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THE CONTINUITY OF THE INVERSION AND THE STRUCTURE OF MAXIMAL SUBGROUPS IN COUNTABLY COMPACT TOPOLOGICAL SEMIGROUPS

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Abstract. We search for conditions on a countably compact (pseudocompact) topological semigroup under which: (i) each maximal subgroup $H(e)$ in S is a (closed) topological subgroup in S; (ii) the Clifford part $H(S)$ (i.e. the union of all maximal subgroups) of the semigroup S is a closed subset in S; (iii) the inversion $\mathrm{inv}: H(S) \to H(S)$ is continuous; and (iv) the projection $\pi: H(S) \to E(S)$, $\pi: x \longmapsto xx^{-1}$, onto the subset of idempotents $E(S)$ of S, is continuous.

In this paper all topological spaces will be assumed to be Hausdorff. We shall follow the terminology of [7, 8, 11]. We shall denote the cardinality of continuum by $\mathfrak{c}$ and the topological closure of subset A in a topological space by $\overline{A}$. We shall call a T_3-space a regular topological space.

A topological space X is said to be *countably compact* if any countable open cover of X contains a finite subcover [11]. A topological space X is called *pseudocompact* if each continuous real-valued function on X is bounded [11].

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A topological space X is said to be *sequential* if each non-closed subset A of X contains a sequence of points $\{x_n\}_{n=1}^{\infty}$ that converges to some point $x \in X \setminus A$. Obviously, a topological space X is sequential if a subset A of X is closed if and only if together with any convergent sequence A contains its limit [11]. A topological space X is called *sequentially compact* if each sequence $\{x_n\}_{n=1}^{\infty} \subset X$ has a convergent subsequence [11].

We recall that the Stone–Čech compactification of a Tychonoff space X is a compact Hausdorff space βX containing X is a dense subspace so that each continuous map $f\colon X \to Y$ to a compact Hausdorff space Y extends to a continuous map $\overline{f}\colon \beta X \to Y$ [11].

A *semigroup* is a set with a binary associative operation. An element e of a semigroup S is called an *idempotent* if $ee = e$. If S is a semigroup, then we denote the subset of all idempotents of S by $E(S)$. A semigroup S is called *inverse* if for any $x \in S$ there exists a unique $y \in S$ such that $xyx = x$ and $yxy = y$. Such an element y is called *inverse* of x and is denoted by x^{-1}. If S is an inverse semigroup, then the map which takes $x \in S$ to the inverse element of x is called the *inversion* and will be denoted by inv.

If S is a semigroup and e is an idempotent in S, then e lies in the maximal subgroup $H(e)$ with the identity e. If a semigroup S is a union of groups then S is called *Clifford*. On a Clifford semigroup $S = \bigcup\{H(e) \mid e \in E(S)\}$ the inversion $\mathrm{inv}\colon S \to S$ is defined which maps each element $x \in H(e)$ to its inverse element x^{-1} in $H(e)$. We also observe that on any Clifford semigroup the *projection* $\pi\colon S \to E(S)$, $\pi(x) = x \cdot x^{-1}$, is defined. For a semigroup S let

$$H(S) = \bigcup_{e \in E(S)} \{H(e) \mid H(e) \text{ is a maximal subgroup in } S \text{ with identity } e\}$$

$$= \{s \in S \mid \text{ there exists } y \in S \text{ such that } xy = yx,\ xyx = x,\ yxy = y\}.$$

A topological space S which is algebraically a semigroup with a continuous semigroup operation is called a *topological semigroup*. A *topological inverse semigroup* is a topological semigroup S that is algebraically an inverse semigroup with continuous inversion. If τ is a topology on a (inverse) semigroup S such that (S,τ) is a topological (inverse) semigroup, then τ is called a (*inverse*) *semigroup topology* on S. By a *paratopological group* we understand a pair (G,τ) consisting of a group G and a topology τ on G making the group operation on G continuous. A paratopological group G with continuous inversion is called a *topological group*.

Finite semigroups and compact topological semigroups have similar properties. For example every finite semigroup and every compact topological semigroup contains idempotents and minimal ideals [26], which are completely simple semigroups [25, 27], and every (0–)simple compact topological (and hence finite) semigroup is completely (0-)simple [20, 25, 27]. Also, a cancellative compact topological (and hence finite) semigroup is a topological group [19].

Compact topological semigroups do not contain the bicyclic semigroup [15]. Gutik and Repovš [14] proved that a countably compact topological inverse semigroup does not contain the bicyclic semigroup. Banakh, Dimitrova and Gutik [2] showed that no topological semigroup S with countably compact square $S \times S$ contains the bicyclic semigroup. They also constructed in [3] a consistent example of a countably compact topological semigroup S which contains the bicyclic semigroup.

It is well known that the closure of a (commutative) subsemigroup of a topological semigroup is a (commutative) subsemigroup [7, Vol. 1, p. 9]. Note that the closure of a subgroup in a topological semigroup is not necesarily a subgroup. But in the case when S is a compact topological semigroup or a topological inverse semigroup, the closure of a subgroup in S is a subgroup, moreover every maximal subgroup of S is closed (see [7, Vol. 1, Theorem 1.11] and [9]).

Also, every compact subgroup of a topological semigroup with induced topology is a topological semigroup. The results when the inversion is continuous in a topological semigroup which is algebraically a group (i.e. a paratopological group) have been extended to some classes of "compact-like" paratopological groups, in particular: regular locally compact paratopological groups [10], regular countably compact paratopological groups [22], quasi-regular pseudo-compact paratopological groups [22], topologically periodic Hausdorff countably compact paratopological groups [5], Čech-complete paratopological groups [6], strongly Baire semitopological groups [16].

On the other hand, Ravsky [21], using a result of Koszmider, Tomita and Watson [17], constructed an MA-example of a Hausdorff countably compact paratopological group failing to be a topological group. Also Grant [13] and Yur'eva [29] showed that a Hausdorff cancellative sequential countably compact topological semigroup is a topological group. Bokalo and Guran [5] established that an analogous theorem is true for cancellative sequentially compact semigroups. Robbie and Svetlichny [23] constructed a CH-example of a countably compact topological semigroup which is not a topological group.

In summary (see [7]), for a compact topological semigroup S the following conditions hold:

(1) each maximal subgroup $H(e)$ in S is a compact topological subgroup in S;

(2) the subset $H(S)$ is closed in S;

(3) the inversion map $\mathrm{inv} : H(S) \to H(S)$ is continuous; and

(4) the projection $\pi : H(S) \to E(S)$ is continuous.

Since sequential compactness, countable compactness and pseudocompactness are generalization of compactness, it is natural to pose the following question: *For which compact-like topological semigroups do the conditions* (1)–(4) *above hold?*

In this paper we shall answer this question by giving sufficient conditions on a countably compact (pseudocompact) topological semigroup under which: (i) each maximal subgroup $H(e)$ in S is a (closed) topological subgroup in S; (ii) the Clifford part $H(S)$ of the semigroup S is a closed subset in S; and (iii) the inversion $\mathrm{inv}: H(S) \to H(S)$ and the projection $\pi: H(S) \to E(S)$ are continuous.

A topological group G is called *totally bounded* if for any open neighbourhood U of the identity e of G there exists a finite subset A in G such that $A \cdot U = G$ (see [28]).

THEOREM 1. *Let S be a Tychonoff topological semigroup with the pseudocompact square $S \times S$. Then S embeds into a compact topological semigroup and the following conditions hold:*

(i) *the inversion* $\mathrm{inv}: H(S) \to H(S)$ *is continuous;*

(ii) *the projection* $\pi: H(S) \to E(S)$ *is continuous; and*

(iii) *for each idempotent $e \in E(S)$ the maximal subgroup $H(e)$ is a totally bounded topological group.*

PROOF. By Theorem 1.3 from [1], for any topological semigroup S with the pseudocompact square $S \times S$ the semigroup operation $\mu: S \times S \to S$ extends to a continuous semigroup operation $\beta\mu: \beta S \times \beta S \to \beta S$, so S is a subsemigroup of the compact topological semigroup βS.

(i) Let

$$\mathrm{Gr_{inv}}\big(H(\beta S)\big) = \big\{(x,y) \in S \times S \mid y = x^{-1}\big\}$$

be the graph of the inversion in $H(\beta S)$. Since βS is a topological semigroup and

$$\mathrm{Gr_{inv}}\big(H(\beta S)\big) = \big\{(x,y) \in S \times S \mid xyx = x,\ yxy = y \text{ and } xy = yx\big\},$$

the graph $\mathrm{Gr_{inv}}\big(H(\beta S)\big)$ is a compact subset of $\beta S \times \beta S$.

Consider the natural projections $\mathrm{pr}_1: \beta S \times \beta S \to \beta S$ and $\mathrm{pr}_2: \beta S \times \beta S \to \beta S$ onto the first and the second coordinates, respectively. It follows from the compactness of $\mathrm{Gr_{inv}}\big(H(\beta S)\big)$ that $\mathrm{pr}_1: \mathrm{Gr_{inv}}\big(H(\beta S)\big) \to H(\beta S)$ and $\mathrm{pr}_2: \mathrm{Gr_{inv}}\big(H(\beta S)\big) \to H(\beta S)$ are homeomorphisms. Consequently, the inversion $\mathrm{inv}|_{H(\beta S)} = \mathrm{pr}_2 \circ (\mathrm{pr}_1)^{-1}: H(\beta S) \to H(\beta S)$ is continuous, being a composition of homeomorphisms. Therefore the inversion $\mathrm{inv}: H(S) \to H(S)$ is continuous as a restriction of a continuous map.

(ii) The projection $\pi: H(S) \to E(S)$ is continuous as a composition of two continuous maps.

(iii) Given an idempotent $e \in E(S)$, consider the maximal subgroup $H_\beta(e)$ in βS containing e. Then by Theorems 1.11 and 1.13 from [7, Vol. 1], $H_\beta(e)$

is a compact topological group and since $H(e)$ is a subgroup of $H_\beta(e)$ the inversion $\mathrm{inv} : H(e) \to H(e)$ is continuous, and $H(e)$ is a totally bounded topological group, see [28]. $\square$

Theorem 1 implies the following:

COROLLARY 2. *If S is a Tychonoff Clifford topological semigroup with the pseudocompact square $S \times S$ then the inversion in S is continuous.*

An element x of a topological semigroup S is called *topologically periodic* if for any open neighbourhood $U(x)$ of x there exists an integer $n \geqq 2$ such that $x^n \in U(x)$. A topological semigroup S is called *topologically periodic* if any element of S is topologically periodic.

REMARK 3. The following observation implies that *an element of any (not necessarily Hausdorff) topological semigroup S is topologically periodic if and only if for any integer $n \geqq 2$ and for any open neighbourhood $U(x)$ of x there exists an integer $m \geqq n$ such that $x^m \in U(x)$.* Let $k \geqq 2$ be an integer such that $x^k \in U(x)$. Then the continuity of the semigroup operation implies that there exists an open neighbourhood $V(x)$ of x such that $\big(V(x)\big)^k \subseteqq U(x)$. Since x is topologically periodic there exists an integer $m \geqq 2$ such that $x^m \in V(x)$. Hence we have $x^{km} \in \big(V(x)\big)^k \subseteqq U(x)$ and $km \geqq 4 = 2^2$. Proceeding by induction, we can find an integer $p \geqq 2^n > n$ such that $x^p \in U(x)$.

THEOREM 4. *Let S be a Hausdorff topological semigroup with the countably compact square $S \times S$. Then:*

(i) *each maximal subgroup $H(e)$ of S is a countably compact topological group; and*

(ii) *the subset $H(S)$ is countably compact.*

PROOF. (i) Let $H(e)$ be any maximal subgroup of S. Since the semigroup operation in S is continuous the subset

$$G = \big\{ (x,y) \in S \times S \mid xy = yx = e,\ xe = ex = x,\ ye = ey = y \big\}$$

is closed in $S \times S$ and Theorem 3.10.4 from [11] implies that G is a countably compact subset in $S \times S$. Consider the natural projection $\mathrm{pr}_1 : S \times S \to S$ onto the first coordinate. Since $\mathrm{pr}_1(G) = H(e)$ and the projection $\mathrm{pr}_1 : S \times S \to S$ is a continuous map, Theorem 3.10.5 from [11] implies that $H(e)$ is a countably compact subspace of S.

Next, we show that $H(e)$ is a topologically periodic paratopological group. Let x be an arbitrary element of the subgroup $H(e)$ and $U(x)$ be any open neighbourhood of x. We consider the sequence $\big\{ (x^{n+1}, x^{-n}) \big\}_{n=1}^{\infty}$ in $H(e) \times H(e) \subseteqq S \times S$. The countable compactness of $S \times S$ guarantees that this sequence has an accumulation point $(a,b) \in S \times S$. Since $x^{n+1} \cdot x^{-n} = x$, the continuity of the semigroup operation on S guarantees that $ab = x$.

Then for any open neighbourhood $U(x)$ of x in S there exist open neighbourhoods $U(a)$ and $U(b)$ of the point a and b in S, respectively, such that $U(a)U(b) \subseteqq U(x)$. Since (a,b) is an accumulation point of the sequence $\big\{(x^{n+1}, x^{-n})\big\}_{n=1}^{\infty}$ in $S \times S$, there exist positive integers m and n such that $x^m \in U(a)$, $x^{-n} \in U(b)$ and $m \geqq n+2$. Hence we get that $x^m \cdot x^{-n} = x^{m-n} \in U(a) \cdot U(b) \subseteqq U(x)$ and $m-n \geqq 2$. Therefore $H(e)$ is a topologically periodic paratopological group. By Bokalo–Guran Theorem (see [5, Theorem 3]) any countably compact paratopological group is a topological group. Consequently, $H(e)$ is a countably compact topological group.

(ii) Since the semigroup operation in S is continuous,

$$H = \big\{(x,y) \in S \times S \mid xyx = x,\ yxy = y,\ xy = yx \in E(S)\big\}$$

is a closed subset in $S \times S$ and Theorem 3.10.4 from [11] implies that H is a countably compact subset in $S \times S$. Consider the natural projection $\mathrm{pr}_1 : S \times S \to S$ onto the first coordinate. Since $\mathrm{pr}_1(H) = H(S)$ and the projection $\mathrm{pr}_1 : S \times S \to S$ is a continuous map, Theorem 3.10.5 from [11] implies that $H(S)$ is a countably compact subspace of S. □

PROPOSITION 5. *Let x be a topologically periodic element of a maximal subgroup $H(e)$ with the unity e in a topological semigroup S. Then the inversion* inv : $H(S) \to H(S)$ *is continuous at x if and only if it is continuous at the idempotent e.*

PROOF. We follow the argument of [4]. Let $U(x^{-1})$ be any open neighbourhood of the inverse element x^{-1} of x in S. Since the semigroup operation in S is continuous there exist open neighbourhoods $V(x^{-1})$ and $V(e)$ of x^{-1} and e in $H(S)$, respectively, such that $V(x^{-1}) \cdot V(e) \subseteqq U(x^{-1})$. Since the inversion is continuous at idempotent e, there exists an open neighbourhood $W(e)$ of e in $H(S)$ such that $\big(W(e)\big)^{-1} \subseteqq V(e)$.

Also, the continuity of the semigroup operation implies that there exists an open neighbourhood $N(x)$ of x in $H(S)$ such that $x^{-1} \cdot N(x) \cdot x^{-1} \subseteqq V(x^{-1})$ and $N(x) \cdot x^{-1} \subseteqq W(e)$. The topological periodicity of x implies that there exists a positive integer n such that $x^{n+2} \in N(x)$. Then we have that

$$x^{n+1} = x^{n+2} \cdot x^{-1} \in N(x) \cdot x^{-1} \subseteqq W(e)$$

and

$$x^{n} = x^{-1} \cdot x^{n+2} \cdot x^{-1} \in x^{-1} \cdot N(x) \cdot x^{-1} \subseteqq V(x^{-1}).$$

Since S is a topological semigroup there exists an open neighbourhood $P(x)$ of x in S such that $\big(H(S) \cap P(x)\big)^{n+1} \subseteqq W(e)$ and $\big(H(S) \cap P(x)\big)^{n} \subseteqq V(x^{-1})$.

Therefore we get

$$\big(H(S)\cap P(x)\big)^{-1} \subseteqq \big(H(S)\cap P(x)\big)^{n} \cdot \Big(\big(H(S)\cap P(x)\big)^{n+1}\Big)^{-1}$$
$$\subseteqq V(x^{-1})\cdot \big(W(e)\big)^{-1} \subseteqq V(x^{-1})\cdot V(e) \subseteqq U(x^{-1}),$$

and hence the inversion is continuous at the point x. $\square$

Proposition 5 implies the following:

COROLLARY 6. *The inversion in a topologically periodic Clifford topological semigroup S is continuous if and only if it is continuous at any idempotent of the semigroup S.*

THEOREM 7. *Let S be a regular topological semigroup with the countably compact square $S\times S$. Then:*
(i) *the inversion* $\mathrm{inv}: H(S)\to H(S)$ *is continuous; and*
(ii) *the projection* $\pi: H(S)\to E(S)$ *is continuous.*

PROOF. (i) By Proposition 5 it is sufficient to show that the inversion $\mathrm{inv}: H(S)\to H(S)$ is continuous at any point of the set $E(S)$.

Fix any $e\in E(S)$. Let $U(e)$ be any open neighbourhood of e in S. Since the topological space of the semigroup S is regular, the continuity of the semigroup operation of S implies that there exists a sequence of open neighbourhoods $\big\{U_i(e)\big\}_{i=1}^{\infty}$ of the idempotent e in S such that $\overline{U_1(e)}\subseteqq U(e)$ and $\overline{\big(U_n(e)\big)^{m}}\subseteqq U_{n-1}(e)$ for any positive integer n and all $m=1,\dots,n$. Let $F=\bigcap_{n=1}^{\infty}\overline{U_n(e)}$.

We shall show that $\big(F\cap H(S)\big)^{-1}\subseteqq F$. Let x be any element of the set $F\cap H(S)$. Since the set F is closed, to prove that $x^{-1}\in F$ it sufficient to show that $V(x^{-1})\cap F\neq\emptyset$ for any open neighbourhood $V(x^{-1})$ of the point x^{-1}. The continuity of the semigroup operation in S and the equality $x^{-1}=x^{-1}\cdot x\cdot x^{-1}$ imply that there exists an open neighbourhood $V(x)$ of the point x in S such that $x^{-1}\cdot V(x)\cdot x^{-1}\subseteqq V(x^{-1})$. By Theorem 4 (i) the element x of S is topologically periodic, and hence there exists a positive integer $n\geqq 2$ such that $x^n\in V(x)$. Then we have

$$x^{n-2}=x^{-1}\cdot x^{n}\cdot x^{-1}\in x^{-1}\cdot V(x)\cdot x^{-1}\subseteqq V(x^{-1})$$

and

$$x^{n-2}\in F^{n-2}\subseteqq \bigcap_{i=n-2}^{\infty}\big(U_i(e)\big)^{n-2}\subseteqq \bigcap_{i=n-2}^{\infty}U_{i-1}(e)\subseteqq F.$$

Hence $V(x^{-1}) \cap F \neq \emptyset$ and since F is a closed subset in S we have that $x^{-1} \in F$. This implies that the inclusion $\big(F \cap H(S)\big)^{-1} \subseteqq F$ holds.

Later we shall show that $\big(U_n(e) \cap H(S)\big)^{-1} \subseteqq U(e) \cap H(S)$ for some positive integer n. Suppose to the contrary that $\big(U_n(e) \cap H(S)\big)^{-1} \nsubseteq U(e) \cap H(S)$ for any positive integer n. Then there exists a sequence $\{x_n\}_{n=1}^{\infty}$ in $H(S)$ such that $x_n \in U_n(e) \setminus \big(U(e)\big)^{-1}$ for all positive integers n.

The countable compactness of the square $S \times S$ implies that the sequence $\big\{(x_n, x_n^{-1})\big\}_{n=1}^{\infty}$ has a cluster point (a, b) in $S \times S$. The continuity of the semigroup operation in S implies that

$$a \cdot b = b \cdot a = f, \quad a \cdot b \cdot a = a, \quad b \cdot a \cdot b = b,$$

and hence $a, b \in H(f)$ for some idempotent f in S. Therefore $b = a^{-1} \in F^{-1} \cap H(S) \subseteqq F$. Then $(a, b) \in F \times F \subseteqq U(e) \times U(e)$. Since (a, b) is a cluster point of the sequence $\big\{(x_n, x_n^{-1})\big\}_{n=1}^{\infty}$, there exists a positive integer n such that $(x_n, x_n^{-1}) \in U(e) \times U(e)$. Therefore we have that $x_n \in \big(U(e)\big)^{-1}$ which contradicts the choice of the sequence $\{x_n\}_{n=1}^{\infty}$. The obtained contradiction implies that $\big(U_n(e) \cap H(S)\big)^{-1} \subseteqq U(e) \cap H(S)$ for some positive integer n, and hence the inversion $\mathrm{inv} : H(S) \to H(S)$ is continuous.

(ii) The projection $\pi : H(S) \to E(S)$ is continuous as a composition of two continuous maps. □

Theorem 7 implies the following corollary generalizing a result of [4].

COROLLARY 8. *The inversion in a regular Clifford topological semigroup with the countably compact square is continuous.*

Let S be a topological semigroup and $e \in E(S)$. We shall say that the semigroup S is *inversely regular at* e if for any open neighbourhood $U(e)$ of e there exists an open neighbourhood $W(e)$ of e such that $\overline{\big(W(e) \cap H(S)\big)^{-1}} \subseteqq \big(U(e) \cap H(S)\big)^{-1}$. A topological semigroup S with non-empty subsets of idempotents is called *inversely regular* if it is inversely regular at each idempotent of S [4].

THEOREM 9. *Let S be a topologically periodic Hausdorff topological semigroup. If S is inversely regular and countably compact, then:*

(i) *the inversion* $\mathrm{inv} : H(S) \to H(S)$ *is continuous;*

(ii) *the projection* $\pi : H(S) \to E(S)$ *is continuous.*

PROOF. (i) Fix any idempotent e in S. Let $U(e)$ be any open neighbourhood of e in S. Since the semigroup operation in S is continuous and S is inversely regular we construct inductively two sequences $\big\{U_n(e)\big\}_{i=1}^{\infty}$ and $\big\{W_n(e)\big\}_{i=1}^{\infty}$ of open neighbourhoods of the idempotent e such that $\big(U_n(e)\big)^i$

$\subseteqq W_{n-1}(e)$ and $\overline{\left(W_n(e)\cap H(S)\right)^{-1}} \subseteqq \left(U_n(e)\cap H(S)\right)^{-1}$ for all positive integers n and $i=1,2,\dots,n$.

Let $F=\bigcap_{n=1}^{\infty}\overline{\left(W_n(e)\cap H(S)\right)^{-1}}$. Then we have that

$$F=\bigcap_{n=1}^{\infty}\overline{\left(W_n(e)\cap H(S)\right)^{-1}} \subseteqq \bigcap_{n=2}^{\infty}\left(U_n(e)\cap H(S)\right)^{-1}$$

$$\subseteqq \bigcap_{n=2}^{\infty}\left(W_{n-1}(e)\cap H(S)\right)^{-1} \subseteqq F.$$

We shall show that $F^{-1}=F$. Let x be an arbitrary element of F. Since the set F is closed it sufficient to prove that $V(x^{-1})\cap F\neq\emptyset$ for any open neighbourhood $V(x^{-1})$ of the point x^{-1}. Since the semigroup S is topologically periodic there exists a positive integer $n\geqq 2$ such that $x^{n-2}\in V(x^{-1})$ (see the proof of Theorem 7). Then we have

$$x^{n-2}\in F^{n-2}\subseteqq \bigcap_{k=n-2}^{\infty}\left(\left(U_k(e)\cap H(S)\right)^{-1}\right)^{n-2}$$

$$\subseteqq \bigcap_{k=n-2}^{\infty}\left(\left(U_k(e)\cap H(S)\right)^{n-2}\right)^{-1} \subseteqq \bigcap_{k=n-2}^{\infty}\left(W_{k-1}(e)\cap H(S)\right)^{-1}\subseteqq F.$$

Hence $V(x^{-1})\cap F\neq\emptyset$ and since F is a closed subset in S we have that $x^{-1}\in F$. This implies that the inclusion $F^{-1}\subseteqq F$ holds. Then after the inversion we get that $F\subseteqq F^{-1}$. Therefore we get that

$$\bigcap_{n=1}^{\infty}\left(W_n(e)\cap H(S)\right)^{-1}=F=F^{-1}=\bigcap_{n=1}^{\infty}\left(\overline{\left(W_n(e)\cap H(S)\right)^{-1}}\right)^{-1}$$

$$\subseteqq \bigcap_{n=1}^{\infty}\left(U_n(e)\cap H(S)\right)\subseteqq U(e).$$

Since the space of the semigroup S is countably compact there exists a positive integer n such that

$$F\subseteqq \left(W_n(e)\cap H(S)\right)^{-1}\subseteqq \overline{\left(W_n(e)\cap H(S)\right)^{-1}}\subseteqq U(e).$$

This implies that the inversion is continuous at the idempotent e.

(ii) The projection $\pi: H(S)\to E(S)$ is continuous as a composition of two continuous maps. □

We recall that a map $f: X \to Y$ between topological spaces is called *sequentially continuous* if $\lim_{n\to\infty} f(x_n) = f\big(\lim_{n\to\infty} x_n\big)$ for any convergent sequence $\{x_n\}_{n=1}^{\infty}$ in X. Obviously a composition of two sequentially continuous maps is a continuous map. A subset F of a topological space X is called *sequentially closed* if no sequence in F converges to a point not in F [12].

THEOREM 10. *Let S be a Hausdorff countably compact topological semigroup. Then the following conditions hold:*

(i) *each maximal subgroup $H(e)$, $e \in E(S)$, is sequentially closed in S;*

(ii) *the subset $H(S)$ is sequentially closed in S;*

(iii) *the inversion* $\mathrm{inv}: H(S) \to H(S)$ *is sequentially continuous; and*

(iv) *the projection $\pi: H(S) \to E(S)$ is sequentially continuous.*

PROOF. (i) Suppose to the contrary that there exists a maximal subgroup $H(e)$ in S which is not a sequentially closed subset in S. Then there exists a sequence $\{x_n\}_{n=1}^{\infty} \subset H(e)$ which converges to $x \notin H(e)$. We put $A = \{x_n\}_{n=1}^{\infty} \cup \{x\}$. Since the sequence $\{x_n\}_{n=1}^{\infty} \subset H(e)$ converges to x the set A with the topology induced from S is a compact space. Then by Corollary 3.10.14 from [11], $A \times S$ is a countably compact space.

The countable compactness of $A \times S$ implies that the sequence $\big\{(x_n, x_n^{-1})\big\}_{n=1}^{\infty}$ has a cluster point (a, b) in $A \times S$. The continuity of the semigroup operation in S implies that $ab = e$, $aba = a$, $bab = b$ and hence $a = b^{-1}$ and $a, b \in H(e)$. The Hausdorff property of S implies that $x = a$ and hence $x \in H(e)$. The obtained contradiction implies assertion (i).

(ii) We argue exactly as in the previous case. Suppose the contrary: $H(S)$ is not a sequentially closed subset in S. Then there exists a sequence $\{x_n\}_{n=1}^{\infty} \subset H(S)$ which converges to $x \notin H(S)$. We put $A = \{x_n\}_{n=1}^{\infty} \cup \{x\}$. Since the sequence $\{x_n\}_{n=1}^{\infty} \subset H(S)$ converges to x, the set A with the topology induced from S is a compact space. Then by Corollary 3.10.14 from [11], $A \times S$ is a countably compact space.

The countable compactness of $A \times S$ implies that the sequence $\big\{(x_n, x_n^{-1})\big\}_{n=1}^{\infty}$ has a cluster point (a, b) in $A \times S$. The continuity of the semigroup operation in S implies that $ab = e$, $aba = a$, $bab = b$ for some idempotent $e \in E(S)$ and hence $a = b^{-1}$ and $a, b \in H(e)$. The Hausdorff property of S implies that $x = a$ and hence $x \in H(e) \subseteqq H(S)$. The obtained contradiction implies assertion (ii).

(iii) The sequential continuity of the inversion $\mathrm{inv}: H(S) \to H(S)$ will follow as soon as we prove that for any countable compactum $C \subset H(S)$ the restriction $\mathrm{inv}|_C$ is continuous. Let

$$\mathrm{Gr_{inv}}(S) = \big\{(x, y) \in S \times S \mid y = x^{-1}\big\}$$

be the graph of the inversion. Since S is a topological semigroup and

$$\mathrm{Gr}_{\mathrm{inv}}(S) = \big\{ (x,y) \in S \times S \mid xyx = x, yxy = y \text{ and } xy = yx \big\},$$

the graph $\mathrm{Gr}_{\mathrm{inv}}(S)$ is a closed subset of $S \times S$.

Since C is a metrizable compactum we can apply Corollary 3.10.14 from [11] to conclude that $C \times S$ is a countably compact space. Then the closedness of $\mathrm{Gr}_{\mathrm{inv}}(S)$ in the space $S \times S$ implies that the space $G = (C \times S) \cap \mathrm{Gr}_{\mathrm{inv}}(S)$ is countably compact and being countable, is compact.

Consider the natural projections $\mathrm{pr}_1 : S \times S \to S$ and $\mathrm{pr}_2 : S \times S \to S$ onto the first and the second coordinates, respectively. It follows from the compactness of G that $\mathrm{pr}_1 : G \to C$ and $\mathrm{pr}_2 : G \to C^{-1}$ are homeomorphisms. Consequently, $\mathrm{inv}|_C = \mathrm{pr}_2 \circ (\mathrm{pr}_1)^{-1} : C \to C^{-1}$ is continuous, being a composition of homeomorphisms.

(iv) The projection $\pi : H(S) \to E(S)$ is sequentially continuous as a composition of two sequentially continuous maps. □

Theorem 10 implies the following:

COROLLARY 11. *Let S be a Hausdorff Clifford countably compact topological semigroup. Then the following conditions hold:*

(i) *each maximal subgroup $H(e)$ is sequentially closed in S;*

(ii) *the inversion $\mathrm{inv} : S \to S$ is sequentially continuous; and*

(iii) *the projection $\pi : S \to E(S)$ is sequentially continuous.*

We observe that any sequentially compact (and hence any sequential countably compact) topological semigroup contains an idempotent (see [2, Theorem 8]). For a sequential countably compact semigroup Theorem 10 implies the following:

COROLLARY 12. *Let S be a Hausdorff sequential, countably compact topological semigroup. Then the following conditions hold:*

(i) *each maximal subgroup $H(e)$ is closed in S;*

(ii) *the subset $H(S)$ is closed in S;*

(iii) *the inversion $\mathrm{inv} : H(S) \to H(S)$ continuous; and*

(iv) *the projection $\pi : H(S) \to E(S)$ is continuous.*

The following example shows that the closure of a subgroup of a countably compact topological semigroup need not be a subgroup.

EXAMPLE 13. Assume $\mathsf{MA}_{\text{countable}}$ holds. Let $(\mathbb{R}, +)$ be the additive topological group of the real numbers with the usual topology and $\mathbb{Z}$ the discrete additive group of integers. Then $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ is a topological group. Let G be the group consisting of all $y \in \mathbb{T}^{\mathfrak{c}}$ such that there exists $\mu \in \mathfrak{c}$ such that $y(\mu)$ is the identity of $\mathbb{T}$ for each $\alpha > \mu$.

There exists $x \in \mathbb{T}^{\mathfrak{c}}$ such that $S = \{nx + y \mid n \in \omega \text{ and } y \in G\}$ is the semigroup with two-sided cancellation but S is not a group, see [23, 24]. Since G

is a dense subgroup in $\mathbb{T}^{\mathfrak{c}}$, G is dense in S. Tomita [24] showed that the existence of an element x in S is independent of ZFC. Also Madariaga-Garcia and Tomita [18] show that the semigroup S can be constructed from $\mathfrak{c}$ selective ultrafilters.

A topological space X is called *quasi-regular* if for any non-empty open subset U in X there exists a non-empty open set $V \subsetneqq U$ in X such that $\overline{V} \subsetneqq U$. The following example shows that there exists a Hausdorff quasi-regular Clifford inverse countably compact topological semigroup S with the discontinuous inversion and discontinuous projection $\pi : S \to E(S)$.

Example 14 [4]. Let ω_1 be the smallest uncountable ordinal and $[0, \omega_1)$ be the well-ordered sets of all countable ordinals, endowed with the natural order topology. It is well-known that $[0, \omega_1)$ is a sequentially compact topological space (see [11], Example 3.10.16) and simple verification shows that the semilattice operation min is continuous on $[0, \omega_1)$.

Let $\mathbb{T} = \{ z \in \mathbb{C} : |z| = 1 \}$ be the unit circle in the complex plane with the usual topology and let $\mathbb{T}$ be endowed with the operation of multiplication of complex numbers. Then by Theorem 3.10.35 from [11] the product $A = [0, \omega_1) \times \mathbb{T}$ is a Hausdorff sequentially compact commutative topological inverse semigroup, as the Cartesian product of a topological semilattice and a commutative topological group.

Let $x = (\omega_1, 1) \notin A$. Put $S = A \cup \{x\}$ and define a topology τ on S letting A be a subspace of S and $U \subset S$ be a neighborhood of x if there are a positive real $\varepsilon > 0$ and a countable ordinal α such that $U \supset U(\alpha, \varepsilon)$ where

$$U(\alpha, \varepsilon) = \{x\} \cup \{ (\beta, e^{i\varphi}) \mid \alpha < \beta < \omega_1,\ 0 < \varphi < \varepsilon \}.$$

Extend the semigroup operation to S by letting $x \cdot x = x$ and $x \cdot a = a \cdot x = a$ for all $a \in A$. It is easy to see that S is a sequentially compact space and the semigroup operation "$\cdot$" is continuous and commutative on S. But the inversion in S is not continuous since $(U(\alpha, \varepsilon))^{-1} \nsubseteqq U(\beta, \delta)$ for all $\alpha, \beta < \omega_1$ and $\varepsilon, \delta \in (0, 1)$.

Observe also that the subsemigroup of idempotents of the semigroup S can be identified with the discrete sum of $[0, \omega_1) \bigsqcup \{\omega_1\}$ and hence is sequentially compact, locally countable, and locally compact.

Also observe that the projective map $\pi : S \to E(S)$ is not continuous.

Remark 15. Example 14 shows that the requirement of regularity in Theorem 7 and Corollary 2 is essential and cannot be replaced by the quasi-regularity. This contrasts with the case of paratopological groups, see [22].

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