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The π-π-Theorem for Manifold Pairs with Boundaries

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Abstract—The surgery obstruction of a normal map to a simple Poincaré pair (X, Y) lies in the relative surgery obstruction group $L_*(\pi_1(Y) \to \pi_1(X))$. A well-known result of Wall, the so-called π-π-theorem, states that in higher dimensions a normal map of a manifold with boundary to a simple Poincaré pair with $\pi_1(X) \cong \pi_1(Y)$ is normally bordant to a simple homotopy equivalence of pairs. In order to study normal maps to a manifold with a submanifold, Wall introduced the surgery obstruction groups LP_* for manifold pairs and splitting obstruction groups LS_*. In the present paper, we formulate and prove for manifold pairs with boundary results similar to the π-π-theorem. We give direct geometric proofs, which are based on the original statements of Wall's results and apply obtained results to investigate surgery on filtered manifolds.

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1. INTRODUCTION

The surgery obstruction groups $L_*(\pi)$ were introduced by Wall in his fundamental paper [1]. Let $(f, b)\colon M \to X$ be a normal map from a closed manifold M^n to a simple Poincaré complex X of formal dimension n, where $b\colon \nu_M \to \xi$ is a bundle map which covers $f\colon M \to X$. Then an obstruction $\theta(b, f) \in L_n(\pi_1(X))$ for the existence of a simple homotopy equivalence in the class of the normal bordism of the map (f, b) is defined.

Indeed, Wall defines L_*-groups and describes surgery theory for the case of manifold n-ads. For example, an obstruction to surgery of a normal map

$$((f, b), (\partial f, \partial b))\colon (M, \partial M) \to (X, \partial X) \tag{1}$$

of manifolds with boundary lies in the relative surgery obstruction group $L_*(\pi_1(\partial X) \to \pi_1(X))$. Hence if the map $\pi_1(\partial X) \to \pi_1(X)$ induced by the inclusion $\partial X \to X$ is an isomorphism, then in higher dimensions the map (1) is normally bordant to a simple homotopy equivalence of pairs (to the s-triangulation of the pair $(X, \partial X)$). In [1, Sec. 4], Wall gives direct geometric proof of this "important special case".

The surgery obstruction groups and natural maps do not depend on the category of manifolds (see [1] and [2]). In the present paper, we shall consider topological manifolds and manifold pairs following the definitions from [2]. All the obtained results can be carried over to *Diff*- and PL-manifolds.

Let (X^n, Y^{n-q}, ξ_Y) be a codimension-q manifold pair [2, p. 570], where ξ_Y is a normal topological block bundle over the submanifold Y. In the given paper, we systematically consider restrictions of bundles to various submanifolds. Hence, for all considered bundles, we add a subscript indicating the

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base of the bundle in the notations. In [2, p. 570–571], the t-triangulation, i.e., the topological normal map

$$((f,b),(g,c))\colon (M,N)\to (X,Y) \tag{2}$$

to a manifold pair (X,Y,ξ_Y), is defined. Note that the set of classes of concordant t-triangulations of a pair (X,Y,ξ_Y) coincides with the set of normal bordism classes of topological normal maps to the manifold X [2, Proposition 7.2.3]. We use also the definition of an s-triangulation of a pair (X,Y,ξ_Y) [2, Sec. 1].

Let

$$F=\begin{pmatrix}\pi_1(S(\xi_Y)) & \longrightarrow & \pi_1(X\setminus Y)\\ \Big\downarrow & & \Big\downarrow\\ \pi_1(Y) & \longrightarrow & \pi_1(X)\end{pmatrix}$$

be the pushout square of fundamental groups with orientations, where $S(\xi_Y)$ is the boundary of the tubular neighborhood of Y in X.

The obstruction groups $LP_*(F)$ to surgery of a t-triangulation (2) to obtain an s-triangulation of the manifold pair (X,Y,ξ_Y) are defined (see [1, Sec. 17E] and [2, Sec. 7.2]). The splitting obstruction groups $LS_*(F)$ are also defined in [1] and [2]. A simple homotopy equivalence $f\colon M\to X$ splits along a submanifold Y if it is homotopy equivalent to an s-triangulation of the manifold pair (X,Y,ξ_Y). The groups $LS_n(F)$ and $LP_n(F)$ depend functorially on the square of fundamental groups F and n mod 4.

For the topological normal map (1), surgery theory is essentially different in two cases. For the case of surgery relative to the boundary (rel$_\partial$-case), we consider topological normal maps (1) with a fixed s-triangulation $\partial f\colon \partial M\to\partial X$ on the boundary. Moreover, the normal bordisms are trivial on the boundary, i.e., have the form

$$\partial f\times \mathrm{Id}\colon \partial M\times I\to \partial X\times I.$$

Note that in the rel$_\partial$-case the obstruction to surgery of the normal map (1) up to simple homotopy equivalence lies in the group $L_n(\pi_1(X))$, where $n=\dim X$. For a manifold pair $(X,\partial X)\supset(Y,\partial Y)$ of codimension-q with boundary (see [2, Sec. 7]) the situation is similar to the rel$_\partial$-case. We consider normal maps (1) which are already split on the boundary, and the concordance is the product on the boundary in this case. The surgery theory for normal maps of pairs of manifolds with boundary in the rel$_\partial$-case is developed in [1] and [2].

In the present paper, we consider surgery on manifold pairs with boundary without fixing maps on the boundary. We need to know the geometric properties of the surgery on such pairs to study the surgery on manifolds with filtrations (see [3]–[5]). This is the special case of the surgery theory on manifold n-ads mentioned by Wall [1, p. 136]. We give the exact statement and proof of π-π-theorem for various maps to a manifold pair with boundary. Then we apply obtained results to study surgery on manifolds with filtrations.

A manifold pair $(X,\partial X)\supset(Y,\partial Y)$ with boundary defines a pair of closed manifolds $\partial Y\subset\partial X$ with a pushout square

$$\Psi=\begin{pmatrix}\pi_1(S(\xi_{\partial Y})) & \longrightarrow & \pi_1(\partial X\setminus \partial Y)\\ \Big\downarrow & & \Big\downarrow\\ \pi_1(\partial Y) & \longrightarrow & \pi_1(\partial X)\end{pmatrix}$$

of fundamental groups for the corresponding splitting problem. Here $\xi_{\partial Y}$ is the restriction of the bundle ξ_Y to the boundary ∂Y of the manifold Y. The natural inclusion $\partial X\to X$ induces a map of $\Delta\colon \Psi\to F$ of squares of fundamental groups.

Now we can formulate the main result of the paper.

Theorem. *Let*

$$(f, \partial f)\colon (M, \partial M) \to (X, \partial X)$$

be a topological normal map (1) *to a pair*

$$(X^n, \partial X) \supset (Y^{n-q}, \partial Y) \tag{3}$$

of manifolds of codimension-q with boundary. Let $n - q \geq 6$ and Δ be an isomorphism of the squares of fundamental groups. Under these conditions, we have the following results.

(A) The normal map $(f, \partial f)$ is normally bordant to an s-triangulation of the pair (3) *of manifolds with boundary.*

(B) If the map $(f, \partial f)$ is a simple homotopy equivalence of the pairs, then it is concordant to an s-triangulation of the pair (3) *of manifolds with boundary.*

(C) Let the normal map $(f, \partial f)$ define a simple homotopy equivalence of the pairs

$$(f|_N, f|_{\partial N})\colon (N, \partial N) \to (Y, \partial Y),$$

where $N = f^{-1}(Y)$, $\partial N = f^{-1}(\partial Y)$ are the transversal preimage. Then the map $(f, \partial f)$ is normally bordant to an s-triangulation of the pair (3) *of manifolds with boundary. Moreover, we can choose a transversal to $Y \times I$ normal bordism*

$$F = (h; g, f_0, f_1)\colon (W; V, M, M') \to (X \times I; \partial X \times I, X \times \{0\}, X \times \{1\})$$

such that the restriction $h_{h^{-1}(Y \times I)}$ is

$$(f|_{(N, \partial N)}) \times \mathrm{Id}\colon (N, \partial N) \times I \to (Y, \partial Y) \times I.$$

In Sec. 2, we give a necessary preliminary material. In Sec. 3, we prove the Theorem and apply obtained results to the surgery on manifolds with filtrations.

2. PRELIMINARIES

We consider a case of topological manifolds and follow notation from [2, Sec. 7.2]. Let (X, Y, ξ_Y) be a codimension-q manifold pair in the sense of Ranicki (see [2, Sec. 7.2]), i. e., a locally flat closed submanifold Y is given with a topological normal block fibration

$$\xi_Y = \xi_{Y \subset X}\colon Y \to \widetilde{BTOP}(q)$$

with the associated (D^q, S^{q-1})-fibration

$$(D^q, S^{q-1}) \to (E(\xi_Y), S(\xi_Y)) \to Y$$

and we have a decomposition of the closed manifold

$$X = E(\xi_Y) \bigcup_{S(\xi_Y)} \overline{X \setminus E(\xi_Y)}.$$

A topological normal map (2) is given by a transversal to Y normal map $(f, b)\colon M \to X$ with $N = f^{-1}(Y)$. Moreover, the pair (M, N) is a codimension-q topological manifold pair with a normal topological block fibration

$$\nu_N\colon N \xrightarrow{f|_N} Y \xrightarrow{\xi_Y} \widetilde{BTOP}(q).$$

Additionally, the following conditions are satisfied:

(i) the restriction

$$(f, b)|_N = (g, c)\colon N \to Y$$

is a normal map;

(ii) the restriction

$$(f,b)|_P = (h,d)\colon (P, S(\nu_N)) \to (Z, S(\xi_Y))$$

is a normal map to the pair $(Z, S(\xi_Y))$, where $P = \overline{M \setminus E(\nu_N)}$ and $Z = \overline{X \setminus E(\xi_Y)}$;

(iii) the restriction

$$(h,d)|_{S(\nu_N)}\colon S(\nu_N) \to S(\xi_Y)$$

coincides with the induced map $(g,c)^!\colon S(\nu_N) \to S(\xi_Y)$ and $(f,b) = (g,c)^! \cup (h,d)$.

In the topological category, an s-triangulation of a manifold pair (X, Y, ξ_Y) [2, p. 571] is given by a t-triangulation for which the maps

$$f\colon M \to X, \qquad g\colon N \to Y, \qquad (P, S(\nu_N)) \to (Z, S(\xi_Y))$$

are simple homotopy equivalences (s-triangulations).

For a codimension-q manifold pair with boundary (3) [2, p. 585], we have a normal fibration $(\xi_Y, \xi_{\partial Y})$ over the pair $(Y, \partial Y)$ and, as in the case of closed manifold, a decomposition

$$(X, \partial X) = \Big(E(\xi_Y) \bigcup_{S(\xi_Y)} Z,\ E(\xi_{\partial Y}) \bigcup_{S(\xi_{\partial Y})} \partial_+ Z\Big),$$

where $(Z; \partial_+ Z, S(\xi_Y); S(\xi_{\partial Y}))$ is a manifold triad. Note here that $\partial_+ Z = \overline{\partial X \setminus E(\xi_{\partial Y})}$.

A topological normal map (1) provides a normal map of a pair $(M, \partial M) \supset (N, \partial N)$ of manifolds with boundary to the pair (3) of manifolds with boundary if it is transversal to $(Y, \partial Y)$ with

$$(N, \partial N) = (f^{-1}(Y), f^{-1}(\partial Y)),$$

and the maps f and ∂f provide topological normal maps of pairs

$$(M, N) \to (X, Y) \qquad \text{and} \qquad (\partial M, \partial N) \to (\partial X, \partial Y),$$

respectively. We note that in this case the topological normal block fibration $(\nu_N, \nu_{\partial N})$ over the pair $(N, \partial N)$ [2, p. 570] defines the decomposition

$$(M, \partial M) = \Big(E(\nu_N) \bigcup_{S(\nu_N)} P,\ E(\nu_{\partial N}) \bigcup_{S(\nu_{\partial N})} \partial_+ P\Big),$$

where $(P; \partial_+ P, S(\nu_N); S(\nu_{\partial N}))$ is a manifold triad.

A topological normal map (1) is an s-triangulation of the manifold pair (3) with boundary if the maps $f\colon M \to X$ and $\partial f\colon \partial M \to \partial X$ are s-triangulations of the pairs (X, Y) and $(\partial X, \partial Y)$, respectively.

Two s-triangulations of a manifold X with a boundary ∂X

$$(f_i, \partial f_i)\colon (M_i, \partial M_i) \to (X, \partial X), \qquad i = 0, 1,$$

are concordant (see [1, Sec. 10] and [2, Sec. 7.1]) if there exists a simple homotopy equivalence of 4-ads

$$(h; g, f_0, f_1)\colon (W; V, M_0, M_1) \to (X \times I; \partial X \times I, X \times \{0\}, X \times \{1\})$$

with $\partial V = \partial M_0 \cup \partial M_1$.

3. PROOF AND COROLLARY

Consider case (A) of the theorem. Without loss of generality, the restriction map $(f, \partial f)$ can be considered transversal to $(Y, \partial Y) \subset (X, \partial X)$. Let $(N, \partial N) = (f^{-1}(Y), (\partial f)^{-1}(\partial Y))$ be the transversal preimage. The map

$$(f|_N, \partial f|_{\partial N})\colon (N, \partial N) \to (Y, \partial Y) \tag{4}$$

is a normal map of the manifolds with boundary. The isomorphism Δ provides an isomorphism $\pi_1(\partial Y) \to \pi_1(Y)$. Thus the normal map (4) satisfies the conditions of π-π-theorem of Wall [1, Sec. 4] for the pair $(Y, \partial Y)$, and hence it is normally bordant to a simple homotopy equivalence of pairs

$$(L, \partial L) \to (Y, \partial Y). \tag{5}$$

By [6, p. 45], we can extend this bordism to obtain a normal bordism with the bottom map $(f, \partial f)$ and with a top normal map

$$(g, \partial g)\colon (K, \partial K) \to (X, \partial X). \tag{6}$$

The map (6) is transversal to $(Y, \partial Y)$, and the pair (K, L) is a topological manifold pair with a topological normal block fibration ν_L. Since the map (5) is a simple homotopy equivalence of pairs, it provides a simple homotopy equivalence of tubular neighborhoods with boundary (see [1, p. 33] and [2, p. 579])

$$\psi = g|_{E(\nu_L)}\colon (E(\nu_L); E(\nu_{\partial L}), S(\nu_L); S(\nu_{\partial L})) \to (E(\xi_Y); E(\xi_{\partial Y}), S(\xi_Y); S(\xi_{\partial Y})).$$

In particular, the restriction of the map ψ to $S(\nu_L)$ is a simple homotopy equivalence of pairs

$$\phi = \psi|_{S(\nu_L)} = g|_{S(\nu_L)}\colon (S(\nu_L), S(\nu_{\partial L})) \to (S(\xi_Y), S(\xi_{\partial Y})). \tag{7}$$

Let

$$P = \overline{K \setminus E(\nu_L)}, \qquad \partial_+ P = \overline{\partial K \setminus E(\nu_{\partial L})}.$$

The restriction of the map g to P provides a normal map of triads

$$\alpha = g|_P\colon (P; \partial_+ P, S(\nu_L); S(\nu_{\partial L})) \to (Z; \partial_+ Z, S(\xi_Y); S(\xi_{\partial Y})), \tag{8}$$

and the restriction of this one to $S(\nu_L)$ is a simple homotopy equivalence ϕ (7) of pairs.

Consider a square of fundamental groups

$$\begin{array}{ccc} \pi_1(S(\xi_{\partial Y})) & \longrightarrow & \pi_1(\partial_+ Z) \\ \big\downarrow & & \big\downarrow \\ \pi_1(S(\xi_Y)) & \longrightarrow & \pi_1(Z) \end{array} \tag{9}$$

of the triad $(Z; \partial_+ Z, S(\xi_Y); S(\xi_{\partial Y}))$.

The isomorphism Δ provides an isomorphism $\pi_1(\partial_+ Z) \to \pi_1(Z)$ in the square (9), and the restriction of the map α to $S(\nu_L)$ coincides with the simple homotopy equivalence ϕ (7) of the pairs. Hence, by the π-π-theorem of Wall for the triad $(Z; \partial_+ Z, S(\xi_Y); S(\xi_{\partial Y}))$, the normal map α (8) is normally bordant relative to $(S(\nu_L), S(\nu_{\partial L}))$ to a simple homotopy equivalence of triads

$$\alpha'\colon (P'; \partial_+ P', S(\nu_L); S(\nu_{\partial L})) \to (Z; \partial_+ Z, S(\xi_Y); S(\xi_{\partial Y}))$$

by means of a normal bordism

$$G\colon W \to Z \times I. \tag{10}$$

The restriction of the map G to the bottom $W_0 = P$ coincides with the map $P \to Z = Z \times \{0\}$ which is defined by the map α, and the restriction of the map G to the top $W_1 = P'$ coincides with the simple homotopy equivalence $P' \to Z = Z \times \{1\}$ which is defined by α'. The side boundary of the manifold W has the form

$$(S(\nu_L) \times I) \bigcup_{S(\nu_{\partial L})} V,$$

where the manifold V provides the bordism between $\partial_+ P$ and $\partial_+ P'$. Denote by F the restriction of the map G to the manifold $S(\nu_L) \times I$, which is a part of the side boundary of the bordism W. The map F has the following form:

$$F = G|_{S(\nu_L)\times I} = \phi \times \mathrm{Id}\colon S(\nu_L) \times I \to S(\xi_Y) \times I. \tag{11}$$

Consider a trivial normal bordism

$$\psi \times \mathrm{Id}\colon E(\nu_L) \times I \to E(\xi_Y) \times I. \tag{12}$$

Gluing together the bordisms $E(\nu_L) \times I$ and W along the general part $S(\nu_L) \times I$ of theirs side boundary, we obtain a bordism

$$W \bigcup_{S(\nu_L)\times I} (E(\nu_L) \times I)$$

with the bottom K and the top

$$M' = P' \bigcup_{S(\nu_L)\times\{1\}} (E(\nu_L) \times \{1\}). \tag{13}$$

The restrictions of the maps G (10) and $\psi \times \mathrm{Id}$ (12) to $S(\nu_L) \times I$ coincide and are equal to the map $F = \phi \times \mathrm{Id}$ (11). Hence we can define a normal bordism

$$\Omega = G \bigcup_{\phi\times\mathrm{Id}} (\psi \times \mathrm{Id})\colon W \bigcup_{S(\nu_L)\times I} (E(\nu_L) \times I) \to X \times I$$

between the normal map $(g, \partial g)$ (6) on the bottom and a normal map

$$(f', \partial f')\colon (M', \partial M') \to (X, \partial X) \tag{14}$$

on the top, where

$$(M', \partial M') = \left(P' \bigcup_{S(\nu_L)\times\{1\}} (E(\nu_L) \times \{1\}),\ \partial_+ P' \bigcup_{S(\nu_{\partial L})\times\{1\}} (E(\nu_{\partial L}) \times \{1\})\right).$$

Since the maps $(f, \partial f)$ (1) and $(g, \partial g)$ (6) are normally bordant, it follows that the map $(f, \partial f)$ (1) is normally bordant to the map $(f', \partial f')$ (14). The restriction of the map f' to $E(\nu_L) \times \{1\} = E(\nu_L)$ gives the simple homotopy equivalence of triads ψ, and the restriction of the map f' to P' gives the simple homotopy equivalence of triads α'. To finish the proof in case (A), it is necessary to verify that the maps $f'\colon M' \to X$ and $\partial f'\colon \partial M' \to \partial X$ are simple homotopy equivalences. According to (13), the space M' is the union of two subspaces P' and $E(\nu_L) \times \{1\}$ meeting in $S(\nu_L) \times \{1\}$. The restrictions of the map f' to these two subspaces and to theirs intersection are simple homotopy equivalences. Hence, according to [7, Theorem 23.1], the map $f'\colon M' \to X$ is a simple homotopy equivalence. For the map $\partial f'$ the situation is similar. Case (A) is proved.

Now consider case (B). The map $(f, \partial f)$ (1) is a simple homotopy equivalence of pairs. Since the map Δ is an isomorphism, by (A) it follows that the map $(f, \partial f)$ is normally bordant to a map

$$(f', \partial f')\colon (M', \partial M') \to (X, \partial X),$$

which is an s-triangulation of the pair (3) of manifolds with boundary. Thus we have a normal bordism

$$\Phi\colon W \to X \times I, \qquad \partial W = W_0 \cup W_1 \cup V,$$

where

$$W_0 = M, \qquad W_1 = M', \qquad V = \Phi^{-1}(\partial X \times I).$$

The bordism (W, Φ) gives a normal map of manifold triads

$$(W; V, W_0 \cup W_1; \partial W_0 \cup \partial W_1) \to (X \times I; \partial X \times I, X \times \{0,1\}; \partial X \times \{0,1\}), \tag{15}$$

which we shall also denote by Φ.

Consider a square of fundamental groupoids

$$\begin{array}{ccc} \pi_1(\partial X \times \{0,1\}) & \longrightarrow & \pi_1(\partial X \times I) \\ \downarrow & & \downarrow \\ \pi_1(X \times \{0,1\}) & \longrightarrow & \pi_1(X \times I) \end{array} \tag{16}$$

of the triad

$$(X \times I; \partial X \times I, X \times \{0,1\}; \partial X \times \{0,1\}).$$

The isomorphism Δ provides the isomorphism $\pi_1(\partial X) \to \pi_1(X)$, and hence the right vertical map $\pi_1(\partial X \times I) \to \pi_1(X \times I)$ in the square (16) is an isomorphism. The restriction of the normal map Φ to the pair $(W_0 \cup W_1, \partial W_0 \cup \partial W_1)$ is a simple homotopy equivalence of pairs

$$(W_0 \cup W_1, \partial W_0 \cup \partial W_1) \to (X \times \{0,1\}, \partial X \times \{0,1\})$$

since

$$\Phi|_{W_0} = f\colon M \to X \qquad \text{and} \qquad \Phi|_{W_1} = f'\colon M' \to X.$$

Therefore, the normal map of triads (15) satisfies the conditions of π-π-theorem for manifold triads relative to the pair $(X \times \{0,1\}, \partial X \times \{0,1\})$.

Thus the map Φ (15) of triads is normally bordant relative to the pair $(X \times \{0,1\}, \partial X \times \{0,1\})$ to a simple homotopy equivalence of triads

$$\Phi'\colon (W'; V', W_0 \cup W_1; \partial W_0 \cup \partial W_1) \to (X \times I; \partial X \times I, X \times \{0,1\}; \partial X \times \{0,1\}),$$

where

$$\Phi'|_{W_0} = f, \qquad \Phi'|_{W_1} = f', \qquad \Phi'|_{\partial W_0} = \partial f, \qquad \Phi'|_{\partial W_1} = \partial f'.$$

By our construction, the map $(f', \partial f')$ is an s-triangulation of the pair $(Y, \partial Y) \subset (X, \partial X)$ (3) of manifolds with boundary, and the map Φ' gives a concordance between $(f, \partial f)$ and $(f', \partial f')$. Case (B) of the theorem is proved.

In case (C), the map f is transversal to $(Y, \partial Y)$ and its restriction $f|_{(N,\partial N)}$ is a simple homotopy equivalence $(N, \partial N) \to (Y, \partial Y)$. Hence the map $f|_{(N,\partial N)}$ induces a simple homotopy equivalence of tubular neighborhoods with boundary (see [1, p. 8] and [2, p. 579])

$$\psi\colon (E(\nu_N); E(\nu_{\partial N}), S(\nu_N); S(\nu_{\partial N})) \to (E(\xi_Y); E(\xi_{\partial Y}), S(\xi_Y); S(\xi_{\partial Y})),$$

which is a simple homotopy equivalence of triads. In particular, the restriction of ψ to $S(\nu_N)$ is the simple homotopy equivalence of pairs

$$\phi\colon (S(\nu_N), S(\nu_{\partial N})) \to (S(\xi_Y), S(\xi_{\partial Y}))$$

whose restriction to $(N, \partial N)$ is the simple homotopy equivalence of pairs $(f|_N, f|_{\partial N})$. Our subsequent reasoning is the same as in case (A). It is sufficient to assume that

$$P = \overline{M \setminus E(\nu_N)}, \qquad \partial_+ P = \overline{\partial M \setminus E(\nu_{\partial N})}$$

and to consider the restriction of the map f to P giving a normal map of triads

$$\alpha = f|_P\colon (P; \partial_+ P, S(\nu_N); S(\nu_{\partial N})) \to (Z; \partial_+ Z, S(\xi_Y); S(\xi_{\partial Y}))$$

whose restriction to $S(\nu_N)$ is a simple homotopy equivalence ϕ of pairs. The theorem is proved. $\square$

Now we apply the obtained results to surgery on filtered manifolds (see [3]–[5] and [8]). First, we recall the necessary definitions.

Let $Z^{n-q-q'} \subset Y^{n-q} \subset X^n$ be a triple of closed topological manifolds (see [2], [4] and [8]). We have the following topological normal bundles: ξ_Y for the submanifold Y in X, η_Z for the submanifold Z in Y, and ν_Z for the submanifold Z in X. Let $(E(\xi_Y), S(\xi_Y))$, $(E(\eta_Z), S(\eta_Z))$ and $(E(\nu_Z), S(\nu_Z))$ be the spaces of associated (D^*, S^{*-1}) fibrations, respectively. We identify the space $E(\nu_Z)$ with the space $E(\xi_{E(\eta_Z)})$ of the restriction of the fibration ξ_Y to the space $E(\eta_Z)$ (see [3] and [4]), in such a way that

$$E(\nu_Z) = E(\xi_{E(\eta_Z)}) \qquad \text{and} \qquad S(\nu_Z) = E(\xi_{S(\eta_Z)}) \cup S(\xi_{E(\eta_Z)}). \tag{17}$$

Let

$$(X_k, \partial X_k) \subset (X_{k-1}, \partial X_{k-1}) \subset \cdots \subset (X_0, \partial X_0) = (X, \partial X) \tag{18}$$

be a filtration of a compact manifold $(X, \partial X)$ by submanifolds with boundary (see [3]–[5]). From now we shall assume that the dimension $\dim X_k \geq 6$.

The filtration (18) defines the filtration

$$\partial X_k \subset \partial X_{k-1} \subset \cdots \subset \partial X_0 = \partial X \tag{19}$$

of ∂X by closed manifolds. Recall that any triple of manifolds from filtrations (18) and (19) satisfy properties that are similar to (17) on the corresponding normal fibrations, and every pair of manifolds with boundary from filtration (18) is a topological manifold pair defined in Sec. 1. The filtration (18) defines a stratified manifold with boundary $(\mathscr{X}, \partial\mathscr{X})$ (see [3]–[5]).

A topological normal map (1) to the filtration $(\mathscr{X}, \partial\mathscr{X})$ (18) is a t-triangulation of this filtration if it is transversal to the all submanifolds of the filtration (18) with the transversal preimages

$$(M_i, \partial M_i) = (f^{-1}(X_i), (\partial f)^{-1}(\partial X_i)),$$

and the constituent maps

$$f|_{(M_j, M_i)} \colon (M_j, M_i) \to (X_j, X_i), \qquad 0 \leq j \leq i \leq k,$$

are t-triangulations of the manifold pairs (X_j, X_i) with boundary. Any topological normal map (1) is normally bordant to a t-triangulation (18) (see [4] and [5]). Thus we can assume that the normal map $(f, \partial f)$ (1) is transversal to all submanifolds $(X_i, \partial X_i)$ of the filtration (18). A topological normal map (1) to the filtration $(\mathscr{X}, \partial\mathscr{X})$ (18) is an s-triangulation of this filtration if the constituent normal maps

$$f|_{(M_j, M_i)} \colon (M_j, M_i) \to (X_j, X_i), \qquad 0 \leq j \leq i \leq k,$$

are s-triangulations of the manifold pairs with boundary (X_j, X_i). The stratified Browder-Quinn groups $L_n^{BQ}(\mathscr{X}, \partial\mathscr{X})$ are defined [3]. These groups are the groups of obstructions to do surgery from a t-triangulation of the filtration $(\mathscr{X}, \partial\mathscr{X})$ to an s-triangulation of the filtration $(\mathscr{X}, \partial\mathscr{X})$.

Let F_i, $1 \leq i \leq k$, denote the square of fundamental groups in the splitting problem for the manifold pair $X_i \subset X_{i-1}$, and Ψ_i be the similar square for the closed manifold pair $\partial X_i \subset \partial X_{i-1}$. The natural inclusions of boundaries induce the maps

$$\Delta_i \colon \Psi_i \to F_i$$

for $1 \leq i \leq k$.

Corollary. *Let all the maps Δ_i, $1 \leq i \leq k$, be isomorphisms. Then every normal map $(f, \partial f)$ (1) in the stratified manifold $(\mathscr{X}, \partial\mathscr{X})$ is normally bordant to an s-triangulation of the pair $(\mathscr{X}, \partial\mathscr{X})$, and hence the group $L_n^{BQ}(\mathscr{X}, \partial\mathscr{X})$ is trivial.*

Proof. Denote by $(f_{k-1}, \partial f_{k-1})$ the restriction of the normal map $(f, \partial f)$ to the submanifold $(M_{k-1}, \partial M_{k-1})$. By item (A) of the theorem, the map $(f_{k-1}, \partial f_{k-1})$ will be normally bordant to an s-triangulation $(g_{k-1}, \partial g_{k-1})$ of the pair

$$(X_{k-1}, \partial X_{k-1}) \supset (X_k, \partial X_k) \tag{20}$$

of the manifolds with boundary. By [6] we can extend this bordism to obtain a bordism $F \colon W \to X \times I$ with a top normal map $(g, \partial g)$ to $(X, \partial X)$, whose restriction $(g_{k-1}, \partial g_{k-1})$ to the transversal preimage (X_{k-1}, X_k) is an s-triangulation of the pair (20) of manifolds with boundary. Without loss of generality, we can assume that the map $(g, \partial g)$ is transversal to all submanifolds $(X_i, \partial X_i)$, $1 \leq i \leq k$, of the filtration (18). For the map $(g, \partial g)$ we denote the transversal preimage of the submanifold $(X_i, \partial X_i)$ by $(M_i, \partial M_i)$ as before.

Consider the restriction of the map $(g, \partial g)$ to the submanifold $(M_{k-2}, \partial M_{k-2})$. By item (C) of the theorem, the map $(g, \partial g)|_{(M_{k-2}, \partial M_{k-2})}$ is normally bordant to an s-triangulation $(g_{k-2}, \partial g_{k-2})$ of the pair

$$(X_{k-2}, \partial X_{k-2}) \supset (X_{k-1}, \partial X_{k-1})$$

of manifolds with boundary and

$$(g_{k-2}, \partial g_{k-2})|_{(M_{k-1}, \partial M_{k-1})} = (g_{k-1}, \partial g_{k-1}).$$

By [6] we can extend this bordism to obtain a bordism $F'\colon W' \to X \times I$ with the top normal map $(g', \partial g')$ transversal to all submanifolds of the filtration (18). Let, as above, $(M_i, \partial M_i)$ denote the transversal preimage of $(X_i, \partial X_i)$ of the map $(g', \partial g')$. The restriction of $(g', \partial g')$ to $(M_{k-2}, \partial M_{k-2})$ coincides with $(g_{k-2}, \partial g_{k-2})$, and the restriction to $(M_{k-1}, \partial M_{k-1})$ coincides with $(g_{k-1}, \partial g_{k-1})$.

Repeating these arguments and applying item (C) of the Theorem, we obtain a map

$$(g_0, \partial g_0)\colon (M', \partial M') \to (X, \partial X)$$

with the following properties. For any $0 \le j \le k-1$ the restriction of $(g_0, \partial g_0)$ to the transversal preimage of $(X_j, \partial X_j)$ is the s-triangulation $(g_j, \partial g_j)$ of the pair

$$(X_j, \partial X_j) \supset (X_{j+1}, \partial X_{j+1})$$

of manifolds with boundary. Now we need the following result.

If a normal map $(f, \partial f)\colon (M, \partial M) \to (X, \partial X)$ is an s-triangulation of the subfiltration

$$(X_{k-1}, \partial X_{k-1}) \subset \cdots \subset (X_1, \partial X_1) \subset (X, \partial X), \tag{21}$$

and the restriction $(f, \partial f)|_{(M_{k-1}, \partial M_{k-1})}$ is an s-triangulation of the pair (20) of manifolds with boundary, then, as follows from [4, Proposition 2.5], the map $(f, \partial f)$ is an s-triangulation of the filtration (18) $(\mathscr{X}, \partial \mathscr{X})$.

Let us apply this result $k-1$ times starting from the subfiltration

$$(X_1, \partial X_1) \subset (X_0, \partial X_0)$$

of the filtration

$$(X_2, \partial X_2) \subset (X_1, \partial X_1) \subset (X_0, \partial X_0)$$

up to the subfiltration (21) of the filtration (18). Corollary is proved. $\square$

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REFERENCES

1. C. T. C. Wall, *Surgery on Compact Manifolds*, Ed. by A. A. Ranicki, 2nd ed. (Amer. Math. Soc., Providence, RI, 1999).
2. A. A. Ranicki, *Exact Sequences in the Algebraic Theory of Surgery* (Mathem. Notes, Princeton, 1981).
3. W. Browder and F. Quinn, "A surgery theory for G-manifolds and stratified spaces," in *Manifolds* (Univ. Tokyo Press, 1975), pp. 27–36.
4. Yu. V. Muranov, D. Repovš, and R. Jimenez, "Surgery spectral sequence and manifolds with filtration," Trudy Moskov. Mat. Obshch. **67**, 294–325 (2006) [Trans. Moscow Math. Soc.].
5. S. Weinberger, *The Topological Classification of Stratified Spaces* (Univ. Chicago Press, Chicago–London, 1994).
6. S. Lopez de Medrano, *Involutions on Manifolds* (Springer-Verlag, Berlin–Heidelberg–New York, 1971).
7. M. M. Cohen, *A Course in Simple-Homotopy Theory* (Springer-Verlag, New York, 1973).
8. Yu. V. Muranov, D. Repovš, and F. Spaggiari, "Surgery on triples of manifolds," Mat. Sb. **8**, 139–160 (2003) [Russian Acad. Sci. Sb. Math. **194** (8), 1251–1271 (2003)].