On monoids of monotone injective partial selfmaps of integers with cofinite domains and images

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Abstract. We study the semigroup $\mathscr{I}_\infty^{\nearrow}(\mathbb{Z})$ of monotone injective partial selfmaps of the set of integers having cofinite domain and image. We show that $\mathscr{I}_\infty^{\nearrow}(\mathbb{Z})$ is bisimple and all of its non-trivial semigroup homomorphisms are either isomorphisms or group homomorphisms. We also prove that every Baire topology τ on $\mathscr{I}_\infty^{\nearrow}(\mathbb{Z})$, such that $(\mathscr{I}_\infty^{\nearrow}(\mathbb{Z}), \tau)$ is a Hausdorff semitopological semigroup, is discrete and we construct a non-discrete Hausdorff inverse semigroup topology τ_W on $\mathscr{I}_\infty^{\nearrow}(\mathbb{Z})$. We show that the discrete semigroup $\mathscr{I}_\infty^{\nearrow}(\mathbb{Z})$ cannot be embedded into some classes of compact-like topological semigroups and that its remainder under the closure in a topological semigroup S is an ideal in S.

Keywords. Topological semigroup, semitopological semigroup, semigroup of bijective partial transformations, symmetric inverse semigroup, ideal, Baire space, semigroup topologization, bicyclic semigroup.

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1 Introduction and preliminaries

In this paper all spaces will be assumed to be Hausdorff. We shall denote the first infinite cardinal by ω and the cardinality of the set A by $|A|$. Also, we shall denote the additive group of integers by $\mathbb{Z}(+)$. We shall identify all sets X with their cardinality $|X|$.

For a topological space X, a family $\{A_s \mid s \in \mathscr{A}\}$ of subsets of X is called *locally finite* if for every point $x \in X$ there exists an open neighborhood U of x in X such that the set $\{s \in \mathscr{A} \mid U \cap A_s \neq \varnothing\}$ is finite. A subset A of X is said to be

- *co-dense* in X if $X \setminus A$ is dense in X;

- an F_σ-*set* in X if A is an intersection of a countable family of open subsets in X.

We recall that a topological space X is said to be

- *compact* if each open cover of X has a finite subcover;

- *countably compact* if each open countable cover of X has a finite subcover;

- *pseudocompact* if each locally finite open cover of X is finite;

- a *Baire space* if for each sequence $A_1, A_2, \ldots, A_i, \ldots$ of nowhere dense sub-sets of X the union $\bigcup_{i=1}^{\infty} A_i$ is a co-dense subset of X;

- *Čech complete* if X is Tychonoff and for every compactification cX of X the remainder $cX \setminus X$ is an F_σ-set in cX;

- *locally compact* if every point of X has an open neighborhood with a compact closure.

According to [14, Theorem 3.10.22], a Tychonoff topological space X is pseu-docompact if and only if each continuous real-valued function on X is bounded.

An algebraic semigroup S is called *inverse* if for any element $x \in S$ there exists a unique $x^{-1} \in S$ such that $x x^{-1} x = x$ and $x^{-1} x x^{-1} = x^{-1}$. The element x^{-1} is called the *inverse of* $x \in S$. If S is an inverse semigroup, then the function inv: $S \to S$ which assigns to every element x of S its inverse element x^{-1} is called an *inversion*.

A *band* is a semigroup of idempotents. If S is a semigroup, then we shall denote the subset of idempotents in S by $E(S)$. If S is an inverse semigroup, then $E(S)$ is closed under multiplication. The semigroup operation on S determines the following partial order $\leqslant$ on $E(S)$: $e \leqslant f$ if and only if $ef = fe = e$. This order is called the *natural partial order* on $E(S)$. A *semilattice* is a commutative semigroup of idempotents. A semilattice E is called *linearly ordered* or a *chain* if its natural order is a linear order. A *maximal chain* of a semilattice E is a chain which is properly contained in no other chain of E.

If $h: S \to T$ is a homomorphism from a semigroup S into a semigroup T then we say that h is

- a *trivial homomorphism* if h is either an isomorphism or an annihilating ho-momorphism;

- a *group homomorphism* if $(S)h$ is a subgroup of T.

If $\mathfrak{C}$ is an arbitrary congruence on a semigroup S, then we denote by $\Phi_{\mathfrak{C}}: S \to S/\mathfrak{C}$ the natural homomorphism from S onto the quotient semigroup $S/\mathfrak{C}$. A con-gruence $\mathfrak{C}$ on a semigroup S is called *non-trivial* if $\mathfrak{C}$ is distinct from universal and identity congruences on S, and a *group congruence* if the quotient semigroup $S/\mathfrak{C}$

is a group. Every inverse semigroup S admits a least (minimal) group congruence $\mathfrak{C}_{\mathrm{mg}}$:

$$a\mathfrak{C}_{\mathrm{mg}}b \text{ if and only if there exists } e \in E(S) \text{ such that } ae = be$$

(see [26, Lemma III.5.2]).

The Axiom of Choice implies the existence of maximal chains in any partially ordered set. According to [26], a chain L is called an ω-chain if L is isomorphic to $\{0, -1, -2, -3, \ldots\}$ with the usual order $\leqslant$. Let E be a semilattice and $e \in E$. We denote $\downarrow e = \{f \in E \mid f \leqslant e\}$ and $\uparrow e = \{f \in E \mid e \leqslant f\}$. By $(\mathscr{P}_{<\omega}(\lambda), \subseteq)$ we shall denote the *free semilattice with identity* over a set of cardinality $\lambda \geqslant \omega$, i.e., $(\mathscr{P}_{<\omega}(\lambda), \subseteq)$ is the set of all finite subsets (including the empty set) of λ with the semilattice operation "union".

If S is a semigroup, then we shall denote the *Green relations* on S by $\mathscr{R}, \mathscr{L}, \mathscr{J}$, $\mathscr{D}$ and $\mathscr{H}$ (see [9, Section 2.1]):

$$a\mathscr{R}b \text{ if and only if } aS^1 = bS^1;$$

$$a\mathscr{L}b \text{ if and only if } S^1a = S^1b;$$

$$a\mathscr{J}b \text{ if and only if } S^1aS^1 = S^1bS^1;$$

$$\mathscr{D} = \mathscr{L} \circ \mathscr{R} = \mathscr{R} \circ \mathscr{L};$$

$$\mathscr{H} = \mathscr{L} \cap \mathscr{R}.$$

A semigroup S is called *simple* if S does not contain any proper two-sided ideals and *bisimple* if S has only one $\mathscr{D}$-class.

A *semitopological* (resp. *topological*) *semigroup* is a Hausdorff topological space together with a separately (resp. jointly) continuous semigroup operation. An inverse topological semigroup with the continuous inversion is called a *topological inverse semigroup*. A Hausdorff topology τ on an (inverse) semigroup S, such that (S, τ) is a topological (inverse) semigroup, is called an *(inverse) semigroup topology*.

If $\alpha \colon X \rightharpoonup Y$ is a partial map, then by $\operatorname{dom} \alpha$ and $\operatorname{ran} \alpha$ we shall denote the domain and the range of α, respectively. Let $\mathscr{I}_\lambda$ denote the set of all partial injective transformations of an infinite set X of cardinality λ together with the following semigroup operation: $x(\alpha\beta) = (x\alpha)\beta$ if $x \in \operatorname{dom}(\alpha\beta) = \{y \in \operatorname{dom} \alpha \mid y\alpha \in \operatorname{dom} \beta\}$, for $\alpha, \beta \in \mathscr{I}_\lambda$. The semigroup $\mathscr{I}_\lambda$ is called the *symmetric inverse semigroup* over the set X (see [9, Section 1.9]). The symmetric inverse semigroup was introduced by Vagner [29] and it plays a major role in the theory of semigroups.

Let $\mathbb{Z}$ be the set of integers with the usual order $\leqslant$. We shall say that a partial map $\alpha \colon \mathbb{Z} \rightharpoonup \mathbb{Z}$ is *monotone* if $n \leqslant m$ implies $(n)\alpha \leqslant (m)\alpha$ for $n, m \in \mathbb{Z}$. By

$\mathscr{I}_{\infty}^{\nearrow}(\mathbb{Z})$ we denote a subsemigroup of injective partial monotone selfmaps of $\mathbb{Z}$ with cofinite domains and images, i.e.,

$$\mathscr{I}_{\infty}^{\nearrow}(\mathbb{Z}) = \{\alpha \in \mathscr{I}_{\omega} \mid \alpha \text{ is monotone, } |\mathbb{Z} \setminus \operatorname{dom}\alpha| < \infty \text{ and } |\mathbb{Z} \setminus \operatorname{ran}\alpha| < \infty\}.$$

Obviously, $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{Z})$ is an inverse submonoid of the semigroup $\mathscr{I}_{\omega}$. We observe that $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{Z})$ is a countable semigroup. Furthermore, we shall denote the identity of the semigroup $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{Z})$ by $\mathbb{I}$ and the group of units of $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{Z})$ by $H(\mathbb{I})$.

Lemma 1.1. *A partial injective monotone map α is an element of the semigroup $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{Z})$ if and only if there exist integers d_α and u_α such that the following conditions hold:*

$$(m-1)\alpha = (m)\alpha - 1 \quad and \quad (n+1)\alpha = (n)\alpha + 1$$

for all integers $m \leqslant d_\alpha$ and $n \geqslant u_\alpha$; and $\alpha \in H(\mathbb{I})$ if and only if $(n+1)\alpha = (n)\alpha + 1$ for any integer n.

Proof. The implication ($\Leftarrow$) is trivial.

($\Rightarrow$) Since for every element α of the semigroup $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{Z})$ the sets $\mathbb{Z} \setminus \operatorname{dom}\alpha$ and $\mathbb{Z} \setminus \operatorname{ran}\alpha$ are finite, we conclude that there exist integers d_α and u_α such that the following condition holds:

$$m, n \in \operatorname{dom}\alpha \cap \operatorname{ran}\alpha \quad \text{for all } m \leqslant d_\alpha \text{ and } n \geqslant u_\alpha. \tag{$*$}$$

Now, since the partial map $\alpha \colon \mathbb{Z} \rightharpoonup \mathbb{Z}$ is monotone, we have that

$$(m-1)\alpha \leqslant (m)\alpha - 1 \quad \text{and} \quad (n)\alpha + 1 \leqslant (n+1)\alpha$$

for all $m \leqslant d_\alpha$ and $n \geqslant u_\alpha$, and hence we get that

$$(m-j)\alpha \leqslant (m-(j-1))\alpha - 1 \leqslant \cdots \leqslant (m)\alpha - j$$

and

$$(n)\alpha + j \leqslant \cdots \leqslant (n+j-1)\alpha + 1 \leqslant (n+j)\alpha$$

for any positive integer j, $m \leqslant d_\alpha$ and $n \geqslant u_\alpha$. Then by condition ($*$) we have that

$$(m-1)\alpha = (m)\alpha - 1 \quad \text{and} \quad (n+1)\alpha = (n)\alpha + 1$$

for all integers $m \leqslant d_\alpha$ and $n \geqslant u_\alpha$.

It is obvious that if $(n+1)\alpha = (n)\alpha + 1$ for any integer n then $\alpha \colon \mathbb{Z} \to \mathbb{Z}$ is a bijective monotone map and hence $\alpha \in H(\mathbb{I})$. Conversely, if $\alpha \in H(\mathbb{I})$ then $\alpha \colon \mathbb{Z} \rightharpoonup \mathbb{Z}$ is a bijective monotone map and the first assertion of lemma implies that $(n+1)\alpha = (n)\alpha + 1$ for any integer n. □

The bicyclic semigroup $\mathscr{C}(p, q)$ is the semigroup with the identity 1 generated by elements p and q subject only to the condition $pq = 1$. The bicyclic semigroup is bisimple and every one of its congruences is either trivial or a group congruence. Moreover, every non-annihilating homomorphism h of the bicyclic semigroup is either an isomorphism or the image of $\mathscr{C}(p, q)$ under h is a cyclic group (see [9, Corollary 1.32]).

The bicyclic semigroup plays an important role in the algebraic theory of semigroups and in the theory of topological semigroups. For example, the well-known result of Andersen [1] states that a (0-)simple semigroup is completely (0-)simple if and only if it does not contain the bicyclic semigroup.

Remark 1.2. Let n be an arbitrary integer. We put $\mathscr{C}(n, +)$ and $\mathscr{C}(n, -)$ to be semigroups which are generated by partial transformations α_n^+ and β_n^+; α_n^- and β_n^-, respectively, of the set of integers $\mathbb{Z}$, defined as follows:

$$(i)\alpha_n^+ = \begin{cases} i, & \text{if } i \leqslant n, \\ i+1, & \text{if } i > n, \end{cases} \qquad (i)\beta_n^+ = \begin{cases} i, & \text{if } i \leqslant n, \\ i-1, & \text{if } i > n+1, \end{cases}$$

$$(i)\alpha_n^- = \begin{cases} i, & \text{if } i \geqslant n, \\ i-1, & \text{if } i < n, \end{cases} \qquad (i)\beta_n^- = \begin{cases} i, & \text{if } i \geqslant n, \\ i+1, & \text{if } i < n-1, \end{cases}$$

$i \in \mathbb{Z}$. We remark that $\mathscr{C}(n, +)$ and $\mathscr{C}(n, -)$ are bicyclic semigroups for every positive integer n. Therefore the semigroup $\mathscr{I}_\infty^{\nearrow}(\mathbb{Z})$ contains infinitely many isomorphic copies of the bicyclic semigroup $\mathscr{C}(p, q)$.

We shall say that a partial map $\alpha\colon \mathbb{Z} \rightharpoonup \mathbb{Z}$ is *almost monotone* if there exists a finite subset F in $\operatorname{dom}\alpha$ such that the restriction $\alpha|_{\operatorname{dom}\alpha\setminus F}\colon \mathbb{Z} \rightharpoonup \mathbb{Z}$ is a monotone partial map. By $\mathscr{I}_\infty^{\leftrightarrow}(\mathbb{Z})$ we denote a subsemigroup of injective partial almost monotone selfmaps of $\mathbb{Z}$ with cofinite domains and images, i.e.,

$$\mathscr{I}_\infty^{\leftrightarrow}(\mathbb{Z}) = \{\alpha \in \mathscr{I}_\omega \mid \alpha \text{ is almost monotone,}$$
$$|\mathbb{Z} \setminus \operatorname{dom}\alpha| < \infty \text{ and } |\mathbb{Z} \setminus \operatorname{ran}\alpha| < \infty\}.$$

Obviously, $\mathscr{I}_\infty^{\leftrightarrow}(\mathbb{Z})$ is an inverse submonoid of the semigroup $\mathscr{I}_\omega$ and $\mathscr{I}_\infty^{\nearrow}(\mathbb{Z})$ is an inverse submonoid of $\mathscr{I}_\infty^{\leftrightarrow}(\mathbb{Z})$. We observe that $\mathscr{I}_\infty^{\leftrightarrow}(\mathbb{Z})$ is a countable semigroup.

It is well known that topological algebra studies the influence of topological properties of its objects on their algebraic properties and the influence of algebraic properties of its objects on their topological properties. There are two main problems in topological algebra: the problem of non-discrete topologization and the problem of embedding into objects with some topological-algebraic properties.

In mathematical literature the question about non-discrete (Hausdorff) topologization was posed by Markov [23]. Pontryagin gave well-known conditions for a base at the unity of a group for its non-discrete topologization (see [19, Theorem 4.5] or [27, Theorem 3.9]). Various authors have refined Markov's question: can a given infinite group G endowed with a non-discrete group topology be embedded into a compact topological group? Again, for an arbitrary Abelian group G the answer is affirmative, but there is a non-Abelian topological group that cannot be embedded into any compact topological group (see [10, Section 9]).

Also, Ol'shanskiy [25] constructed an infinite countable group G such that every Hausdorff group topology on G is discrete. Eberhart and Selden [13] showed that every Hausdorff semigroup topology on the bicyclic semigroup $\mathcal{C}(p,q)$ is discrete. Bertman and West [7] proved that every Hausdorff topology τ on $\mathcal{C}(p,q)$, such that $(\mathcal{C}(p,q), \tau)$ is a semitopological semigroup, is also discrete. Taimanov [28] gave sufficient conditions on a commutative semigroup to have a non-discrete semigroup topology.

Many mathematicians have studied the problems of embeddings of topological semigroups into compact or compact-like topological semigroups (see [8]). Neither stable nor Γ-compact topological semigroups can contain a copy of the bicyclic semigroup [3, 20]. Also, the bicyclic semigroup cannot be embedded into any countably compact topological inverse semigroup [17]. Moreover, the conditions were given in [5,6] when a countably compact or pseudocompact topological semigroup cannot contain the bicyclic semigroup.

However, Banakh, Dimitrova and Gutik [6] have constructed (assuming the Continuum Hypothesis or the Martin Axiom) an example of a Tychonoff countably compact topological semigroup which contains the bicyclic semigroup. The problems of topologization of semigroups of partial transformations and their embeddings into compact-like semigroup were studied in [15, 16].

We showed in [18] that the semigroup $\mathcal{I}_{\infty}^{\nearrow}(\mathbb{N})$ of partial cofinite monotone injective transformations of the set of positive integers $\mathbb{N}$ has algebraic properties similar to those of the bicyclic semigroup: it is bisimple and all of its non-trivial semigroup homomorphisms are either isomorphisms or group homomorphisms. We proved that every locally compact topology τ on $\mathcal{I}_{\infty}^{\nearrow}(\mathbb{N})$, such that $(\mathcal{I}_{\infty}^{\nearrow}(\mathbb{N}), \tau)$ is a topological inverse semigroup, is discrete and we described the closure of $(\mathcal{I}_{\infty}^{\nearrow}(\mathbb{N}), \tau)$ in a topological semigroup.

In this paper we shall describe Green relations on $\mathcal{I}_{\infty}^{\nearrow}(\mathbb{Z})$, show that $\mathcal{I}_{\infty}^{\nearrow}(\mathbb{Z})$ is bisimple and all of its non-trivial semigroup homomorphisms are either isomorphisms or group homomorphisms. We shall also prove that every Baire topology τ on $\mathcal{I}_{\infty}^{\nearrow}(\mathbb{Z})$, such that $(\mathcal{I}_{\infty}^{\nearrow}(\mathbb{Z}), \tau)$ is a Hausdorff semitopological semigroup, is discrete and construct a non-discrete Hausdorff semigroup inverse topology τ_W on

$\mathscr{I}_{\infty}^{\nearrow}(\mathbb{Z})$. We shall show that the discrete semigroup $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{Z})$ cannot be embedded into some classes of compact-like topological semigroups and that its remainder under the closure in a topological semigroup S is an ideal in S.

2   Algebraic properties of the semigroup $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{Z})$

Proposition 2.1. *The following assertions hold:*

(i) *An element α of the semigroup $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{Z})$ is an idempotent if and only if $(x)\alpha = x$ for every $x \in \operatorname{dom}\alpha$.*

(ii) *If $\varepsilon, \iota \in E(\mathscr{I}_{\infty}^{\nearrow}(\mathbb{Z}))$, then $\varepsilon \leqslant \iota$ if and only if $\operatorname{dom}\varepsilon \subseteq \operatorname{dom}\iota$.*

(iii) *The semilattice $E(\mathscr{I}_{\infty}^{\nearrow}(\mathbb{Z}))$ is isomorphic to $(\mathcal{P}_{<\omega}(\mathbb{Z}), \subseteq)$ under the mapping $(\varepsilon)h = \mathbb{Z} \setminus \operatorname{dom}\varepsilon$.*

(iv) *Every maximal chain in $E(\mathscr{I}_{\infty}^{\nearrow}(\mathbb{Z}))$ is an ω-chain.*

(v) *$\alpha \mathcal{R} \beta$ in $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{Z})$ if and only if $\operatorname{dom}\alpha = \operatorname{dom}\beta$.*

(vi) *$\alpha \mathcal{L} \beta$ in $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{Z})$ if and only if $\operatorname{ran}\alpha = \operatorname{ran}\beta$.*

(vii) *$\alpha \mathcal{H} \beta$ in $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{Z})$ if and only if $\operatorname{dom}\alpha = \operatorname{dom}\beta$ and $\operatorname{ran}\alpha = \operatorname{ran}\beta$.*

(viii) *$\mathscr{I}_{\infty}^{\nearrow}(\mathbb{Z})$ is a simple semigroup and hence $\mathcal{J} = \mathscr{I}_{\infty}^{\nearrow}(\mathbb{Z}) \times \mathscr{I}_{\infty}^{\nearrow}(\mathbb{Z})$.*

(ix) *For all idempotents $\varepsilon, \varphi \in \mathscr{I}_{\infty}^{\nearrow}(\mathbb{Z})$ there exist infinitely many elements $\alpha, \beta \in \mathscr{I}_{\infty}^{\nearrow}(\mathbb{Z})$ such that $\alpha \cdot \beta = \varepsilon$ and $\beta \cdot \alpha = \varphi$.*

Proof. Statements (i)–(iv) are trivial and they follow from the definition of the semigroup $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{Z})$.

The proofs of (v)–(vii) follow trivially from the fact that $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{Z})$ is a regular semigroup, and by [21, Proposition 2.4.2, Exercise 5.11.2].

(viii) Note that every cofinite subset of $\mathbb{Z}$ is order-isomorphic to $\mathbb{Z}$. Let $\varphi, \gamma \in \mathscr{I}_{\infty}^{\nearrow}(\mathbb{Z})$ be arbitrary. Since the sets $\mathbb{Z} \setminus \operatorname{dom}\varphi$, $\mathbb{Z} \setminus \operatorname{dom}\gamma$ and $\mathbb{Z} \setminus \operatorname{ran}\varphi$ are finite and the sets $\operatorname{dom}\varphi$, $\operatorname{dom}\gamma$ and $\operatorname{ran}\varphi$ are order-isomorphic to $\mathbb{Z}$, we conclude that there exist bijective monotone maps $\varphi_{\operatorname{dom}}: \operatorname{dom}\varphi \to \mathbb{Z}$, $\gamma_{\operatorname{dom}}: \operatorname{dom}\gamma \to \mathbb{Z}$ and $\varphi_{\operatorname{ran}}: \operatorname{ran}\varphi \to \mathbb{Z}$. We put $\operatorname{dom}\kappa = \operatorname{dom}\gamma$, $\operatorname{ran}\kappa = \operatorname{dom}\varphi$ and $\kappa = \gamma_{\operatorname{dom}} \cdot (\varphi_{\operatorname{dom}})^{-1}$. Then $\kappa: \mathbb{Z} \rightharpoonup \mathbb{Z}$ is a monotone partial map as a composition of monotone partial maps. We define an injective partial map $\chi: \mathbb{Z} \rightharpoonup \mathbb{Z}$ in the following way: $\operatorname{dom}\chi = \mathbb{Z}$, $\operatorname{ran}\chi = \operatorname{ran}\gamma$ and $(n)\chi = (n)((\varphi_{\operatorname{ran}})^{-1} \cdot \varphi^{-1} \cdot \kappa \cdot \gamma)$ for $n \in \mathbb{Z}$. Then $\chi: \mathbb{Z} \rightharpoonup \mathbb{Z}$ is a monotone partial map, being a composition of monotone partial maps. We put $\operatorname{dom}\xi = \operatorname{ran}\varphi$, $\operatorname{ran}\xi = \operatorname{ran}\gamma$ and $\xi = \varphi_{\operatorname{ran}} \cdot \chi$. Then $\xi: \mathbb{Z} \rightharpoonup \mathbb{Z}$

is a monotone partial map, being a composition of monotone partial maps. Hence $\gamma = \kappa \cdot \varphi \cdot \xi$ and so $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{Z})$ is simple.

(ix) Let $\varepsilon, \varphi \in E(\mathscr{I}_{\infty}^{\nearrow}(\mathbb{Z}))$ be arbitrary. Then by statement (i) we have that $\operatorname{dom} \varepsilon = \operatorname{ran} \varepsilon$ and $\operatorname{dom} \varphi = \operatorname{ran} \varphi$. Since the sets $\mathbb{Z} \setminus \operatorname{dom} \varepsilon$ and $\mathbb{Z} \setminus \operatorname{dom} \varphi$ are finite and the sets $\operatorname{dom} \varepsilon$ and $\operatorname{dom} \varphi$ are order-isomorphic to $\mathbb{Z}$, we conclude that there exist bijective monotone maps $\varepsilon_{\operatorname{dom}} \colon \operatorname{dom} \varepsilon \to \mathbb{Z}$ and $\varphi_{\operatorname{dom}} \colon \operatorname{dom} \varphi \to \mathbb{Z}$. Also, we note that for every integer k the translation $\sigma_k \colon \mathbb{Z} \to \mathbb{Z} \colon n \mapsto n + k$ is a bijective monotone map. Now we define for any integer i

$$\alpha_i = \varepsilon_{\operatorname{dom}} \cdot \sigma_i \cdot (\varphi_{\operatorname{dom}})^{-1}.$$

Then we have that

$$
\begin{aligned}
\alpha_i \cdot \alpha_i^{-1} &= \varepsilon_{\operatorname{dom}} \cdot \sigma_i \cdot (\varphi_{\operatorname{dom}})^{-1} \cdot \varphi_{\operatorname{dom}} \cdot \sigma_i^{-1} \cdot (\varepsilon_{\operatorname{dom}})^{-1} \\
&= \varepsilon_{\operatorname{dom}} \cdot \sigma_i \cdot \mathbb{I} \cdot \sigma_i^{-1} \cdot (\varepsilon_{\operatorname{dom}})^{-1} = \varepsilon_{\operatorname{dom}} \cdot \sigma_i \cdot \sigma_i^{-1} \cdot (\varepsilon_{\operatorname{dom}})^{-1} \\
&= \varepsilon_{\operatorname{dom}} \cdot \mathbb{I} \cdot (\varepsilon_{\operatorname{dom}})^{-1} = \varepsilon_{\operatorname{dom}} \cdot (\varepsilon_{\operatorname{dom}})^{-1} = \varepsilon
\end{aligned}
$$

and

$$
\begin{aligned}
\alpha_i^{-1} \cdot \alpha_i &= \varphi_{\operatorname{dom}} \cdot \sigma_i^{-1} \cdot (\varepsilon_{\operatorname{dom}})^{-1} \cdot \varepsilon_{\operatorname{dom}} \cdot \sigma_i \cdot (\varphi_{\operatorname{dom}})^{-1} \\
&= \varphi_{\operatorname{dom}} \cdot \sigma_i^{-1} \cdot \mathbb{I} \cdot \sigma_i \cdot (\varphi_{\operatorname{dom}})^{-1} = \varphi_{\operatorname{dom}} \cdot \sigma_i^{-1} \cdot \sigma_i \cdot (\varphi_{\operatorname{dom}})^{-1} \\
&= \varphi_{\operatorname{dom}} \cdot \mathbb{I} \cdot (\varphi_{\operatorname{dom}})^{-1} = \varphi_{\operatorname{dom}} \cdot (\varphi_{\operatorname{dom}})^{-1} = \varphi
\end{aligned}
$$

for every integer i. This completes the proof of the assertion.   □

Proposition 2.2. *The group of units $H(\mathbb{I})$ of the semigroup $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{Z})$ is isomorphic to $\mathbb{Z}(+)$.*

Proof. Let α be an arbitrary element of $H(\mathbb{I})$. Then α is a bijective monotone map from $\mathbb{Z}$ onto $\mathbb{Z}$. We fix arbitrary $n \in \mathbb{Z}$. Then the monotonicity of α implies that $(n)\alpha < (n + 1)\alpha$. If $(n)\alpha + 1 < (n + 1)\alpha$ then there exists an integer m such that $(m)\alpha = (n)\alpha + 1$. But if $m > n + 1$ or $m < n$ this contradicts the monotonicity of α. Therefore we get that $(n)\alpha + 1 = (n + 1)\alpha$. Similarly we get that $(n)\alpha - 1 = (n - 1)\alpha$. Hence every $\alpha \in H(\mathbb{I})$ is a shift of the set of integers. We define the map $h \colon H(\mathbb{I}) \to \mathbb{Z}(+)$ by the formula $(\alpha)h = (n)\alpha - n$. Since α is a shift of the set of integers, we conclude that the definition of the map h is correct. Simple verifications show that $h \colon H(\mathbb{I}) \to \mathbb{Z}(+)$ is a group isomorphism.   □

Since $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{Z})$ is an inverse semigroup, Proposition 2.1 (ix) and [24, Lemma 1.1] imply the following proposition.

Proposition 2.3. $\mathscr{I}_\infty^{\nearrow}(\mathbb{Z})$ *is a bisimple semigroup.*

Propositions 2.2 and 2.3, and [9, Theorem 2.20] imply the following corollary.

Corollary 2.4. *Every maximal subgroup of the semigroup $\mathscr{I}_\infty^{\nearrow}(\mathbb{Z})$ is isomorphic to $\mathbb{Z}(+)$.*

Proposition 2.5. *For all $\alpha, \beta \in \mathscr{I}_\infty^{\nearrow}(\mathbb{Z})$, both sets $\{\chi \in \mathscr{I}_\infty^{\nearrow}(\mathbb{Z}) \mid \alpha \cdot \chi = \beta\}$ and $\{\chi \in \mathscr{I}_\infty^{\nearrow}(\mathbb{Z}) \mid \chi \cdot \alpha = \beta\}$ are finite.*

Proof. We denote

$$A = \{\chi \in \mathscr{I}_\infty^{\nearrow}(\mathbb{Z}) \mid \alpha \cdot \chi = \beta\}, \quad B = \{\chi \in \mathscr{I}_\infty^{\nearrow}(\mathbb{Z}) \mid \alpha^{-1} \cdot \alpha \cdot \chi = \alpha^{-1} \cdot \beta\}.$$

Then $A \subseteq B$ and the restriction of any partial map $\chi \in B$ to $\mathrm{dom}(\alpha^{-1} \cdot \alpha)$ coincides with the partial map $\alpha^{-1} \cdot \beta$. Since every partial map from $\mathscr{I}_\infty^{\nearrow}(\mathbb{Z})$ is monotone, we conclude that the set B is finite and hence so is A. □

Lemma 2.6. *Let S be an arbitrary semigroup and $h: \mathscr{I}_\infty^{\nearrow}(\mathbb{Z}) \to S$ a semigroup homomorphism. If there exist distinct idempotents $\varepsilon, \varphi \in \mathscr{I}_\infty^{\nearrow}(\mathbb{Z})$ such that $(\varepsilon)h = (\varphi)h$ then $(\psi)h = (\mathbb{I})h$ for all $\psi \in E(\mathscr{I}_\infty^{\nearrow}(\mathbb{Z}))$.*

Proof. Since $(\varepsilon)h = (\varphi)h = (\varphi \cdot \varphi)h = (\varphi)h \cdot (\varphi)h = (\varphi)h \cdot (\varepsilon)h = (\varphi \cdot \varepsilon)h$, we can assume without loss of generality that $\varepsilon \leqslant \varphi$ in $E(\mathscr{I}_\infty^{\nearrow}(\mathbb{Z}))$. Therefore, if ι is an idempotent of the semigroup $\mathscr{I}_\infty^{\nearrow}(\mathbb{Z})$ such that $\varepsilon \leqslant \iota \leqslant \varphi$, then $(\varepsilon)h = (\iota)h$. Hence Proposition 2.1 (ii) implies that we can assume without loss of generality that $|\mathrm{dom}\,\varphi \setminus \mathrm{dom}\,\varepsilon| = 1$.

Let ψ be an arbitrary idempotent of the semigroup $\mathscr{I}_\infty^{\nearrow}(\mathbb{Z})$ and let $n_0 = \min\{\mathbb{Z} \setminus \mathrm{dom}\,\psi\} - 1$. Let $\sigma: \mathbb{Z} \rightharpoonup \mathbb{Z}$ be a partial order preserving injective map which maps $\mathrm{dom}\,\varphi$ onto $\mathbb{Z}$ and $n = \mathbb{Z} \setminus (\mathrm{dom}\,\varepsilon)\sigma$. Without loss of generality we can assume that $n = n_0$. Then $\tilde\varphi = \sigma^{-1} \circ \varphi \circ \sigma: \mathbb{Z} \to \mathbb{Z}$ is an identity map and $\tilde\varepsilon = \sigma^{-1} \circ \varepsilon \circ \sigma$ is an identity map from $\mathbb{Z} \setminus \{n_0\}$ onto $\mathbb{Z} \setminus \{n_0\}$. Then $\tilde\varepsilon$ is a unit of the semigroup $\mathscr{I}_\infty^{\nearrow}(\mathbb{Z})$. Since $\sigma \in \mathscr{I}_\infty^{\nearrow}(\mathbb{Z})$ and $\mathrm{dom}\,\varepsilon \subsetneqq \mathrm{dom}\,\varphi$, we have that

$$(\tilde\varphi)h = (\sigma \cdot \varphi \cdot \sigma)h = (\sigma)h \cdot (\varphi)h \cdot (\sigma)h = (\sigma)h \cdot (\varepsilon)h \cdot (\sigma)h = (\tilde\varepsilon)h,$$

$$\tilde\varepsilon \cdot \tilde\varphi = (\sigma^{-1} \cdot \varepsilon \cdot \sigma) \cdot (\sigma^{-1} \cdot \varphi \cdot \sigma) = \sigma^{-1} \cdot \varepsilon \cdot (\sigma \cdot \sigma^{-1}) \cdot \varphi \cdot \sigma$$

$$= \sigma^{-1} \cdot \varepsilon \cdot \varphi \cdot \varphi \cdot \sigma = \sigma^{-1} \cdot \varepsilon \cdot \varphi \cdot \sigma = \sigma^{-1} \cdot \varepsilon \cdot \sigma = \tilde\varepsilon,$$

$$\tilde\varphi \cdot \tilde\varepsilon = (\sigma^{-1} \cdot \varphi \cdot \sigma) \cdot (\sigma^{-1} \cdot \varepsilon \cdot \sigma) = \sigma^{-1} \cdot \varphi \cdot (\sigma \cdot \sigma^{-1}) \cdot \varepsilon \cdot \sigma$$

$$= \sigma^{-1} \cdot \varphi \cdot \varphi \cdot \varepsilon \cdot \sigma = \sigma^{-1} \cdot \varphi \cdot \varepsilon \cdot \sigma = \sigma^{-1} \cdot \varepsilon \cdot \sigma = \tilde\varepsilon,$$

and hence $\tilde\varepsilon \leqslant \tilde\varphi$.

We observe that $\tilde{\varphi}, \tilde{\varepsilon} \in \mathcal{C}(n_0, +)$. Since $(\tilde{\varphi})h = (\tilde{\varepsilon})h$, [9, Corollary 1.32] implies that $(\tilde{\varphi})h = (\chi)h$ for every idempotent $\chi \in \mathcal{C}(n_0, +)$. Since $i > n_0$ for all $i \in \mathbb{Z} \setminus \operatorname{dom} \psi$, the definition of the semigroup $\mathscr{I}_\infty^{\nearrow}(\mathbb{Z})$ implies that there exists an idempotent $\varepsilon_0 \in \mathcal{C}(n_0, +)$ such that $\varepsilon_0 \leqslant \psi \leqslant \tilde{\varphi}$. Therefore we have that $(\psi)h = (\varepsilon_0)h = (\tilde{\varphi})h$. This completes the proof of the lemma.   □

Theorem 2.7. *Let S be a semigroup and $h: \mathscr{I}_\infty^{\nearrow}(\mathbb{Z}) \to S$ a non-annihilating homomorphism. Then either h is a monomorphism or $(\mathscr{I}_\infty^{\nearrow}(\mathbb{Z}))h$ is a subgroup of S.*

Proof. Suppose that $h: \mathscr{I}_\infty^{\nearrow}(\mathbb{Z}) \to S$ is not a monomorphism. Then $(\alpha)h = (\beta)h$, for some distinct $\alpha, \beta \in \mathscr{I}_\infty^{\nearrow}(\mathbb{Z})$. We consider two cases:

  (i) α and β are not $\mathcal{H}$-equivalent;

  (ii) α and β are $\mathcal{H}$-equivalent.

Suppose that case (i) holds. Since $\mathscr{I}_\infty^{\nearrow}(\mathbb{Z})$ is an inverse semigroup, we have that either $\alpha \cdot \alpha^{-1} \neq \beta \cdot \beta^{-1}$ or $\alpha^{-1} \cdot \alpha \neq \beta^{-1} \cdot \beta$. Suppose that $\alpha \cdot \alpha^{-1} \neq \beta \cdot \beta^{-1}$. In the other case the proof is similar. Since $\mathscr{I}_\infty^{\nearrow}(\mathbb{Z})$ is an inverse semigroup, we conclude that

$$(\alpha^{-1})h = ((\alpha)h)^{-1} = ((\beta)h)^{-1} = (\beta^{-1})h$$

and hence $(\alpha \cdot \alpha^{-1})h = (\alpha)h \cdot (\alpha^{-1})h = (\beta)h \cdot (\beta^{-1})h = (\beta \cdot \beta^{-1})h$. Therefore the assertion of Lemma 2.6 holds. Since every homomorphic image of an inverse semigroup is an inverse semigroup, we conclude that $(\mathscr{I}_\infty^{\nearrow}(\mathbb{Z}))h$ is a subgroup of S.

Suppose that $\alpha \mathcal{H} \beta$. Then by [9, Theorem 2.20] there exist distinct $\alpha_0, \beta_0 \in H(\mathbb{I})$ such that $(\alpha_0)h = (\beta_0)h$. Therefore we have that $(\mathbb{I})h = (\gamma)h$ for $\gamma = \alpha_0^{-1} \cdot \beta_0 \in H(\mathbb{I})$ and $\gamma \neq \mathbb{I}$. We fix an arbitrary integer i. Let $\iota: \mathbb{Z} \setminus \{i\} \to \mathbb{Z} \setminus \{i\}$ be an identity map. Then $(\iota)h = (\iota \cdot \mathbb{I})h = (\iota \cdot \gamma)h$. Hence ι is an idempotent of the semigroup $\mathscr{I}_\infty^{\nearrow}(\mathbb{Z})$ and $\operatorname{ran} \iota \neq \operatorname{ran}(\iota \cdot \gamma)$. Therefore by Proposition 2.1 (vii) the elements ι and $\iota \cdot \gamma$ are not $\mathcal{H}$-equivalent in the semigroup $\mathscr{I}_\infty^{\nearrow}(\mathbb{Z})$. This implies that there exist distinct non-$\mathcal{H}$-equivalent elements α, β in $\mathscr{I}_\infty^{\leftrightarrow}(\mathbb{Z})$ such that $(\alpha)h = (\beta)h$ and hence case (i) holds. Therefore we get that $(\mathscr{I}_\infty^{\leftrightarrow}(\mathbb{Z}))h$ is a subgroup of S.   □

Proposition 2.8. *Let $\mathfrak{C}_{\mathrm{mg}}$ be a least group congruence on the semigroup $\mathscr{I}_\infty^{\nearrow}(\mathbb{Z})$. Then the quotient semigroup $\mathscr{I}_\infty^{\nearrow}(\mathbb{Z})/\mathfrak{C}_{\mathrm{mg}}$ is isomorphic to the direct product $\mathbb{Z}(+) \times \mathbb{Z}(+)$.*

Proof. Let α and β be $\mathfrak{C}_{\mathrm{mg}}$-equivalent elements of the semigroup $\mathscr{I}_\infty^{\nearrow}(\mathbb{Z})$. Then by [26, Lemma III.5.2] there exists an idempotent ε_0 in $\mathscr{I}_\infty^{\nearrow}(\mathbb{Z})$ such that $\alpha \cdot \varepsilon_0 = \beta \cdot \varepsilon_0$.

Since $\mathscr{I}_\infty^\nearrow(\mathbb{Z})$ is an inverse semigroup, we conclude that $\alpha \cdot \varepsilon = \beta \cdot \varepsilon$ for all $\varepsilon \in E(\mathscr{I}_\infty^\nearrow(\mathbb{Z}))$ such that $\varepsilon \leqslant \varepsilon_0$. Then Lemma 1.1 implies that there exist integers d and u such that

$$(m-1)\alpha = (m)\alpha - 1, \qquad (n+1)\alpha = (n)\alpha + 1,$$
$$(m-1)\beta = (m)\beta - 1, \qquad (n+1)\beta = (n)\beta + 1$$

for all integers $m \leqslant d$ and $n \geqslant u$. We put $D = \min\{(d)\alpha, (d)\beta\}$ and $U = \max\{(u)\alpha, (u)\beta\}$. Let ε_1 be an identity map from $\mathbb{Z} \setminus \{D, D+1, \ldots, U\}$ onto itself. Then $\varepsilon^0 = \varepsilon_1 \circ \varepsilon_0 \leqslant \varepsilon_0$ and hence we have that $\alpha \cdot \varepsilon^0 = \beta \cdot \varepsilon^0$. Therefore we have showed that if the elements α and β of the semigroup $\mathscr{I}_\infty^\nearrow(\mathbb{Z})$ are $\mathfrak{C}_{mg}$-equivalent, then there exist integers d and u such that

$$(m)\alpha = (m)\beta \quad \text{and} \quad (n)\alpha = (n)\beta$$

for all integers $m \leqslant d$ and $n \geqslant u$.

Conversely, suppose that there exist integers d and u such that

$$(m)\alpha = (m)\beta \quad \text{and} \quad (n)\alpha = (n)\beta$$

for all integers $m \leqslant d$ and $n \geqslant u$. Then we have that $d \leqslant u$. If $d = u$ or $d = u-1$ then $\alpha = \beta$ in $\mathscr{I}_\infty^\nearrow(\mathbb{Z})$ and hence α and β are $\mathfrak{C}_{mg}$-equivalent. If $d < u - 1$ then we put ε_0 to be the identity map of the set $\mathbb{Z} \setminus \{(d+1)\alpha, \ldots, (u-1)\alpha\}$. Then we get that $(n)(\alpha \circ \varepsilon_0) = (n)(\beta \circ \varepsilon_0)$ for any $n \in \mathbb{Z} \setminus \{d+1, \ldots, u-1\}$ and therefore $\alpha \cdot \varepsilon_0 = \beta \cdot \varepsilon_0$. Hence [26, Lemma III.5.2] implies that α and β are $\mathfrak{C}_{mg}$-equivalent elements of the semigroup $\mathscr{I}_\infty^\nearrow(\mathbb{Z})$.

Now we define the map $h\colon \mathscr{I}_\infty^\nearrow(\mathbb{Z}) \to \mathbb{Z}(+) \times \mathbb{Z}(+)$ by the formula

$$(\alpha)h = \big((d_\alpha)\alpha - d_\alpha, (u_\alpha)\alpha - u_\alpha\big),$$

where the integers d_α and u_α are defined in Lemma 1.1.

We observe that

$$(d_\alpha - n)\alpha = (d_\alpha)\alpha - n \quad \text{and} \quad (u_\alpha + n)\alpha = (u_\alpha)\alpha + n$$

for any positive integer n. Hence we have that

$$(m)\alpha - m = (d_\alpha)\alpha - d_\alpha \quad \text{and} \quad (n)\alpha - n = (u_\alpha)\alpha - u_\alpha$$

for all integers $m \leqslant d_\alpha$ and $n \geqslant u_\alpha$.

Lemma 1.1 implies that there exist integers d^0 and u^0 such that

$$
\begin{aligned}
(m-1)\alpha &= (m)\alpha - 1, & (n+1)\alpha &= (n)\alpha + 1, \\
(m-1)\beta &= (m)\beta - 1, & (n+1)\beta &= (n)\beta + 1, \\
(m-1)(\alpha \cdot \beta) &= (m)(\alpha \cdot \beta) - 1, & (n+1)(\alpha \cdot \beta) &= (n)(\alpha \cdot \beta) + 1
\end{aligned}
$$

for all integers $m \leqslant d^0$ and $n \geqslant u^0$. Hence for all integers $m \leqslant d^0$ and $n \geqslant u^0$ we have that

$$(m)(\alpha \cdot \beta) - m = (m)(\alpha \cdot \beta) - (m)\alpha + (m)\alpha - m = ((d_\beta)\beta - d_\beta) + ((d_\alpha)\alpha - d_\alpha),$$

$$(n)(\alpha \cdot \beta) - n = (n)(\alpha \cdot \beta) - (n)\alpha + (n)\alpha - n = ((u_\beta)\beta - u_\beta) + ((u_\alpha)\alpha - u_\alpha).$$

This implies that the map $h \colon \mathscr{I}_\infty^{\nearrow}(\mathbb{Z}) \to \mathbb{Z}(+) \times \mathbb{Z}(+)$ is a homomorphism. □

Theorem 2.7 and Proposition 2.8 imply the following theorem.

Theorem 2.9. *Let S be a semigroup and $h \colon \mathscr{I}_\infty^{\nearrow}(\mathbb{Z}) \to S$ a non-annihilating homomorphism. Then either h is a monomorphism or $(\mathscr{I}_\infty^{\nearrow}(\mathbb{Z}))h$ is a homomorphic image of the group $\mathbb{Z}(+) \times \mathbb{Z}(+)$.*

3   Some remarks on the semigroup $\mathscr{I}_\infty^{\looparrowright}(\mathbb{Z})$

In this section we shall denote the identity of the semigroup $\mathscr{I}_\infty^{\looparrowright}(\mathbb{Z})$ by $\mathbb{I}$ and the group of units of $\mathscr{I}_\infty^{\looparrowright}(\mathbb{Z})$ by $H(\mathbb{I})$. The proof of the following proposition is similar to corresponding propositions in Section 2.

Proposition 3.1. *The following assertions hold:*

(i) $E(\mathscr{I}_\infty^{\looparrowright}(\mathbb{Z})) = E(\mathscr{I}_\infty^{\nearrow}(\mathbb{Z}))$.

(ii) $\alpha \mathcal{R} \beta$ *in* $\mathscr{I}_\infty^{\looparrowright}(\mathbb{Z})$ *if and only if* $\operatorname{dom}\alpha = \operatorname{dom}\beta$.

(iii) $\alpha \mathcal{L} \beta$ *in* $\mathscr{I}_\infty^{\looparrowright}(\mathbb{Z})$ *if and only if* $\operatorname{ran}\alpha = \operatorname{ran}\beta$.

(iv) $\alpha \mathcal{H} \beta$ *in* $\mathscr{I}_\infty^{\looparrowright}(\mathbb{Z})$ *if and only if* $\operatorname{dom}\alpha = \operatorname{dom}\beta$ *and* $\operatorname{ran}\alpha = \operatorname{ran}\beta$.

(v) $\mathscr{I}_\infty^{\looparrowright}(\mathbb{Z})$ *is a simple semigroup and hence* $\mathcal{J} = \mathscr{I}_\infty^{\looparrowright}(\mathbb{Z}) \times \mathscr{I}_\infty^{\looparrowright}(\mathbb{Z})$.

(vi) *For all idempotents* $\varepsilon, \varphi \in \mathscr{I}_\infty^{\looparrowright}(\mathbb{Z})$ *there exist infinitely many elements* $\alpha, \beta \in \mathscr{I}_\infty^{\looparrowright}(\mathbb{Z})$ *such that* $\alpha \cdot \beta = \varepsilon$ *and* $\beta \cdot \alpha = \varphi$.

(vii) $\mathscr{I}_\infty^{\looparrowright}(\mathbb{Z})$ *is a bisimple semigroup.*

(viii) *For all* $\alpha, \beta \in \mathscr{I}_\infty^{\looparrowright}(\mathbb{Z})$, *both sets* $\{\chi \in \mathscr{I}_\infty^{\looparrowright}(\mathbb{Z}) \mid \alpha \cdot \chi = \beta\}$ *and* $\{\chi \in \mathscr{I}_\infty^{\looparrowright}(\mathbb{Z}) \mid \chi \cdot \alpha = \beta\}$ *are finite.*

Proposition 3.2. *For every $\alpha \in \mathcal{I}_\infty^{\looparrowright}(\mathbb{Z})$ there exist idempotents $\varepsilon_l, \varepsilon_r, \varepsilon$ in $\mathcal{I}_\infty^{\looparrowright}(\mathbb{Z})$ such that $\varepsilon_l \cdot \alpha, \alpha \cdot \varepsilon_r, \varepsilon \cdot \alpha \cdot \varepsilon \in \mathcal{I}_\infty^{\nearrow}(\mathbb{Z})$.*

Proof. The definition of the semigroup $\mathcal{I}_\infty^{\looparrowright}(\mathbb{Z})$ implies that for every element α of $\mathcal{I}_\infty^{\looparrowright}(\mathbb{Z})$ there exists a smallest finite (or empty) subset F_α such that the restriction $\alpha|_{\operatorname{dom}\alpha \setminus F_\alpha} \colon \mathbb{Z} \rightharpoonup \mathbb{Z}$ is a monotone partial map. We put $\varepsilon_l = \operatorname{id}_{\operatorname{dom}\alpha \setminus F_\alpha}$ to be the identity map from $\operatorname{dom}\alpha \setminus F_\alpha$ onto $\operatorname{dom}\alpha \setminus F_\alpha$. Also we set $\varepsilon_r = \operatorname{id}_{(\operatorname{dom}\alpha \setminus F_\alpha)\alpha}$ and $\varepsilon = \varepsilon_l \cdot \varepsilon_r$. Then we have that $\varepsilon_l \cdot \alpha, \alpha \cdot \varepsilon_r, \varepsilon \cdot \alpha \cdot \varepsilon \in \mathcal{I}_\infty^{\nearrow}(\mathbb{Z})$. □

We denote by $\mathsf{S}_\infty(\mathbb{Z})$ the group of all bijective transformations of $\mathbb{Z}$ with finite supports (i.e., $\alpha \in \mathsf{S}_\infty(\mathbb{Z})$ if and only if the set $\{x \in \mathbb{Z} \mid (x)\alpha \neq x\}$ is finite). We observe that $\mathsf{S}_\infty(\mathbb{Z})$ is a subgroup of the group of units $H(\mathbb{I})$ of the semigroup $\mathcal{I}_\infty^{\looparrowright}(\mathbb{Z})$ and since every element α in $H(\mathbb{I})$ is an almost monotone bijective self-map of the set of integers, we get that for every $\alpha \in H(\mathbb{I})$ there exists an integer n_α such that the set $\{i \in \mathbb{Z} \mid (i)\alpha + n_\alpha \neq i\}$ is finite. This observation implies that $\mathsf{S}_\infty(\mathbb{Z})$ is a normal subgroup of $H(\mathbb{I})$. Moreover, we have that every element of the group of units $H(\mathbb{I})$ has a unique representation $\alpha = \sigma \cdot \beta$ by the formula $(n)\alpha = (n)\sigma + \beta$, $n \in \mathbb{Z}$, where $\sigma \in \alpha \in \mathsf{S}_\infty(\mathbb{Z})$ and $\beta \in \mathbb{Z}(+)$. Hence we have that $H(\mathbb{I}) = \mathsf{S}_\infty(\mathbb{Z}) \cdot \mathbb{Z}(+)$ and it is obvious that $\mathsf{S}_\infty(\mathbb{Z}) \cap \mathbb{Z}(+) = \{\mathbb{I}\}$. Thus the group $\mathbb{Z}(+)$ acts on $\mathsf{S}_\infty(\mathbb{Z})$ by the conjugation action in $H(\mathbb{I})$ and hence it follows by [12, Exercise 2.5.3] that the group $H(\mathbb{I})$ is isomorphic to the semidirect product $\mathsf{S}_\infty(\mathbb{Z}) \rtimes \mathbb{Z}(+)$ (or split extension of $\mathsf{S}_\infty(\mathbb{Z})$ by $\mathbb{Z}(+)$). Also, we observe that since the action of the group $\mathbb{Z}(+)$ on $\mathsf{S}_\infty(\mathbb{Z})$ is not the identity map, we conclude that the group $H(\mathbb{I})$ is not isomorphic to the direct product $\mathsf{S}_\infty(\mathbb{Z}) \times \mathbb{Z}(+)$. We put $\widetilde{\mathbb{Z}}(+) = \mathsf{S}_\infty(\mathbb{Z}) \rtimes \mathbb{Z}(+)$. Therefore we have proved the following result.

Proposition 3.3. *The group of units $H(\mathbb{I})$ of the semigroup $\mathcal{I}_\infty^{\looparrowright}(\mathbb{Z})$ is isomorphic to $\widetilde{\mathbb{Z}}(+)$.*

Proposition 3.1 (vii) and [9, Theorem 2.20] imply the following corollary.

Corollary 3.4. *Every maximal subgroup of the semigroup $\mathcal{I}_\infty^{\looparrowright}(\mathbb{Z})$ is isomorphic to $\widetilde{\mathbb{Z}}(+)$.*

Theorem 3.5. *Let S be a semigroup and $h \colon \mathcal{I}_\infty^{\looparrowright}(\mathbb{Z}) \to S$ a non-annihilating homomorphism. Then either h is a monomorphism or $(\mathcal{I}_\infty^{\looparrowright}(\mathbb{Z}))h$ is a subgroup of S.*

Proof. Suppose that $h \colon \mathcal{I}_\infty^{\looparrowright}(\mathbb{Z}) \to S$ is not an monomorphism. Then $(\alpha)h = (\beta)h$, for some distinct $\alpha, \beta \in \mathcal{I}_\infty^{\looparrowright}(\mathbb{Z})$.

Suppose that α and β are not $\mathscr{H}$-equivalent. Since $\mathcal{I}_\infty^{\looparrowright}(\mathbb{Z})$ is an inverse semigroup, we have that either $\alpha \cdot \alpha^{-1} \neq \beta \cdot \beta^{-1}$ or $\alpha^{-1} \cdot \alpha \neq \beta^{-1} \cdot \beta$. Suppose

that $\alpha \cdot \alpha^{-1} \neq \beta \cdot \beta^{-1}$. Since $\mathscr{I}_\infty^{\looparrowright}(\mathbb{Z})$ is an inverse semigroup, we conclude that $(\alpha^{-1})h = (\beta^{-1})h$ and hence $(\alpha \cdot \alpha^{-1})h = (\beta \cdot \beta^{-1})h$. Therefore the assertion of Lemma 2.6 holds for the subsemigroup $\mathscr{I}_\infty^{\nearrow}(\mathbb{Z})$ of the semigroup $\mathscr{I}_\infty^{\looparrowright}(\mathbb{Z})$. Now by Proposition 3.1 (i) and since every homomorphic image of an inverse semigroup is an inverse semigroup, it follows that $(\mathscr{I}_\infty^{\looparrowright}(\mathbb{Z}))h$ is a subgroup of S.

Suppose that $\alpha \mathscr{H} \beta$. Then by [9, Theorem 2.20] there exist distinct $\alpha_0, \beta_0 \in H(\mathbb{I})$ such that $(\alpha_0)h = (\beta_0)h$. Then we have that $(\mathbb{I})h = (\gamma)h$ for $\gamma = \alpha_0^{-1} \cdot \beta_0 \in H(\mathbb{I})$ and $\gamma \neq \mathbb{I}$. We fix an arbitrary integer i. Let $\iota \colon \mathbb{Z} \setminus \{i\} \to \mathbb{Z} \setminus \{i\}$ be the identity map. Hence $(\iota)h = (\iota \cdot \mathbb{I})h = (\iota \cdot \gamma)h$. Then ι is an idempotent of the semigroup $\mathscr{I}_\infty^{\looparrowright}(\mathbb{Z})$ and $\operatorname{ran}\iota \neq \operatorname{ran}(\iota \cdot \gamma)$. Therefore by Proposition 3.1 (iv) the elements ι and $\iota \cdot \gamma$ are not $\mathscr{H}$-equivalent in the semigroup $\mathscr{I}_\infty^{\looparrowright}(\mathbb{Z})$. This implies that there exist distinct non-$\mathscr{H}$-equivalent elements α, β in $\mathscr{I}_\infty^{\looparrowright}(\mathbb{Z})$ such that $(\alpha)h = (\beta)h$ and hence $(\mathscr{I}_\infty^{\looparrowright}(\mathbb{Z}))h$ is a subgroup of S.   □

Proposition 3.6. *Let $\mathfrak{C}_{\mathrm{mg}}$ be a least group congruence on the semigroup $\mathscr{I}_\infty^{\looparrowright}(\mathbb{Z})$. Then the quotient semigroup $\mathscr{I}_\infty^{\looparrowright}(\mathbb{Z})/\mathfrak{C}_{\mathrm{mg}}$ is isomorphic to the direct product $\mathbb{Z}(+) \times \mathbb{Z}(+)$.*

Proof. Let α and β be $\mathfrak{C}_{\mathrm{mg}}$-equivalent elements of the semigroup $\mathscr{I}_\infty^{\looparrowright}(\mathbb{Z})$. Then by [26, Lemma III.5.2] there exists an idempotent ε_0 in $\mathscr{I}_\infty^{\looparrowright}(\mathbb{Z})$ such that $\alpha \cdot \varepsilon_0 = \beta \cdot \varepsilon_0$. By Proposition 3.2 we can assume without loss of generality that $\alpha \cdot \varepsilon_0$, $\beta \cdot \varepsilon_0 \in \mathscr{I}_\infty^{\nearrow}(\mathbb{Z})$. Then similarly as in the proof of Proposition 2.8 we can show that elements α and β of the semigroup $\mathscr{I}_\infty^{\looparrowright}(\mathbb{Z})$ are $\mathfrak{C}_{\mathrm{mg}}$-equivalent if and only if there exist integers d and u such that

$$(m)\alpha = (m)\beta \quad \text{and} \quad (n)\alpha = (n)\beta$$

for all integers $m \leqslant d$ and $n \geqslant u$.

Let $\alpha_0 = \alpha \cdot \varepsilon_0$. Then the map $h \colon \mathscr{I}_\infty^{\looparrowright}(\mathbb{Z}) \to \mathbb{Z}(+) \times \mathbb{Z}(+)$ defined by the formula

$$(\alpha)h = \big((d_{\alpha_0})\alpha - d_{\alpha_0}, (u_{\alpha_0})\alpha - u_{\alpha_0}\big),$$

where the integers d_{α_0} and u_{α_0} are defined for element α_0 of the semigroup $\mathscr{I}_\infty^{\nearrow}(\mathbb{Z})$ in Lemma 1.1, is a natural homomorphism which is generated by the least group congruence $\mathfrak{C}_{\mathrm{mg}}$ on the semigroup $\mathscr{I}_\infty^{\looparrowright}(\mathbb{Z})$.   □

Theorem 3.5 and Proposition 3.6 imply the following result.

Theorem 3.7. *Let S be a semigroup and $h \colon \mathscr{I}_\infty^{\looparrowright}(\mathbb{Z}) \to S$ a non-annihilating homomorphism. Then either h is a monomorphism or $(\mathscr{I}_\infty^{\looparrowright}(\mathbb{Z}))h$ is a homomorphic image of the group $\mathbb{Z}(+) \times \mathbb{Z}(+)$.*

4   On topologizations of the semigroup $\mathscr{I}_\infty^{\nearrow}(\mathbb{Z})$

Theorem 4.1. *Every Baire topology τ on $\mathscr{I}_\infty^{\nearrow}(\mathbb{Z})$, such that $(\mathscr{I}_\infty^{\nearrow}(\mathbb{Z}), \tau)$ is a Hausdorff semitopological semigroup, is discrete.*

Proof. If no point in $\mathscr{I}_\infty^{\nearrow}(\mathbb{Z})$ is isolated, then since $(\mathscr{I}_\infty^{\nearrow}(\mathbb{Z}), \tau)$ is Hausdorff, it follows that $\{\alpha\}$ is nowhere dense for all $\alpha \in \mathscr{I}_\infty^{\nearrow}(\mathbb{Z})$. But, if this is the case, then since $\mathscr{I}_\infty^{\nearrow}(\mathbb{Z})$ is countable, it cannot be a Baire space. Hence $\mathscr{I}_\infty^{\nearrow}(\mathbb{Z})$ contains an isolated point μ. If $\gamma \in \mathscr{I}_\infty^{\nearrow}(\mathbb{Z})$ is arbitrary, then by Proposition 2.1 (viii), there exist $\alpha, \beta \in \mathscr{I}_\infty^{\nearrow}(\mathbb{Z})$ such that $\alpha \cdot \gamma \cdot \beta = \mu$. The map $f : \chi \mapsto \alpha \cdot \chi \cdot \beta$ is continuous and so $(\{\mu\}) f^{-1}$ is open. By Proposition 2.5, $(\{\mu\}) f^{-1}$ is finite and since $(\mathscr{I}_\infty^{\nearrow}(\mathbb{Z}), \tau)$ is Hausdorff, $\{\gamma\}$ is open, and hence isolated. $\square$

Since every Čech complete space (and hence every locally compact space) is Baire, Theorem 4.1 implies Corollaries 4.2 and 4.3.

Corollary 4.2. *Every Hausdorff Čech complete (locally compact) topology τ on $\mathscr{I}_\infty^{\nearrow}(\mathbb{Z})$, such that $(\mathscr{I}_\infty^{\nearrow}(\mathbb{Z}), \tau)$ is a Hausdorff semitopological semigroup, is discrete.*

Corollary 4.3. *Every Hausdorff Baire topology (and hence Čech complete or locally compact topology) τ on $\mathscr{I}_\infty^{\nearrow}(\mathbb{Z})$, such that $(\mathscr{I}_\infty^{\nearrow}(\mathbb{Z}), \tau)$ is a Hausdorff topological semigroup, is discrete.*

The following example shows that there exists a non-discrete Tychonoff topology τ_W on the semigroup $\mathscr{I}_\infty^{\nearrow}(\mathbb{Z})$ such that $(\mathscr{I}_\infty^{\nearrow}(\mathbb{Z}), \tau_W)$ is a topological inverse semigroup.

Example 4.4. We define a topology τ_W on the semigroup $\mathscr{I}_\infty^{\nearrow}(\mathbb{Z})$ as follows. For every $\alpha \in \mathscr{I}_\infty^{\nearrow}(\mathbb{Z})$ we define a family

$$\mathscr{B}_W(\alpha) = \{U_\alpha(F) \mid F \text{ is a finite subset of } \operatorname{dom}\alpha\},$$

where

$$U_\alpha(F) = \{\beta \in \mathscr{I}_\infty^{\nearrow}(\mathbb{Z}) \mid \operatorname{dom}\beta \subseteq \operatorname{dom}\alpha \text{ and } (x)\beta = (x)\alpha \text{ for all } x \in F\}.$$

It is straightforward to verify that $\{\mathscr{B}_W(\alpha)\}_{\alpha \in \mathscr{I}_\infty^{\nearrow}(\mathbb{Z})}$ forms a basis for a topology τ_W on the semigroup $\mathscr{I}_\infty^{\nearrow}(\mathbb{Z})$.

Proposition 4.5. $(\mathscr{I}_\infty^{\nearrow}(\mathbb{Z}), \tau_W)$ *is a Tychonoff topological inverse semigroup.*

Proof. Let α and β be arbitrary elements of the semigroup $\mathscr{I}_\infty^{\nearrow}(\mathbb{Z})$. We put $\gamma = \alpha \cdot \beta$ and let $F = \{n_1, \ldots, n_i\}$ be a finite subset of dom γ. We denote $m_1 = (n_1)\alpha, \ldots, m_i = (n_i)\alpha$ and $k_1 = (n_1)\gamma, \ldots, k_i = (n_i)\gamma$. Then we get that $(m_1)\beta = k_1, \ldots, (m_i)\beta = k_i$. Hence we have that

$$U_\alpha(\{n_1, \ldots, n_i\}) \cdot U_\beta(\{m_1, \ldots, m_i\}) \subseteq U_\gamma(\{n_1, \ldots, n_i\})$$

and

$$\left(U_\gamma(\{n_1, \ldots, n_i\})\right)^{-1} \subseteq U_{\gamma^{-1}}(\{k_1, \ldots, k_i\}).$$

Therefore the semigroup operation and the inversion are continuous in $(\mathscr{I}_\infty^{\nearrow}(\mathbb{Z}), \tau_W)$.

Let $Z = \mathbb{Z} \cup \{a\}$ for some $a \notin \mathbb{Z}$. Then Z^Z with the operation composition is a semigroup and the map $\Psi \colon \mathscr{I}_\infty^{\nearrow}(\mathbb{Z}) \to Z^Z$ defined by the formula

$$(x)(\alpha)\Psi = \begin{cases} (x)\alpha, & \text{if } x \in \text{dom } \alpha, \\ a, & \text{if } x \notin \text{dom } \alpha \end{cases}$$

is a monomorphism. Hence Z^Z is a topological semigroup with the product topology if Z has the discrete topology. Obviously, this topology generates topology τ_W on $\mathscr{I}_\infty^{\nearrow}(\mathbb{Z})$. Therefore by [14, Theorem 2.3.11] the topological space Z^Z is Tychonoff and hence by [14, Theorem 2.1.6] so is $(\mathscr{I}_\infty^{\nearrow}(\mathbb{Z}), \tau_W)$. This completes the proof of the proposition.   □

Theorem 4.6. *Let S be a topological semigroup which contains $\mathscr{I}_\infty^{\nearrow}(\mathbb{Z})$ as a dense discrete subsemigroup. If $I = S \setminus \mathscr{I}_\infty^{\nearrow}(\mathbb{Z}) \neq \varnothing$ then I is an ideal of S.*

Proof. Suppose that I is not an ideal of S. Then at least one of the following conditions holds:

$$I \cdot \mathscr{I}_\infty^{\nearrow}(\mathbb{Z}) \nsubseteq I, \quad \mathscr{I}_\infty^{\nearrow}(\mathbb{Z}) \cdot I \nsubseteq I, \quad \text{or} \quad I \cdot I \nsubseteq I.$$

Since $\mathscr{I}_\infty^{\nearrow}(\mathbb{Z})$ is a dense discrete subspace of S, [14, Theorem 3.5.8] implies that $\mathscr{I}_\infty^{\nearrow}(\mathbb{Z})$ is an open subspace of S. Suppose there exist $\alpha \in \mathscr{I}_\infty^{\nearrow}(\mathbb{Z})$ and $\beta \in I$ such that $\beta \cdot \alpha = \gamma \notin I$. Since $\mathscr{I}_\infty^{\nearrow}(\mathbb{Z})$ is a dense open discrete subspace of S, the continuity of the semigroup operation in S implies that there exists an open neighborhood $U(\beta)$ of β in S such that $U(\beta) \cdot \{\alpha\} = \{\gamma\}$. Hence we have that $(U(\beta) \cap \mathscr{I}_\infty^{\nearrow}(\mathbb{Z})) \cdot \{\alpha\} = \{\gamma\}$ and the set $U(\beta) \cap \mathscr{I}_\infty^{\nearrow}(\mathbb{Z})$ is infinite. But by Proposition 2.5, the equation $\chi \cdot \alpha = \gamma$ has finitely many solutions in $\mathscr{I}_\infty^{\nearrow}(\mathbb{Z})$. This contradicts the assumption that $\beta \in S \setminus \mathscr{I}_\infty^{\nearrow}(\mathbb{Z})$. Therefore $\beta \cdot \alpha = \gamma \in I$ and hence $I \cdot \mathscr{I}_\infty^{\nearrow}(\mathbb{Z}) \subseteq I$. The proof of the inclusion $\mathscr{I}_\infty^{\nearrow}(\mathbb{Z}) \cdot I \subseteq I$ is similar.

Suppose there exist $\alpha, \beta \in I$ such that $\alpha \cdot \beta = \gamma \notin I$. Since $\mathscr{I}_\infty^\nearrow(\mathbb{Z})$ is a dense open discrete subspace of S, the continuity of the semigroup operation in S implies that there exist open neighborhoods $U(\alpha)$ and $U(\beta)$ of α and β in S, respectively, such that $U(\alpha) \cdot U(\beta) = \{\gamma\}$. Hence we have that

$$\big(U(\beta) \cap \mathscr{I}_\infty^\nearrow(\mathbb{Z})\big) \cdot \big(U(\alpha) \cap \mathscr{I}_\infty^\nearrow(\mathbb{Z})\big) = \{\gamma\}$$

and the sets $U(\beta) \cap \mathscr{I}_\infty^\nearrow(\mathbb{Z})$ and $U(\alpha) \cap \mathscr{I}_\infty^\nearrow(\mathbb{Z})$ are infinite. But by Proposition 2.5, the equations $\chi \cdot \beta = \gamma$ and $\alpha \cdot \kappa = \gamma$ have finitely many solutions in $\mathscr{I}_\infty^\nearrow(\mathbb{Z})$. This contradicts the assumption that $\alpha, \beta \in S \setminus \mathscr{I}_\infty^\nearrow(\mathbb{Z})$. Therefore $\alpha \cdot \beta = \gamma \in I$ and hence $I \cdot I \subseteq I$. $\qquad\square$

Proposition 4.7. *Let S be a Hausdorff topological semigroup which contains $\mathscr{I}_\infty^\nearrow(\mathbb{Z})$ as a dense discrete subsemigroup. Then for every $\gamma \in \mathscr{I}_\infty^\nearrow(\mathbb{Z})$ the set*

$$D_\gamma = \{(\chi, \varsigma) \in \mathscr{I}_\infty^\nearrow(\mathbb{Z}) \times \mathscr{I}_\infty^\nearrow(\mathbb{Z}) \mid \chi \cdot \varsigma = \gamma\}$$

is a closed-and-open subset of $S \times S$.

Proof. Since $\mathscr{I}_\infty^\nearrow(\mathbb{Z})$ is a discrete subspace of S, we have that D_γ is an open subset of $S \times S$.

Suppose that there exists $\gamma \in \mathscr{I}_\infty^\nearrow(\mathbb{Z})$ such that D_γ is a non-closed subset of $S \times S$. Then there exists an accumulation point $(\alpha, \beta) \in S \times S$ of the set D_γ. The continuity of the semigroup operation in S implies that $\alpha \cdot \beta = \gamma$. But $\mathscr{I}_\infty^\nearrow(\mathbb{Z}) \times \mathscr{I}_\infty^\nearrow(\mathbb{Z})$ is a discrete subspace of $S \times S$ and hence by Theorem 4.6, the points α and β belong to the ideal $I = S \setminus \mathscr{I}_\infty^\nearrow(\mathbb{Z})$ and hence $\alpha \cdot \beta \in S \setminus \mathscr{I}_\infty^\nearrow(\mathbb{Z})$ cannot be equal to γ. $\qquad\square$

Theorem 4.8. *If a Hausdorff topological semigroup S contains $\mathscr{I}_\infty^\nearrow(\mathbb{Z})$ as a dense discrete subsemigroup then the square $S \times S$ cannot be pseudocompact.*

The proof of Theorem 4.8 is similar to that of [6, Theorem 5.1 (3)].

Recall that a topological semigroup S is called Γ-compact if for every $x \in S$ the closure of the set $\{x, x^2, x^3, \ldots\}$ is a compactum in S (see [20]). We recall that the Stone–Čech compactification of a Tychonoff space X is a compact Hausdorff space βX containing X as a dense subspace so that each continuous map $f \colon X \to Y$ to a compact Hausdorff space Y extends to a continuous map $\overline{f} \colon \beta X \to Y$ (see [14]).

Corollary 4.9. *If a topological semigroup S satisfies one of the following conditions below, then S does not contain the semigroup $\mathscr{I}_\infty^\nearrow(\mathbb{Z})$ (and hence the semigroup $\mathscr{I}_\infty^{\leftrightarrow}(\mathbb{Z})$):*

(i) *S is compact;*

(ii) *S is Γ-compact;*

(iii) *the square S × S is countably compact;*

(iv) *S is a countably compact topological inverse semigroup; or*

(v) *the square S × S is a Tychonoff pseudocompact space.*

Proof. By [22, Theorem 2] every compact topological semigroup is stable. But by [3, Corollary 3.1] a stable semigroup cannot contain the bicyclic semigroup. Since by Remark 1.2, for any positive integer n the semigroup $\mathscr{I}_\infty^{\nearrow}(\mathbb{Z})$ contains the semigroup $\mathscr{C}(n, +)$ which is isomorphic to the bicyclic semigroup, we conclude that any compact topological semigroup cannot contain the semigroup $\mathscr{I}_\infty^{\nearrow}(\mathbb{Z})$.

Similarly by [20, Proposition 5.3] no Γ-compact topological semigroup can contain the bicyclic semigroup. Also the proof of [5, Theorem 10] implies that every topological semigroup S with countably compact square $S \times S$ cannot contain the bicyclic semigroup and by [17, Theorem 1] any countably compact topological inverse semigroup cannot contain the bicyclic semigroup, either. Next we apply Remark 1.2.

By [4, Theorem 1.3] for any topological semigroup S with the pseudocompact square $S \times S$ the semigroup operation $\mu \colon S \times S \to S$ extends to a continuous semigroup operation $\beta\mu \colon \beta S \times \beta S \to \beta S$, so S is a subsemigroup of the compact topological semigroup βS. Therefore if S contains the bicyclic semigroup then βS also contains the bicyclic semigroup which is a contradiction. ☐

The proofs of the following three theorems are similar to the proofs of Theorems 4.1, 4.6 and 4.8, respectively.

Theorem 4.10. *Every Baire topology τ on $\mathscr{I}_\infty^{\leftrightarrow}(\mathbb{Z})$, such that $(\mathscr{I}_\infty^{\leftrightarrow}(\mathbb{Z}), \tau)$ is a Hausdorff semitopological semigroup, is discrete.*

Theorem 4.11. *Let S be a topological semigroup which contains $\mathscr{I}_\infty^{\leftrightarrow}(\mathbb{Z})$ as a dense discrete subsemigroup. If $I = S \setminus \mathscr{I}_\infty^{\leftrightarrow}(\mathbb{Z}) \neq \varnothing$ then I is an ideal of S.*

Theorem 4.12. *If a Hausdorff topological semigroup S contains $\mathscr{I}_\infty^{\leftrightarrow}(\mathbb{Z})$ as a dense discrete subsemigroup then the square $S \times S$ cannot be pseudocompact.*

Remark 4.13. We observe that the topology $\tau_W^{\leftrightarrow}$ on the semigroup $\mathscr{I}_\infty^{\leftrightarrow}(\mathbb{Z})$ which is generated by the family

$$\mathscr{B}_W^{\leftrightarrow}(\alpha) = \left\{ U_\alpha(F) \mid F \text{ is a finite subset of } \operatorname{dom}\alpha \right\},$$

where

$$U_\alpha(F) = \{\beta \in \mathscr{I}_\infty^{\looparrowright}(\mathbb{Z}) \mid \operatorname{dom}\beta \subseteq \operatorname{dom}\alpha \text{ and } (x)\beta = (x)\alpha \text{ for all } x \in F\},$$

is a non-discrete inverse semigroup topology. The proof of continuity of the semigroup operation and inversion in $(\mathscr{I}_\infty^{\looparrowright}(\mathbb{Z}), \tau_W^{\looparrowright})$ is similar to the proof of Proposition 4.5 and obviously the topology $\tau_W^{\looparrowright}$ induces the topology τ_W on the subsemigroup $\mathscr{I}_\infty^{\nearrow}(\mathbb{Z})$.

The following example shows that there exists a non-discrete Tychonoff topology $\tau_H^{\looparrowright}$ on the semigroup $\mathscr{I}_\infty^{\looparrowright}(\mathbb{Z})$ such that $(\mathscr{I}_\infty^{\looparrowright}(\mathbb{Z}), \tau_H^{\looparrowright})$ is a topological inverse semigroup, every H-class in $\mathscr{I}_\infty^{\looparrowright}(\mathbb{Z})$ is an open subset in $(\mathscr{I}_\infty^{\looparrowright}(\mathbb{Z}), \tau_H^{\looparrowright})$ and the topology $\tau_H^{\looparrowright}$ is finer than the topology $\tau_W^{\looparrowright}$.

Example 4.14. We define a topology $\tau_H^{\looparrowright}$ on the semigroup $\mathscr{I}_\infty^{\looparrowright}(\mathbb{Z})$ as follows. For every $\alpha \in \mathscr{I}_\infty^{\looparrowright}(\mathbb{Z})$ we define a family

$$\mathscr{B}_H^{\looparrowright}(\alpha) = \{W_\alpha(F) \mid F \text{ is a finite subset of } \operatorname{dom}\alpha\},$$

where

$$W_\alpha(F) = \{\beta \in \mathscr{I}_\infty^{\looparrowright}(\mathbb{Z}) \mid \beta \mathscr{H} \alpha \text{ and } (x)\beta = (x)\alpha \text{ for all } x \in F\}.$$

It is straightforward to verify that $\{\mathscr{B}_H^{\looparrowright}(\alpha)\}_{\alpha \in \mathscr{I}_\infty^{\looparrowright}(\mathbb{Z})}$ forms a basis for a topology $\tau_H^{\looparrowright}$ on the semigroup $\mathscr{I}_\infty^{\looparrowright}(\mathbb{Z})$.

Proposition 4.15. $(\mathscr{I}_\infty^{\looparrowright}(\mathbb{Z}), \tau_H^{\looparrowright})$ *is a Tychonoff topological inverse semigroup.*

Proof. The proof of continuity of the semigroup operation and inversion in $(\mathscr{I}_\infty^{\looparrowright}(\mathbb{Z}), \tau_H^{\looparrowright})$ is similar to the proof of Proposition 4.5. Also the definition of the topology $\tau_H^{\looparrowright}$ implies that all H-classes are open subsets in $(\mathscr{I}_\infty^{\looparrowright}(\mathbb{Z}), \tau_H^{\looparrowright})$ and the group of units $H(\mathbb{I})$ of the semigroup $\mathscr{I}_\infty^{\looparrowright}(\mathbb{Z})$ with the induced topology from $(\mathscr{I}_\infty^{\looparrowright}(\mathbb{Z}), \tau_H^{\looparrowright})$ is a non-discrete topological group, and hence by [19, Theorem II.8.4] the topological subspace $H(\mathbb{I})$ is Tychonoff. Hence since every H-class in an arbitrary topological inverse semigroup S is a closed subset in S (see [13]), Proposition 3.1, Corollary 3.4 and [9, Theorem 2.20] imply that the topological space $(\mathscr{I}_\infty^{\looparrowright}(\mathbb{Z}), \tau_H^{\looparrowright})$ is homeomorphic to the topological sum of countable many topological spaces $H(\mathbb{I})$, and hence $(\mathscr{I}_\infty^{\looparrowright}(\mathbb{Z}), \tau_H^{\looparrowright})$ is a Tychonoff space. □

We observe that the topology $\tau_H^{\looparrowright}$ on the semigroup $\mathscr{I}_\infty^{\looparrowright}(\mathbb{Z})$ induces the discrete topology on its subsemigroup $\mathscr{I}_\infty^{\nearrow}(\mathbb{Z})$.

Recall [11] that a *Bohr compactification* of a topological semigroup S is a pair $(\beta, B(S))$ such that $B(S)$ is a compact topological semigroup, $\beta \colon S \to B(S)$ is a

continuous homomorphism, and if $g\colon S \to T$ is a continuous homomorphism of S into a compact semigroup T, then there exists a unique continuous homomorphism $f\colon B(S) \to T$ such that the diagram

$$
\begin{array}{ccc}
S & \xrightarrow{\quad\beta\quad} & B(S) \\
& \searrow{\scriptstyle g} \quad \swarrow{\scriptstyle f} & \\
& T &
\end{array}
$$

commutes. Then Theorems 2.9 and 3.7, and [2, Proposition 2] imply the following corollary.

Corollary 4.16. *The Bohr compactifications of the discrete semigroups $\mathscr{I}_\infty^{\nearrow}(\mathbb{Z})$ and $\mathscr{I}_\infty^{\looparrowright}(\mathbb{Z})$ are topologically isomorphic to the Bohr compactification of the discrete group $\mathbb{Z}(+) \times \mathbb{Z}(+)$.*

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