in $\mathbb{R}^n$ is *slippery* if for each Cantor set D in $\mathbb{R}^n$ and for each $\epsilon > 0$, there is a homeomorphism $h\colon \mathbb{R}^n \to \mathbb{R}^n$, within ϵ of the identity, with $h(C) \cap D = \emptyset$.

Željko [**28**] defines the genus of a Cantor set X in $\mathbb{R}^3$ and the local genus of points in X. A *defining sequence* for a Cantor set $X \subset \mathbb{R}^n$ is a sequence (M_i) of compact n-manifolds with boundary such that $M_{i+1} \subset \operatorname{int} M_i$ and $X = \bigcap_i M_i$. Let $\mathcal{D}(X)$ be the set of all defining sequences for X. For a disjoint union of handlebodies $M = \bigsqcup_{\lambda\in\Lambda} M_\lambda$, we define $g(M) = \sup\{\text{genus}(M_\lambda) : \lambda \in \Lambda\}$.

For any subset $A \subset X$, and for $(M_i) \in \mathcal{D}(X)$ we denote by M_i^A the union of those components of M_i which intersect A. The *genus of the Cantor set X with respect to the subset A*, $g_A(X) = \inf\{g_A(X;(M_i)) : (M_i) \in \mathcal{D}(X)\}$, where $g_A(X;(M_i)) = \sup\{g(M_i^A) : i \geq 0\}$. For $A = \{x\}$ we call the number $g_{\{x\}}(X)$ the *local genus of the Cantor set X at the point x* and denote it by $g_x(X)$. For $A = X$ we call the number $g_X(X)$ the *genus of the Cantor set X* and denote it by $g(X)$.

The problems

Antoine [**2**] produced the first example of a wild Cantor set in $\mathbb{R}^3$, the well-known Antoine's necklace. Blankinship [**6**] extended Antoine's construction to

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higher dimensions, producing wild Cantor sets in Euclidean spaces of dimensions ≥ 4. Daverman [**8**] produced an example of a strongly homogenously embedded Cantor set if $\mathbb{R}^n$ for $n \geq 5$. His example relied on decomposition theory results that only applied in high dimensions and on the existence of non simply connected homology spheres in dimensions ≥ 3.

1443? **Question 1.** *Is there a strongly homogeneously embedded wild Cantor set in $\mathbb{R}^3$ or $\mathbb{R}^4$, or are such sets necessarily tame?*

The Antoine construction can be carefully done with sufficiently many tori at each stage so as to produce wild Cantor sets that are geometrically self similar and are Lipschitz homogenously embedded in $\mathbb{R}^3$. See [**15, 12, 29**] for definitions and details. It is not clear that the Blankinship construction in higher dimensions can be done so as to produce geometrically self similar Cantor sets.

1444? **Question 2.** *Is there a geometrically self similar wild Cantor set in $\mathbb{R}^4$ or in higher dimensions?*

1445? **Question 3.** *Are there Lipschitz homogenously embedded wild Cantor sets in $\mathbb{R}^4$ or in higher dimensions?*

Rushing [**18**] produced examples in $\mathbb{R}^3$ of wild Cantor sets of each possible Hausdorff dimension. At the end of his paper, he stated that a modification of the Blankinship construction would allow similar results in higher dimensions. Because of the difficulty in producing a self similar Blankinship construction, it is not clear how the generalization to higher dimensions would work.

1446? **Question 4.** *Are there wild Cantor sets in $\mathbb{R}^n$, $n \geq 4$ of arbitrary possible Hausdorff dimension?*

DeGryse and Osborne [**11**] produced an example of a wild Cantor set in $\mathbb{R}^3$ with simply connected complement. Later, Skora [**20**] produced such Cantor sets using a different construction. Rigid wild Cantor sets in $\mathbb{R}^3$ and in higher dimensions were produced by Wright [**24**] using variations on the Antoine and Blankinship constructions. Garity, Repovš, and Željko [**13**] recently produced examples of rigid wild Cantor sets in $\mathbb{R}^3$ that also had simply connected complement. However the latter examples necessarily used tori of arbitrarily high genus in the construction.

1447? **Question 5.** *Is there a rigid Cantor set in $\mathbb{R}^3$ with simply connected complement that has local genus n or less at every point, for some fixed n?*

Bing–Whitehead Cantor sets are a generalization of the Cantor sets produced by DeGryse and Osborne. Ancel and Starbird [**1**] and later Wright [**26**] characterized which Bing–Whitehead constructions actually yield Cantor sets.

1448? **Question 6.** *Is there a modification of the Bing–Whitehead Cantor set construction that yields rigid Cantor sets with simply connected complements?*

1449? **Question 7.** *Are Bing–Whitehead Cantor sets with infinite differences in the number of Whitehead constructions inequivalently embedded?*

Sher in [**19**] showed that two equivalent Antoine Cantor sets necessarily had the same number of components in each stage of their defining sequences. In [**12**], the authors and Željko show that Antoine Cantor sets with the same number of components at each stage can be inequivalent. Knot theory techniques are used in the proof. This leads to the following question.

Question 8. *Is it possible to completely classify Antoine Cantor sets using knot theory invariants?* 1450?

The following questions deal with the possibility of classifying wild Cantor sets in $\mathbb{R}^3$ using various properties.

Question 9. *Is there a way of classifying wild Cantor sets in $\mathbb{R}^3$ using local genus and other geometric properties?* 1451?

Question 10. *Can one use the volume of the hyperbolic 3-manifolds $M^3 = S^3 \setminus X$ where X is a wild Cantor set to distinguish between classes of wild Cantor sets?* 1452?

The following questions are about the relationship of Hausdorff dimension to various types of Cantor sets.

Question 11. *Can two rigid Cantor sets have different Hausdorff dimensions? How does Hausdorff dimension detect rigidity of Cantor sets?* 1453–1454?

Question 12. *Is there a rigid Cantor set of minimal Hausdorff dimension?* 1455?

Question 13. *Can two Cantor sets of different genus have the same Hausdorff dimension? How are Hausdorff dimension and genus of Cantor sets related?* 1456–1457?

The final few questions deal with homotopy groups of the complement of wild Cantor sets.

Question 14. *Can two different (rigid) Cantor sets have the same fundamental groups of the complement?* 1458?

Question 15. *Which groups can occur as the fundamental groups of (rigid) wild Cantor set complements?* 1459?

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