Order (2010) 27:69–81
DOI 10.1007/s11083-010-9140-x

On Chains in *H*-Closed Topological Pospaces

Oleg Gutik · Dušan Pagon · Dušan Repovš

Received: 16 August 2009 / Accepted: 29 October 2009 / Published online: 23 January 2010

Abstract We study chains in an H-closed topological partially ordered space. We give sufficient conditions for a maximal chain L in an H-closed topological partially ordered space (H-closed topological semilattice) under which L contains a maximal (minimal) element. We also give sufficient conditions for a linearly ordered topological partially ordered space to be H-closed. We prove that a linearly ordered H-closed topological semilattice is an H-closed topological pospace and show that in general, this is not true. We construct an example of an H-closed topological pospace with a non-H-closed maximal chain and give sufficient conditions under which a maximal chain of an H-closed topological pospace is an H-closed topological pospace.

Keywords H-closed topological partially ordered space · Chain · Maximal chain · Topological semilattice · Regularly ordered pospace · MCC-chain · Scattered space

Mathematics Subject Classifications (2000) Primary 06B30 · 54F05; Secondary 06F30 · 22A26 · 54G12 · 54H12

O. Gutik
Department of Mechanics and Mathematics,
Ivan Franko Lviv National University,
Universytetska 1, Lviv, 79000, Ukraine
e-mail: o_gutik@franko.lviv.ua, ovgutik@yahoo.com

D. Pagon
Institute of Mathematics, Physics and Mechanics,
Jadranska 19, Ljubljana, 1000, Slovenia
e-mail: dusan.pagon@uni-mb.si

D. Repovš (✉)
Faculty of Mathematics and Physics, University of Ljubljana,
Jadranska 19, Ljubljana 1000, Slovenia
e-mail: dusan.repovs@guest.arnes.si

1 Introduction

In this paper all topological spaces will be assumed to be Hausdorff. We shall follow the terminology of [3, 4, 7–10, 14, 17]. If A is a subset of a topological space X, then we denote the *closure* of the set A in X by $\mathrm{cl}_X(A)$. By a *partial order* on a set X we mean a reflexive, transitive and anti-symmetric binary relation $\leqslant$ on X. If the partial order $\leqslant$ on a set X satisfies the following linearity law

$$\text{if } x, y \in X, \quad \text{then } x \leqslant y \text{ or } y \leqslant x,$$

then it is said to be a *linear order*. We write $x < y$ if $x \leqslant y$ and $x \neq y$, $x \geqslant y$ if $y \leqslant x$, and $x \nleqslant y$ if the relation $x \leqslant y$ is false. Obviously, if $\leqslant$ is a partial order or a linear order on a set X then so is $\geqslant$. A set endowed with a partial order (resp. linear order) is called a *partially ordered* (resp. *linearly ordered*) set. If $\leqslant$ is a partial order on X and A is a subset of X then we denote

$$\downarrow A = \{y \in X \mid y \leqslant x \text{ for some } x \in A\} \quad \text{and}$$
$$\uparrow A = \{y \in X \mid x \leqslant y \text{ for some } x \in A\}.$$

For any elements a, b of a partially ordered set X such that $a \leqslant b$ we denote $\uparrow a = \uparrow\{a\}$, $\downarrow a = \downarrow\{a\}$, $[a, b] = \uparrow a \cap \downarrow b$ and $[a, b) = [a, b] \setminus \{b\}$. A subset A of a partially ordered set X is called *increasing* (resp. *decreasing*) if $A = \uparrow A$ (resp. $A = \downarrow A$).

A partial order $\leqslant$ on a topological space X is said to be *lower* (resp. *upper*) *semicontinuous* provided that whenever $x \nleqslant y$ (resp. $y \nleqslant x$) in X, then there exists an open set $U \ni x$ such that if $a \in U$ then $a \nleqslant y$ (resp. $y \nleqslant a$). A partial order is called *semicontinuous* if it is both upper and lower semicontinuous. Next, it is said to be *continuous* or *closed* provided that whenever $x \nleqslant y$ in X, there exist open sets $U \ni x$ and $V \ni y$ such that if $a \in U$ and $b \in V$ then $a \nleqslant b$. Clearly, the statement that the partial order $\leqslant$ on X is semicontinuous is equivalent to the assertion that $\uparrow a$ and $\downarrow a$ are closed subsets of X for each $a \in X$. A topological space equipped with a continuous partial order is called a *topological partially ordered space* or shortly *topological pospace*. A partial order $\leqslant$ on a topological space X is continuous if and only if the graph of $\leqslant$ is a closed subset in $X \times X$ [17, Lemma 1]. Also, a semicontinuous linear order on a topological space is continuous [17, Lemma 3].

A *chain* of a partially ordered set X is a subset of X which is linearly ordered with respect to the partial order. A *maximal chain* is a chain which is properly contained in no other chain. The Axiom of Choice implies the existence of maximal chains in any partially ordered set. Every maximal chain in a topological pospace is a closed set [17, Lemma 4].

An element y of a partially ordered set X is called *minimal* (resp. *maximal*) in X whenever $x \leqslant y$ (resp. $y \leqslant x$) in X implies $y \leqslant x$ (resp. $x \leqslant y$). Let X and Y be partially ordered sets. A map $f\colon X \to Y$ is called *monotone* (or *partial order preserving*) if $x \leqslant y$ implies $f(x) \leqslant f(y)$ for every $x, y \in X$.

A Hausdorff topological space X is called *H-closed* if X is a closed subspace of every Hausdorff space in which it is contained [1, 2]. A Hausdorff pospace X is called *H-closed* if X is a closed subspace of every Hausdorff pospace in which it is contained. It is obvious that the notion of H-closedness is a generalization of compactness. For any element x of a compact topological pospace X there exists a minimal element $y \in X$ and a maximal element $z \in X$ such that $y \leqslant x \leqslant z$ (cf. [10]). Every maximal chain in a compact topological pospace is a compact subset and hence it contains minimal and maximal elements. Also, for any point x of a compact topological pospace X there exists a base at x which consists of open order-convex subsets [14]. (A non-empty set A of a partially ordered set is called *order-convex* if A is an intersection of increasing and decreasing subsets.) We are interested in the following question: *Under which conditions does an H-closed topological pospace have properties similar to those of a compact topological pospace?*

In this paper we study chains in an arbitrary H-closed topological partially ordered space. We give sufficient conditions for a maximal chain L in an H-closed topological partially ordered space (H-closed topological semilattice) under which L contains a maximal (minimal) element. Also, we give sufficient conditions for a linearly ordered topological partially ordered space to be H-closed. We prove that a linearly ordered H-closed topological semilattice is an H-closed topological pospace and show that in general, this is not true. We construct an example of an H-closed topological pospace with a non-H-closed maximal chain and give sufficient conditions under which a maximal chain of an H-closed topological pospace is an H-closed topological pospace.

2 On Maximal and Minimal Elements of Maximal Chains in *H*-Closed Topological Pospaces

A subset A of a partially ordered set X is called *down-directed* (resp. *up-directed*) if and only if $\uparrow A = X$ (resp. $\downarrow A = X$). A topological pospace X is called *upper point separated* (resp. *lower point separated*) if for every $x \in X$ such that $\uparrow x \neq X$ (resp. $\downarrow x \neq X$) there exist an open non-empty decreasing (resp. increasing) subset V in X and a neighbourhood $U(x)$ of x such that $a \nleqslant b$ (resp. $b \nleqslant a$) for each $a \in U(x)$ and $b \in V$.

Theorem 2.1 *If an upper (lower) point separated H-closed topological pospace X contains a down-directed (up-directed) chain, then X has a minimum (maximum) element.*

Proof Suppose to the contrary, that X does not contain a minimum element. Let $x \notin X$. We put $X^* = X \cup \{x\}$ and extend the partial order $\leqslant$ from X onto X^* as follows:

$$x \leqslant y \qquad \text{for all} \quad y \in X^*.$$

Let τ be the topology on X and $\mathscr{D}$ the set of all non-empty decreasing open subsets of X. The Hausdorff topology τ^* on X^* is generated by the base $\tau \cup \left\{\{x\} \cup U \mid U \in \mathscr{D}\right\}$. Since X does not contain a minimum element the definition of the family τ implies that x is not an isolated point in X^*. Also, since X is an upper point separated topological pospace, $\leqslant$ is a closed partial order on X^*. Therefore X is a dense subspace of X^*, a contradiction. This implies the assertion of the theorem. □

Theorem 2.1 implies the following:

Corollary 2.2 *Every down-directed (up-directed) chain of an upper (lower) point separated H-closed topological pospace X contains a minimum (maximum) element.*

Proposition 2.3 *Every locally compact topological pospace is upper (lower) point separated.*

Proof Let X be a locally compact topological pospace and $x \in X$ a point such that $\uparrow x \neq X$. Fix any $y \in X \setminus \uparrow x$. Local compactness of X implies that there exists an open neighbourhood $U(y)$ of y such that $U(y) \subseteq \mathrm{cl}_X(U(y)) \subseteq X \setminus \uparrow x$ and the set $\mathrm{cl}_X(U(y))$ is compact. Proposition VI-1.6(*ii*) of [10] implies that $\uparrow \mathrm{cl}_X(U(y))$ is a closed subset of X. Hence $V = X \setminus \uparrow \mathrm{cl}_X(U(y))$ is an open decreasing subset of X and $a \nleqslant b$ for each $a \in U(y)$ and $b \in V$. This completes the proof of the proposition. □

Theorem 2.1 and Proposition 2.3 imply the following:

Corollary 2.4 *If a locally compact H-closed topological pospace X contains a down-directed (up-directed) chain, then X has a minimum (maximum) element.*

Also, Corollary 2.2 and Proposition 2.3 imply the following:

Corollary 2.5 *Every down-directed (up-directed) chain of a locally compact H-closed topological pospace X contains a minimum (maximum) element.*

A subset F of topological pospace X is said to be *upper* (resp. *lower*) *separated* if and only if for each $a \in X \setminus \uparrow F$ (resp. $a \in X \setminus \downarrow F$) there exist disjoint open neighbourhoods U of a and V of F such that U is decreasing (resp. increasing) and V is increasing (resp. decreasing) in X. We shall say that a subset A of a topological pospace X *has the DS-property* (resp. US-property) if for any $x \in X$ such that $A \setminus \uparrow x \neq \varnothing$ (resp. $A \setminus \downarrow x \neq \varnothing$) there exist a neighbourhood $U(x)$ of x and an open decreasing (resp. increasing) set V such that $V \cap U(x) = \varnothing$ and $V \cap A \neq \varnothing$.

Theorem 2.6 *Every upper (lower) separated maximal chain with the DS-property (resp. US-property) of an H-closed topological pospace contains a minimum (resp. maximum) element.*

Proof Suppose to the contrary, that there exists an H-closed topological pospace X with the DS-property and a maximal upper separated chain L in X such that L does not contain a minimum element.

Let $x \notin X$. We extend the partial order $\leqslant$ from X onto $X^* = X \cup \{x\}$ as follows:

$$x \leqslant x \qquad \text{and} \qquad x \leqslant y \quad \text{if} \quad y \in \uparrow L.$$

Let $\mathscr{U}_L$ be the set of all open increasing subsets in X which contain the chain L. We denote the set of all open decreasing subsets which intersect L by $\mathscr{D}_L$. If τ is the topology on X then we define the Hausdorff topology τ^* as the one which is generated by the pseudobase $\tau \cup \left\{\{x\} \cup U \mid U \in \mathscr{D}_L \cup \mathscr{U}_L\right\}$. Since L is an upper separated maximal chain with the DS-property, we conclude that the partial order $\leqslant$ is continuous on X^*. Therefore X is a dense subspace of X^*, a contradiction. This implies the assertion of the theorem. □

Proposition 2.7 *Every subset of a locally compact topological pospace has the DS- and the US-properties.*

Proof Let X be a locally compact topological pospace. Let $A \subset X$ and $x \in X$ be such that $A \setminus \uparrow x \neq \varnothing$. Fix any $y \in A \setminus \uparrow x$. Since $x \nleqslant y$ there exist neighbourhoods $U(x)$ and $U(y)$ of x and y, respectively, such that $a \nleqslant b$ for all $a \in U(x)$ and $b \in U(y)$. Local compactness of X implies that there exists an open neighbourhood $V(x)$ of x such that $V(x) \subseteq \mathrm{cl}_X(V(x)) \subseteq U(x)$ and the set $\mathrm{cl}_X(V(x))$ is compact. Proposition VI-1.6(*ii*) of [10] implies that $\uparrow \mathrm{cl}_X(V(x))$ is a closed subset of X. Hence $V = X \setminus \uparrow \mathrm{cl}_X(V(x))$ is an open decreasing subset of X such that $V \cap A \neq \varnothing$. This completes the proof of the proposition. □

Theorem 2.6 and Proposition 2.7 imply the following:

Corollary 2.8 *Every upper (lower) separated maximal chain of an H-closed locally compact topological pospace contains a minimum (maximum) element.*

Similarly to [13, 15] we shall say that a topological pospace X is a *C_i-space* (resp. *C_d-space*) if whenever a subset F of X is closed, the set $\uparrow F$ (resp. $\downarrow F$) is closed in X. A maximal chain L of a topological pospace X is called an *MCC_i-chain* (resp. an *MCC_d-chain*) in X if $\uparrow L$ (resp. $\downarrow L$) is a closed subset in X. Obviously, if a topological pospace X is a C_i-space (resp. C_d-space) then any maximal chain in X is an MCC_i-chain (resp. MCC_d-chain) in X. A topological pospace X is said to be *upper* (resp. *lower*) *regularly ordered* if and only if for each closed increasing (resp. decreasing) subset F in X and each element $a \notin F$, there exist disjoint open neighbourhoods U of a and V of F such that U is decreasing (resp. increasing) and V is increasing (resp. decreasing) in X [5, 11]. A topological pospace X is *regularly ordered* if it is upper and lower regularly ordered.

Theorem 2.6 implies Corollaries 2.9 and 2.10:

Corollary 2.9 *Every maximal MCC_i-chain with the US-property of an H-closed upper regularly ordered topological pospace X contains the least element which is a minimal element of X. Consequently, if in an H-closed upper regularly ordered C_i-space X every maximal chain has the US-property, then X contains a collection M of minimal elements such that $\uparrow M = X$.*

Corollary 2.10 *Every maximal MCC_d-chain with the DS-property of an H-closed lower regularly ordered topological pospace X contains the greatest element which is a maximal element of X. Consequently, if in an H-closed lower regularly ordered C_d-space X every maximal chain has the DS-property, then X contains a collection M of maximal elements such that $\downarrow M = X$.*

3 On *H*-Closed Topological Semilattices

A topological space S which is algebraically a semigroup with a continuous semigroup operation is called a *topological semigroup*. A *semilattice* is a semigroup with a commutative idempotent semigroup operation. A *topological semilattice* is a topological semigroup which is algebraically a semilattice. If E is a semilattice, then the semilattice operation on E determines the partial order $\leqslant$ on E:

$$e \leqslant f \quad \text{if and only if} \quad ef = fe = e.$$

This order is called *natural*. A semilattice E is called *linearly ordered* if the semilattice operation admits a linear natural order on E. The natural order on a Hausdorff topological semilattice E admits the structure of topological pospace on E (cf. [10, Proposition VI-1.14]). Obviously, if S is a topological semilattice then $\uparrow e$ and $\downarrow e$ are closed subsets in S for every $e \in S$.

A topological semilattice S is called *H-closed* if it is a closed subset in any topological semilattice which contains S as a subsemilattice. Properties of H-closed topological semilattices were established in [6, 12, 16].

Theorem 3.1 *Every upper point separated H-closed topological semilattice contains the smallest idempotent.*

Proof Suppose to the contrary, that there exists an upper point separated H-closed topological semilattice E which does not contain the smallest idempotent. Let $x \notin E$. We put $E^* = E \cup \{x\}$ and extend semilattice operation from E onto E^* as follows:

$$xx = xe = ex = x \qquad \text{for all} \quad e \in E.$$

Let τ be the topology on E and $\mathscr{D}$ the set of all non-empty decreasing open subsets of E. The Hausdorff topology τ^* on E^* is generated by the base $\tau \cup \big\{\{x\} \cup U \mid U \in \mathscr{D}\big\}$. The continuity of the semilattice operation at x follows from the definition of the topology τ^*. Since E is upper point separated we conclude that (E^*, τ^*) is a Hausdorff topological space. Therefore E is a dense subspace of E^*, a contradiction. This implies the assertion of the theorem. □

Theorem 3.2 *Let S be a topological semilattice which is an H-closed topological pospace. Then every maximal chain of S has a maximum element. Consequently, every topological semilattice S which is an H-closed topological pospace has a collection M of maximal elements such that* $\downarrow M = S$.

Proof Let L be a maximal chain of S. Fix any $x \in L$. If x is a maximum element of L, the proof is complete. If x is not a maximum element of L, then there exists $y \in L$ such that $x < y$. Let $U(x)$ and $U(y)$ be open neighbourhoods of x and y, respectively, such that $a \nleqslant b$ for all $b \in U(x)$ and $a \in U(y)$. The continuity of the semilattice operation and Hausdorffness of S imply that there exist open neighbourhoods $V(x)$ and $V(y)$ of x and y, respectively, such that

$$V(x) \cdot V(y) = V(y) \cdot V(x) \subseteq U(x), \quad V(x) \subseteq U(x), \quad V(y) \subseteq U(y) \quad \text{and}$$

$$V(x) \cap V(y) = \varnothing.$$

Therefore $\uparrow V(y) \cap V(x) = \varnothing$. By Proposition VI-1.13 of [10], $\uparrow V(y)$ is an open subset of S and hence the chain L has the US-property. Therefore by Theorem 2.6, the chain L contains a maximum element. □

We observe that every Hausdorff topological semilattice which is an H-closed topological pospace is obviously an H-closed topological semilattice. However, there exists an H-closed Hausdorff topological semilattice which is not an H-closed topological pospace (cf. Example 3.6). Simple verifications establish the following:

Proposition 3.3 *Every linearly ordered topological pospace admits a structure of a topological semilattice.*

Since the closure of a chain in a topological pospace is again a chain, Proposition 3.3 implies the following:

Proposition 3.4 *A linearly ordered topological semilattice is H-closed if and only if it is H-closed as a topological pospace.*

A linearly ordered set E is called *complete* if every non-empty subset of S has an inf and a sup. Propositions 3.3 and 3.4, and Theorem 2 of [12] imply the following:

Corollary 3.5 *A linearly ordered topological pospace X is H-closed if and only if the following conditions hold:*

(i) *X is a complete set with respect to the partial order on X;*
(ii) *$x = \sup A$ for $A = \downarrow A \setminus \{x\}$ implies $x \in \mathrm{cl}_X A$, whenever $A \neq \varnothing$; and*
(iii) *$x = \inf B$ for $B = \uparrow B \setminus \{x\}$ implies $x \in \mathrm{cl}_X B$, whenever $B \neq \varnothing$.*

A semilattice S is called *algebraically closed* (or *absolutely maximal*) if S is a closed subsemilattice in any topological semilattice which contains S as a subsemilattice [16]. Stepp [16] proved that a semilattice S is algebraically closed if and only if every chain in S is finite. Therefore an algebraically closed semilattice S is an H-closed topological semilattice with any Hausdorff topology τ such that (S, τ) is a topological semilattice.

A partially ordered set A is called a *tree* if $\downarrow a$ is a chain for any $a \in A$. Example 3.6 shows that there exists an algebraically closed (and hence H-closed) topological semilattice X which is a tree but X is not an H-closed topological pospace.

Example 3.6 Let X be a discrete infinite space of cardinality τ and let $\mathscr{A}(\tau)$ be the one-point Alexandroff compactification of X. We put $\{\alpha\} = \mathscr{A}(\tau) \setminus X$ and fix $\beta \in X$. On $\mathscr{A}(\tau)$ we define a partial order $\leqslant$ as follows:

$$x \leqslant x, \quad \beta \leqslant x \quad \text{and} \quad x \leqslant \alpha \quad \text{for all} \quad x \in \mathscr{A}(\tau).$$

The partial order $\leqslant$ induces a semilattice operation '$*$' on $\mathscr{A}(\tau)$:

(1) $x * x = x$, $\beta * x = x * \beta = \beta$ and $\alpha * x = x * \alpha = x$ for all $x \in \mathscr{A}(\tau)$; and
(2) $x * y = y * x = \beta$ for all distinct $x, y \in X$.

Since X is a discrete subspace of $\mathscr{A}(\tau)$, X with the semilattice operation induced from $\mathscr{A}(\tau)$ is a topological semilattice. By [16, Theorem 9], X is an algebraically closed semilattice, and hence it is an H-closed topological semilattice. Simple verifications show that for every $a, b \in \mathscr{A}(\tau)$ such that $a \nleqslant b$ there exist an open increasing neighbourhood $V(a)$ of a and an open decreasing neighbourhood $V(b)$ of b such that $V(a) \cap V(b) = \varnothing$. Therefore $\mathscr{A}(\tau)$ is a compact (and hence normally orderable) topological pospace. However, X is a dense subspace of $\mathscr{A}(\tau)$ and hence X is not an H-closed topological pospace.

4 Linearly Ordered H-Closed Topological Pospaces

Let C be a maximal chain of a topological pospace X. Then $C = \bigcap_{x \in C}({\downarrow}x \cup {\uparrow}x)$, and hence C is a closed subspace of X. Therefore we get the following:

Lemma 4.1 *Let K be a linearly ordered subspace of a topological pospace X. Then $\mathrm{cl}_X(K)$ is a linearly ordered subspace of X.*

Since the conditions (i)–(iii) of Corollary 3.5 are preserved by continuous monotone maps, we have the following:

Theorem 4.2 *Any continuous monotone image of a linearly ordered H-closed topological pospace into a topological pospace is an H-closed topological pospace.*

Also, Proposition 4.3 follows from Corollary 3.5.

Proposition 4.3 *Let (X, τ_X) be a non-empty H-closed sub-pospace of a linearly ordered topological pospace (T, τ_T). Then the set ${\uparrow}x \cap X$ (${\downarrow}x \cap X$) contains a minimal (maximal) element for any $x \in T$.*

Let L be a subset of a linearly ordered set X. A subset A of X is called an *L-chain* in X if $A \subseteq L$ and A is order convex (i. e., ${\uparrow}x \cap {\downarrow}y \subseteq L$ for any $x, y \in A$, $x \leqslant y$).

Theorem 4.4 *Let X be a linearly ordered topological pospace and L a subspace of X such that L is an H-closed topological pospace and any maximal $X \setminus L$-chain in X is an H-closed topological pospace. Then X is an H-closed topological pospace.*

Proof Suppose to the contrary, that the topological pospace X is not H-closed. Then by Lemma 4.1, there exists a linearly ordered topological pospace Y which contains X as a non-closed subspace. Without loss of generality we may assume that X is a dense subspace of a linearly ordered topological pospace Y.

Let $x \in Y \setminus X$. The assumptions of the theorem imply that the set $X \setminus L$ is a disjoint union of maximal $X \setminus L$-chains L_α, $\alpha \in \mathscr{A}$, which are H-closed topological pospaces. Therefore any open neighbourhood of the point x intersects infinitely many sets $L_\alpha, \alpha \in \mathscr{A}$.

Since any maximal $X\setminus L$-chain in X is an H-closed topological pospace, one of the following conditions holds:

$$\uparrow x \cap L \neq \varnothing \qquad \text{or} \qquad \downarrow x \cap L \neq \varnothing.$$

We consider the case when the sets $\uparrow x \cap L$ and $\downarrow x \cap L$ are nonempty. The proofs in the other cases are similar.

By Proposition 4.3, the set $\uparrow x \cap L$ contains a minimal element x_m and the set $\downarrow x \cap L$ contains a maximal element x_M. Then the sets $\uparrow x_m$ and $\downarrow x_M$ are closed in Y and, obviously, $L \subset \downarrow x_M \cup \uparrow x_m$. Let $U(x)$ be an open neighbourhood of the point x in Y. We put

$$V(x) = U(x) \setminus (\downarrow x_M \cup \uparrow x_m).$$

Then $V(x)$ is an open neighbourhood of the point x in Y which intersects at most one maximal $S\setminus L$-chain L_α, a contradiction. Therefore X is an H-closed topological pospace. □

Corollary 4.5 *Let X be a linearly ordered topological pospace and L a subspace of X such that L is a compact topological pospace and any maximal $X\setminus L$-chain in X is a compact topological pospace. Then X is an H-closed topological pospace.*

Example 4.6 Let $\mathbb{N}$ be the set of all positive integers. Let $\{x_n\}$ be an increasing sequence in $\mathbb{N}$. Put $\mathbb{N}^* = \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}\}$ and let $\leqslant$ be the usual order on $\mathbb{N}^*$. We put $U_n(0) = \{0\} \cup \{\frac{1}{x_k} \mid k \geqslant n\}, n \in \mathbb{N}$. A topology τ on $\mathbb{N}^*$ is defined as follows:

a) any point $x \in \mathbb{N}^* \setminus \{0\}$ is isolated in $\mathbb{N}^*$; and
b) $\mathscr{B}(0) = \{U_n(0) \mid n \in \mathbb{N}\}$ is the base of the topology τ at the point $0 \in \mathbb{N}^*$.

It is easy to see that $(\mathbb{N}^*, \leqslant, \tau)$ is a countable linearly ordered σ-compact locally compact metrizable topological pospace and if $x_{k+1} > x_k + 1$ for every $k \in \mathbb{N}$, then $(\mathbb{N}^*, \leqslant, \tau)$ is a non-compact topological pospace.

By Corollary 4.5, $(\mathbb{N}^*, \leqslant, \tau)$ is an H-closed topological pospace. Also, $(\mathbb{N}^*, \leqslant, \tau)$ is a normally ordered (or monotone normal) topological pospace, i.e. for any closed subset $A = \downarrow A$ and $B = \uparrow B$ in X such that $A \cap B = \varnothing$, there exist open subsets $U = \downarrow U$ and $V = \uparrow V$ in X such that $A \subseteq U$, $B \subseteq V$, and $U \cap V = \varnothing$ [14]. Therefore for any disjoint closed subsets $A = \downarrow A$ and $B = \uparrow B$ in X, there exists a continuous monotone function $f\colon X \to [0, 1]$ such that $f(A) = 0$ and $f(B) = 1$ (cf. [14]).

Example 4.6 implies negative answers to the following questions:

(i) Is every closed subspace of an H-closed topological pospace H-closed?
(ii) Has every locally compact topological pospace a subbasis of open decreasing and open increasing subsets?

Example 4.7 shows that there exists a countably compact topological pospace, whose space is H-closed. This example also shows that there exists a countably compact totally disconnected scattered topological pospace which is not embeddable into any locally compact topological pospace.

Example 4.7 Let the set $X = [0, \omega_1)$ be equipped with the order topology (cf. [9, Example 3.10.16]), and let $Y = \{0\} \cup \{\frac{1}{n} \mid n = 1, 2, 3, \ldots\}$ have the natural topology. We consider $S = X \times Y$ equipped with the product topology τ_p and the partial order $\preccurlyeq$:

$$(x_1, y_1) \preccurlyeq (x_2, y_2) \quad \text{if and only if} \quad x_2 \leqslant_X x_1 \text{ and } y_2 \leqslant_Y y_1,$$

where $\leqslant_X$ and $\leqslant_Y$ are the usual linear orders on X and Y, respectively. We extend the partial order $\preccurlyeq$ onto $S^* = S \cup \{\alpha\}$, where $\alpha \notin S$, as follows: $\alpha \preccurlyeq \alpha$ and $\alpha \preccurlyeq x$ for all $x \in S$, and define a topology τ on S^* as follows. The bases of topologies τ and τ_p at the point $x \in S$ coincide and the family $\mathscr{B}(\alpha) = \{U_\beta(\alpha) \mid \beta \in \omega_1\}$ is the base of the topology τ at the point $\alpha \in S^*$, where

$$U_\beta(\alpha) = \{\alpha\} \cup \big([\beta, \omega_1) \times \{1/n \mid n = 1, 2, 3, \ldots\}\big).$$

Since $\mathrm{cl}_{S^*}(U_\beta(\alpha)) \nsubseteq U_\gamma(\alpha)$ for any $\beta, \gamma \in \omega_1$, Propositions 1.5.2 and 1.5.5 of [9] imply that $(S^*, \preccurlyeq, \tau)$ is a Hausdorff non-regular topological pospace. Therefore by Theorem 2.1.6 [9], the topological space $(S^*, \preccurlyeq, \tau)$ does not embed into any regular topological space, and hence by Theorem 3.3.1 [9] neither into any locally compact space. Proposition 3.12.5 of [9] implies that (S^*, τ) is an H-closed topological space. By Corollary 3.10.14 of [9] and Theorem 3.10.8 of [9], the topological space (S^*, τ) is countably compact. Since every point of (S^*, τ) has a singleton component, the topological space (S^*, τ) is totally disconnected.

Let A be a closed subset of $(S^*, \preccurlyeq, \tau)$ such that $A \neq \{\alpha\}$. Then there exists $x \in [0, \omega_1)$ such that $\tilde{A} = A \cap ([0, x] \times Y) \neq \varnothing$. Since $[0, x] \times Y$ is a compactum, $\tilde{A}$ is a compact topological pospace and hence $\tilde{A}$ contains a maximal element of $\tilde{A}$. Let x_m be a maximal element of $\tilde{A}$. The definition of the topology τ on S^* implies that $\uparrow x_m$ is an open subset in (S^*, τ). Then $\uparrow x_m \cap \tilde{A} = x_m$ and hence x_m is an isolated point of the space $\tilde{A}$ with the induced topology from (S^*, τ). Therefore every closed subset of (S^*, τ) contains an isolated point and hence (S^*, τ) is a scattered topological space.

Remark 4.8 The topological pospace $(\mathbb{N}^*, \leqslant, \tau)$ from Example 4.6 admits the structure of a topological semilattice:

$$ab = \min\{a, b\}, \quad \text{for} \quad a, b \in \mathbb{N}^*.$$

Also, the topological pospace $(S^*, \preccurlyeq, \tau)$ from Example 4.7 admits the continuous semilattice operation

$$(x_1, y_1) \cdot (x_2, y_2) = (\max\{x_1, x_2\}, \max\{y_1, y_2\}) \quad \text{and} \quad (x_1, y_1) \cdot \alpha = \alpha \cdot (x_1, y_1) = \alpha,$$

for $x_1, x_2 \in X$ and $y_1, y_2 \in Y$.

The following example shows that there exists a countable H-closed scattered totally disconnected topological pospace which has a non-H-closed maximal chain.

Example 4.9 Let $\mathbb{N}$ be the set of all positive integers with the discrete topology, and consider $Y = \{0\} \cup \left\{\frac{1}{n} \mid n = 1, 2, 3, \ldots\right\}$ equipped with the natural topology. We define $T = \mathbb{N} \times Y$ with the product topology τ_T and the partial order $\preccurlyeq$:

$$(x_1, y_1) \preccurlyeq (x_2, y_2) \quad \text{if and only if} \quad x_2 \leqslant x_1 \text{ and } y_2 \leqslant y_1,$$

where $\leqslant$ is the usual linear order induced from $\mathbb{R}$ on $\mathbb{N}$ and Y, respectively. We extend the partial order $\preccurlyeq$ to $T^* = T \cup \{\alpha\}$, where $\alpha \notin T$, as follows: $\alpha \preccurlyeq \alpha$ and $\alpha \preccurlyeq x$ for all $x \in T$. We define a topology τ^* on T^* as follows: the bases of topologies τ^* and τ_T at the point $x \in T$ coincide and the family $\mathscr{B}(\alpha) = \{U_k(\alpha) \mid k \in \{1, 2, 3, \dots\}\}$ is the base of the topology τ^* at the point $\alpha \in T^*$, where

$$U_k(\alpha) = \{\alpha\} \cup \left(\{k, k+1, k+2, \dots\} \times \left\{\frac{1}{n} \mid n = 1, 2, 3, \dots\right\}\right).$$

It is obvious that $(T^*, \preccurlyeq, \tau^*)$ is a Hausdorff non-regular topological pospace. Proposition 3.12.5 of [9] implies that (T^*, τ^*) is an H-closed topological space. Since every point of (T^*, τ^*) has a singleton component, the topological space (T^*, τ^*) is totally disconnected. The proof that (T^*, τ^*) is a scattered topological pospace is similar to the proof of the scatteredness of the topological pospace $(S^*, \preccurlyeq, \tau)$ in Example 4.7.

We observe that the set $L = (\mathbb{N} \times \{0\}) \cup \{\alpha\}$ with the induced partial order from the topological pospace $(T^*, \preccurlyeq, \tau^*)$ is a maximal chain in T^*. The topology τ^* induces the discrete topology on L. Corollary 3.5 implies that L is not an H-closed topological pospace.

Theorem 4.10 gives sufficient conditions for a maximal chain of an H-closed topological pospace to be H-closed. We shall say that a chain L of a partially ordered set P has the $\downarrow\cdot$ max-*property* (resp. $\uparrow\cdot$ min-*property*) in P if for every $a \in P$ such that ${\downarrow}a \cap L \neq \varnothing$ (resp. ${\uparrow}a \cap L \neq \varnothing$) the chain ${\downarrow}a \cap L$ (${\uparrow}a \cap L$) has a maximal (resp. minimal) element. If the chain of a partially ordered set P has the $\downarrow\cdot$ max- and the $\uparrow\cdot$ min-properties, then we shall say that L has the $\updownarrow\cdot$ m-*property*.

Similarly to [13, 15] we shall say that a topological pospace X is a CC_i-*space* (resp. CC_d-*space*) if whenever a chain F of X is closed, ${\uparrow}F$ (resp. ${\downarrow}F$) is a closed subset in X.

Theorem 4.10 *Let X be an H-closed topological pospace. If X satisfies the following properties:*

(i) *X is regularly ordered;*
(ii) *X is a CC_i-space; and*
(iii) *X is a CC_d-space,*

then every maximal chain in X with the $\updownarrow\cdot$ m-property is an H-closed topological pospace.

Proof Suppose to the contrary, that there is a non-H-closed maximal chain L with the $\updownarrow\cdot$ m-property in X. Then by Corollary 3.5, at least one of the following conditions holds:

(I) the set L is not a complete semilattice with the induced partial order from X;
(II) there exists a non-empty subset A in L with $x = \inf A$ such that $A = {\uparrow}A \setminus \{x\}$ and $x \notin \mathrm{cl}_L(A)$;
(III) there exists a non-empty subset B in L with $y = \sup B$ such that $B = {\downarrow}B \setminus \{y\}$ and $y \notin \mathrm{cl}_L(B)$.

Suppose that condition (I) holds. Since a topological space X with the order dual to $\leqslant$ is a topological pospace, we can assume without loss generality that there exists a subset S of L which does not have a sup in L. Then the set $\downarrow S \cap L$ does not have a sup in L either. Hence the set $I = L \setminus \downarrow S$ does not have an inf in L. We observe that the maximality of L implies that there exist no lower bound b of I and no upper bound a of S such that $a \leqslant b$. Also, we observe that properties (ii)–(iii) of X and Corollaries 2.9 and 2.10 imply that $I \neq \varnothing$. Otherwise, if $I = \varnothing$ then by Corollary 2.10 the chain S has a sup in X, which contradicts the maximality of the chain L. We observe that the dual argument shows that $S \neq \varnothing$, when there exists a subset I in L which does not have an inf in L. Therefore we can assume without loss of generality that $S = \downarrow S \cap L$, $I = \uparrow I \cap L$ and L is the disjoint union of S and I.

Since the set S does not have a sup in L we conclude that $\bigcap_{x \in S} \uparrow x$ is a closed subset of X and $\bigcap_{x \in S} \uparrow x \cap S = \varnothing$. Hence S is an open subset in L. A dual argument shows that I is an open subset in L. Therefore S and I are clopen subsets of L.

Let $x \notin X$. We extend the partial order $\leqslant$ from X onto $X^* = X \cup \{x\}$ by setting $a \leqslant b$ in X^* if and only if one of the following conditions holds:

1) $a, b \in X$ and $a \leqslant b$ in X;
2) $a = x$ and $b \in \uparrow_X I$;
3) $a \in \downarrow_X S$ and $b = x$.

Let $\mathscr{U}_S$ be the set of all increasing open subsets of X which intersect S and let $\mathscr{D}_I$ be the set of all decreasing open subsets of X which intersect I. Let τ be the topology of X and let τ^* be the topology generated by the pseudobase

$$\tau \cup \left\{\{x\} \cup U \mid U \in \mathscr{U}_S\right\} \cup \left\{\{x\} \cup U \mid U \in \mathscr{D}_I\right\}.$$

Since the chain L has the $\updownarrow\cdot$ m-property and conditions (i)–(iii) hold we conclude that X^* is a topological pospace which contains X as a dense subspace, a contradiction.

Suppose that the statement (II) holds, i. e. that there exists an open neighbourhood $O(x)$ of $x = \inf A$ such that $O(x) \cap A = \varnothing$. We can assume without loss of generality that $\uparrow A = L \cap A$. By Corollary 2.9, the chain L has a minimum element and hence $B = L \setminus A \neq \varnothing$ and $x \in B$. Since $\bigcap_{y \in B} \downarrow y$ is a closed subset in X we conclude that A is an open subset of L. Since for any $y \in B \setminus \{x\}$ we have that $X \setminus \uparrow x$ is an open neighbourhood of y and there exists an open neighbourhood $O(x)$ of x such that $O(x) \cap A = \varnothing$, we obtain that A is a closed subset of L. The maximality of L implies that A is a closed subset of X.

Let $p \notin X$. We extend the partial order $\leqslant$ from X onto $X^\dagger = X \cup \{p\}$ by setting $a \leqslant b$ in $X^\dagger$ if and only if one of the following conditions holds:

1) $a, b \in X$ and $a \leqslant b$ in X;
2) $a = p$ and $b \in \uparrow_X A$;
3) $a \in \downarrow_X B$ and $b = p$.

Let $\mathscr{U}_A$ be the set of all increasing open subsets of X which contain A and let $\mathscr{D}_A$ be the set of all decreasing open subsets of X which intersect A. Let τ be the topology of X and let $\tau^\dagger$ be the topology generated by the pseudobase

$$\tau \cup \left\{\{p\} \cup U \mid U \in \mathscr{U}_A\right\} \cup \left\{\{p\} \cup U \mid U \in \mathscr{D}_A\right\}.$$

Since the chain L has the $\updownarrow\cdot$ m-property and conditions (i)–(iii) hold we conclude that $X^\dagger$ is a topological pospace. Therefore we get that $(X^\dagger, \tau^\dagger, \leqslant)$ is a topological pospace and X is a dense subspace of $(X^\dagger, \tau^\dagger, \leqslant)$. This contradicts the assumption that X is an H-closed pospace.

In case (III) we get a similar contradiction as in (II). These contradictions imply the assertion of the theorem. □

Remark 4.11 We observe that the topological pospace $(T^*, \preccurlyeq, \tau^*)$ from Example 4.9 is not regularly ordered and is not a CC_i-space. Also, the topological pospace $(T^*, \preccurlyeq, \tau^*)$ admits the continuous semilattice operation

$$(x_1, y_1) \cdot (x_2, y_2) = (\max\{x_1, x_2\}, \max\{y_1, y_2\}) \quad \text{and} \quad (x_1, y_1) \cdot \alpha = \alpha \cdot (x_1, y_1) = \alpha,$$

for $x_1, x_2 \in X$ and $y_1, y_2 \in Y$. Therefore a maximal chain of an H-closed topological semilattice is not necessarily an H-closed topological semilattice.

Acknowledgements This research was supported by the Slovenian Research Agency grants P1-0292-0101-04, J1-9643-0101 and BI-UA/07-08/001. We thank the referees for several comments and suggestions.

References

1. Alexandroff, P., Urysohn, P.: Sur les espaces topologiques compacts. Bull. Intern. Acad. Pol. Sci. Sér. A, 5–8 (1923)
2. Alexandroff, P., Urysohn, P.: Mémoire sur les espaces topologiques compacts. Vehr. Akad. Wetensch. Amsterdam **14**, 1–96 (1929)
3. Carruth, J.H., Hildebrant, J.A., Koch, R.J.: The Theory of Topological Semigroups, vol. I. Marcel Dekker, New York (1983)
4. Carruth, J.H., Hildebrant, J.A., Koch, R.J.: The Theory of Topological Semigroups, vol. II. Marcel Dekker, New York (1986)
5. Choe, T.H., Park, Y.S.: Embedding ordered topological spaces into topological semilattices. Semigroup Forum **17**, 189–199 (1979)
6. Chuchman, I., Gutik, O.: On H-closed topological semigroups and semilattices. Algebra Discrete Math. **1**, 13–23 (2007)
7. Clifford, A.H., Preston, G.B.: The Algebraic Theory of Semigroups, vol. I. Amer. Math. Soc. Surveys 7. American Mathematical Society, Providence (1961)
8. Clifford, A.H., Preston, G.B.: The Algebraic Theory of Semigroups, vol. II. Amer. Math. Soc., Surveys 7. American Mathematical Society, Providence (1967)
9. Engelking, R.: General Topology, 2nd edn. Heldermann, Berlin (1989)
10. Gierz, G., Hofmann, K.H., Keimel, K., Lawson, J.D., Mislove, M.W., Scott, D.S.: Continuous Lattices and Domains. Cambridge Univ. Press, Cambridge (2003)
11. Green, M.D.: A locally convex topology on a preordered space. Pac. J. Math. **26**(3), 487–491 (1968)
12. Gutik, O., Repovš, D.: On linearly ordered H-closed topological semilattices. Semigroup Forum **77**(3), 474–481 (2008)
13. McCartan, S.D.: Bicontinuous preordered topological spaces. Pac. J. Math. **38**(2), 523–529 (1971)
14. Nachbin, L.: Topology and Order. van Nostrand Company, Princeton (1965)
15. Priestley, H.A.: Ordered topological spaces and the representation of distributive lattices. Proc. Lond. Math. Soc. **24**(3), 507–520 (1972)
16. Stepp, J.W.: Algebraic maximal semilattices. Pac. J. Math. **58**(1), 243–248 (1975)
17. Ward, L.E. Jr.: Partially ordered topological spaces. Proc. Am. Math. Soc. **5**(1), 144–161 (1954)