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# GEOMETRIC TOPOLOGY OF GENERALIZED 3-MANIFOLDS

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Abstract. In this paper, we describe the history and the present status of one of the main classical problems in low-dimensional geometric topology—the recognition of topological 3-manifolds in the class of all generalized 3-manifolds (i.e., ANR homology 3-manifolds). This problem naturally splits into the cell-like resolution problem for 3-manifolds by means of homology 3-manifolds and the general-position problem for topological 3-manifolds. We have also included some open problems.

## 1. Introduction

We shall study the problem of finding geometric properties that are relatively easy to verify and that distinguish topological 3-manifolds in the class of topological spaces. More generally, Cannon [9] conjectured that manifolds can be characterized as generalized manifolds satisfying a minimal amount of general position. The need for such geometric criteria arises, e.g., in decomposition theory when one must decide whether a cell-like decomposition of a given topological manifold is again a manifold (necessarily of the same dimension). Several results are already known (see, e.g., [14, 15, 24]). Although the focus will rest on dimension 3, comparison with results obtained for dimensions $\geq 5$ provide useful insight as well as motivation. It is convenient to explain this now in terms of *generalized n-manifolds* $X^n$, namely, locally compact, locally contractible, finite-dimensional metric spaces with the local relative homology of $\mathbb{R}^n$, i.e., the group $H_*(X^n, X^n \setminus \{x\}; \mathbb{Z})$ is isomorphic to $H_*(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}; \mathbb{Z})$ for all $x \in X^n$.

In this paper, manifolds and generalized manifolds are assumed to have no boundary, unless otherwise specified. For $n \leq 2$, the $n$-manifolds coincide with the generalized $n$-manifolds, but for $n > 2$, the situation is much more complex. Upon making the obvious observations that $n$-manifolds are generalized $n$-manifolds and that the latter are defined in terms of elementary properties, one sees why the goal in Cannon's conjecture is to recognize genuine manifolds among generalized manifolds.

If $f\colon M \to X$ is a proper, cell-like, surjective map defined on an $n$-manifold and $\dim X < \infty$, then $X$ is a generalized $n$-manifold, but examples like Bing's famous dogbone space [1] reveal that $X$ need not be a topological manifold. Cell-like maps form the primary source for nonmanifold examples (for the topology of cell-like maps and homology manifolds, we refer the reader, for example, to [12]). Thus, a generalized $n$-manifold $X$ is *resolvable* if there exists a proper, cell-like, surjective map $f\colon M \to X$ defined on some $n$-manifold $M$, in which case the map $f$ is called a (cell-like) *resolution* of $X$.

Except for certain 3-dimensional examples whose existence depends on the hypothetical failure of the 3-dimensional Poincaré conjecture, the generalized manifolds explicitly described in the literature are all known to be resolvable. According to Quinn [22], the existence of a resolution for a given generalized $n$-manifold $X$ ($n \geq 5$) reduces to an integer-valued, algebraic obstruction $i(X)$ to a local surgery problem. This obstruction has the intriguing feature of being locally defined and locally constant. Consequently, if $X$ is connected and the obstruction vanishes on some open subset of $X$ (for instance, if some open subset is a manifold), then $X$ is resolvable.

Examples of generalized $n$-manifolds $X$ ($n \geq 6$) with nonvanishing Quinn's obstruction $i(X) \neq 1$ are known to exist: they were constructed by Bryant, Ferry, Mio, and Weinberger [7]. Note that no such connected example can have a manifold neighborhood at any point. Further constructions of $4k$-dimensional generalized manifolds ($k > 1$), which have no resolution, can be found in [11]. However, if the 3-dimen-

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sional Poincaré conjecture is false, then there is a nonresolvable generalized 3-manifold $X^3$; moreover, $X^3$ contains a point $x_0$ such that $X^3 \setminus \{x_0\}$ is a 3-manifold (cf. [23]).

The central 3-dimensional resolution problem is as follows: *Under the assumption that the Poincaré conjecture is true, do all generalized 3-manifolds have resolutions?* Thickstun has supplied an affirmative answer for generalized 3-manifolds whose singular set has "general-position dimension one" (earlier, he settled the 0-dimensional singular-set case in [26]).

## 2. General-Position Properties

In dimensions greater than four, the work of Edwards [18] provides means for detecting genuine manifolds among resolvable generalized manifolds in terms of the following simple general-position property. A metric space $X$ is said to have the *disjoint disks property* (briefly DDP) if any pair of maps from the standard 2-disk $I^2$ ($I = [0, 1]$) to $X$ can be approximated, arbitrarily closely, by a pair of maps with disjoint images. Recall that a *near-homeomorphism* is a map $X \to Y$ onto a metric space, which is the uniform limit of surjective homeomorphisms.

**Theorem 2.1** ([18])**.** *Let $p\colon M \to X$ be a cell-like resolution of a generalized $n$-manifold $X$ ($n \geq 5$). Then the map $p$ is a near-homeomorphism if and only if $X$ has the disjoint disks property.*

The first recognition theorem is due to Bing—he considered a special, cellular, upper semicontinuous decomposition $X$ of $\mathbb{R}^3$ called the dogbone space. In order to show that $X$ is not a 3-manifold, he demonstrated that $X$ fails to possess a certain disjoint disks property which is true in $\mathbb{R}^3$. His example is very intriguing since it turns out to be a factor of a 4-manifold.

**Theorem 2.2** ([1])**.**

(1) *$X$ is not a topological 3-manifold;*
(2) *the product $X \times \mathbb{R}$ is homeomorphic to $\mathbb{R}^4$.*

The first partially similar results of Edwards' kind in dimension 3 are due to Lambert and Sher [20] and, more than a decade later to Repovš and Lacher [24]. Finally, Daverman and Repovš proved in [14,15] 3-dimensional versions of Edwards' theorem. Part of the difficulty has been to produce appropriate general-position properties. The DDP is clearly inappropriate, being possessed by neither 3-manifolds nor 4-manifolds, and so some alternative property must be set forth. They investigated several concepts, chiefly described in terms of what we call simplicial approximation properties.

**Definition 2.3.** A space $X$ is said to have the *weak simplicial approximation property* (WSAP) if for each map $\mu\colon I^2 \to X$ and each $\varepsilon > 0$, there exists a map $\psi\colon I^2 \to X$ such that $\mathrm{dist}_X(\mu, \psi) < \varepsilon$ and $\psi(I^2)$ is contained in a finite union of 2-cells $B_i \subset X$, each 1-LCC embedded in $X$.

Moreover, $X$ is said to have the *simplicial approximation property* (SAP) if for each $\mu\colon I^2 \to X$ and each $\varepsilon > 0$, there exist a map $\psi\colon I^2 \to X$ and a finite topological 2-complex $K_\psi \subset X$ such that:

(1) $\mathrm{dist}_X(\psi, \mu) < \varepsilon$;
(2) $\psi(I^2) \subset K_\psi$;
(3) $X \setminus K_\psi$ is 1-FLG in $X$.

Finally, $X$ is said to have the *spherical simplicial approximation property* (SSAP) if the same holds when $I^2$ is replaced by $S^2$ throughout.

The 1-FLG condition is known to characterize tamely embedded 2-complexes $K_\psi$ in 3-manifolds $M^3$ (for many concepts of peripheral acyclicity and local homotopical properties, we refer the reader to [10,13]).

There are three elementary but significant observations to make. First, for a 2-complex $K_\psi \subset X$ having no local cut points, $X \setminus K_\psi$ is 1-FLG in $X$ if and only if each 2-simplex in $K_\psi$ is 1-LCC embedded in $X$. Second, SSAP implies SAP and, in turn, SAP implies WSAP. Third, manifolds of dimension $n \geq 3$ have all of these approximation properties.

**Definition 2.4.** A metric space $(X,\rho)$ is said to have the *light map separation property* (LMSP) if for every $\varepsilon > 0$, every $k \in \mathbb{N}$, and every map $f\colon B \to X$ of a collection of $k$ standard 2-cells $B = \coprod_{i=1}^{k} B_i^2$ into $X$ such that

(1) $N_f \subset \operatorname{Int} B$, where $N_f = \{y \in B \mid f^{-1}(f(y)) \neq y\}$;
(2) $\dim N_f \leq 0$;
(3) $\dim Z_f \leq 0$, where $Z_f = \{x \in X \mid x \in f(B_i^2) \cap f(B_j^2),\ \text{for some } i \neq j\}$,

there exists a map $F\colon B \to X$ such that

(1) $\rho(F, f) < \varepsilon$;
(2) $F|\partial B = f|\partial B$;
(3) for every $i \neq j$, $F(B_i^2) \cap F(B_j^2) = \varnothing$.

For a strengthening of the LMSP, see [2].

The main results proved in [14, 15], were two recognition theorems for 3-manifolds, which depend on these terms. The first of them is the following assertion.

**Theorem 2.5** ([14])**.** *A resolvable, generalized* 3*-manifold* $X$ *is a topological* 3*-manifold if and only if* $X$ *possesses both the WSAP and LMSP.*

The second result is as follows.

**Theorem 2.6** ([15])**.** *A resolvable, generalized* 3*-manifold is a topological* 3*-manifold if and only if it possesses the SSAP.*

As a corollary, a resolvable, generalized 3-manifold with nowhere dense nonmanifold set is a 3-manifold if and only if it has the SAP.

**Problem 2.7.** Is every resolvable, generalized 3-manifold with the WSAP (SAP, LMSP) a 3-manifold?

Here are two natural properties for singular disks in resolvable, generalized 3-manifolds $X$ that depend on an explicit resolution $f\colon M \to X$ of $X$.

**Definition 2.8.** A space $X$ is said to have the *resolution disjoint disks property* (RDDP) if for every $\varepsilon > 0$, every $k \in \mathbb{N}$, and every collection of $k$ pairwise disjoint, tame embeddings $f_i\colon B^2 \to M$, there exist maps $g_i\colon B^2 \to X$ satisfying the following conditions:

(1) $\rho(g_i, \pi f_i) < \varepsilon$;
(2) for every $i \neq j$, we have $g_i(B^2) \cap g_j(B^2) = \varnothing$.

Next, $X$ is said to have the *resolution embedding disk property* (REDP) if for every $\varepsilon > 0$ and every tame embedding $f\colon B^2 \to M$, there exists an embedding $g\colon B^2 \to X$ such that $\rho(g, \pi f) < \varepsilon$.

It is somewhat surprising that the following is still unsettled.

**Problem 2.9.** Is every cellular resolution of a generalized 3-manifold with RDDP (REDP) a near-homeomorphism?

**Problem 2.10.** Is every generalized 3-manifold with SSAP resolvable (hence a 3-manifold)?

## 3. Cell-Like Resolutions

The resolution conjecture in dimension three implies the Poincaré conjecture. The demonstration (only sketched here) relies on the following construction: "replace" each of a null sequence of pairwise-disjoint polyhedral 3-balls in an arbitrary closed 3-manifold $M$ by a compact, contractible 3-manifold. (We call a space obtained in this way a 3-*near manifold*.) Of course, if the classical Poincaré conjecture is valid, then every compact contractible 3-manifold is homeomorphic to the standard 3-ball and hence the construction yields only $M$. However, if the Poincaré conjecture fails (and infinitely many of the replacements are made with fake 3-balls), then the resulting space is a nonresolvable, generalized 3-manifold. More subtle examples of nonresolvable, generalized 3-manifolds (assuming failure of the Poincaré conjecture) were constructed in [4, 5]. These examples contain no fake 3-balls but admit so-called "near" resolutions (defined below). In light of the above examples, the following conjecture is a more reasonable reformulation of the resolution conjecture (in the sense that, given the Poincaré conjecture, it is equivalent to the resolution conjecture but does not imply the Poincaré conjecture).

**Near-Resolution Conjecture 3.1.** Any generalized 3-manifold is the cell-like image of a 3-near manifold.

**Definition 3.2.** We will call such a map a *near resolution* of the generalized 3-manifold. Furthermore, if $X$ is a generalized 3-manifold, then the *singular set* of $X$ is defined as

$$S(X) = \{p \in X \mid p \text{ has no Euclidean neighborhoods}\}.$$

The *manifold set* of $X$ is defined as $M(X) = X \setminus S(X)$.

**Remarks.** The construction of 3-near manifolds was generalized by Jakobsche to produce, given a counterexample to the Poincarè conjecture, a counterexample to the Bing–Borsuk conjecture (which states that any 3-dimensional, homogeneous ANR is a 3-manifold). An example is the (carefully controlled) inverse limit of 3-manifolds containing progressively more fake 3-balls. These spaces are completely singular, generalized 3-manifolds but are cell-like images of 3-near manifolds [Thickstun, unpublished] and hence compatible with the near-resolution conjecture.

The principal thrust of work on the resolution conjecture (and near-resolution conjecture) has produced a sequence of resolution theorems, each of which takes as an additional hypothesis (beyond those of the resolution conjecture itself), some restriction on the "size" (and in some cases "complexity") of the singular set of the generalized 3-manifold considered. We state these theorems in their chronological order (which is also roughly in the order of increasing generality; see the remarks below). In each of these statements, $X$ is a closed, generalized 3-manifold and any needed definitions are appended.

**Theorem 3.3** ([8])**.** *If $S(X)$ is 0-dimensional and 1-LCC, then $X$ is a closed 3-near manifold (and hence the identity map is a near resolution).*

**Remark.** The statement in [8] assumes that $M(X)$ contains at most finitely many fake 3-balls and concludes that $X$ is a 3-manifold. The above statement is an easy generalization.

**Theorem 3.4** ([4])**.** *If $S(X)$ is 0-dimensional and toral, then, modulo the Poincaré conjecture, $X$ has a resolution.*

**Definition 3.5.** The singular set $S(X)$ is *toral* if for any neighborhood $U$ of $S(X)$ in $X$, there exists a compact neighborhood $V$ of $S(X)$ with $V \subset U$ such that the frontier of $V$ is the disjoint union of finitely many tori.

**Theorem 3.6** ([5])**.** *If $S(X)$ is 0-dimensional and has arbitrarily tight neighborhoods with torsion-free fundamental groups, then, modulo the Poincaré conjecture, $X$ has a resolution.*

**Definition 3.7.** The singular set $S(X)$ has *arbitrarily tight neighborhoods with torsion-free fundamental group* if for any neighborhood $U$ of $S(X)$, there exists a neighborhood $V$ of $S(X)$ with $V \subset U$ such that each component of $V$ has torsion-free fundamental group.

**Theorem 3.8** ([25])**.** *If $S(X)$ is 0-dimensional, then $X$ has a near resolution.*

**Theorem 3.9** ([27])**.** *If $S(X)$ has general-position dimension one in $X$, then $X$ has a near resolution.*

**Definition 3.10.** If $X$ is a generalized 3-manifold and $A$ is a compact subspace of $X$, we say that $A$ has *general-position dimension one* in $X$ if any map $f\colon B^2 \to X$ can be approximated (to within any preassigned $\varepsilon > 0$) by a map $g\colon B^2 \to X$ such that $g(B^2) \cap A$ is 0-dimensional. (*Caution*: In the first version of [27], this property was called "embedding dimension one.")

**Remarks.** Obviously, Theorem 3.9 implies Theorem 3.8 and Theorem 3.8 is stronger than each of Theorems 3.3, 3.4, and 3.6. The relationships among Theorems 3.3, 3.4, and 3.6 is less clear-cut. Of course, it follows from the conclusion of Theorem 3.3 that any generalized 3-manifold (as in the hypothesis) has "spherical" singular set and, therefore, Theorem 3.4 can be applied, a posteriori, to a larger class of generalized 3-manifolds. It is unclear (and unknown to these authors) whether the hypothesis of Theorem 3.4 guarantees torsion-free neighborhoods of $S(X)$ and hence whether or not Theorem 3.6 is stronger than Theorem 3.4. On the other hand, there seems to exist no example $X$ of a generalized 3-manifold such that $\dim S(X) = 0$ and $X$ fails to satisfy the torsion-free hypothesis of Theorem 3.6.

Generalized 3-manifolds satisfying the hypothesis of Theorem 3.9 but not Theorem 3.8 are easily constructed. To construct one especially transparent class of such examples proceed as follows. Let $K$ be a tame 1-polyhedron in a closed 3-manifold $M$ and let $\{C_i\}_{i\in\mathbb{N}}$ be a null sequence of pairwise-disjoint, cell-like, noncellular compacta in $M$ such that, for each $i$, $C_i \cap K$ is a singleton and $\bigcup_{i=1}^{\infty}(C_i \cap K)$ is dense in $K$. The decomposition space arising from the decomposition of $M$ consisting of $\{C_i\}_{i\in\mathbb{N}}$ and singletons is the desired example $X$.

## 4. The Virtual Loop Theorem

The principal tool in the proof of each of these theorems is either the loop theorem [19] of classical 3-manifold topology or some extension or version thereof (although the proof of Theorem 3.4 as found in [4] must be considerably recast to conform to this mold). The significance of extensions of the loop theorem and their role in proving the resolution conjecture (or at least special cases of it) was hinted at by Brin in his doctoral thesis [3]. Subsequently, Brin proved a loop theorem (not stated here), taking for its hypothesis a certain compact singular surface (rather than a disk). That loop theorem led directly to Theorem 3.6 and the torsion-free hypothesis of Theorem 3.6 is inherited from Brin's loop theorem. The following loop theorem (the virtual loop theorem or VLT) is the main tool in the proof of Theorem 3.9. First, we introduce one term and some notation. Given $\alpha\colon S^1 \to Y$ (where $Y$ is connected), we denote by $[\alpha]$ the union of the two conjugacy classes in $\pi_1(Y)$ represented by $\alpha$. A map $g\colon B^2 \to X$, where $X$ is a generalized 3-manifold, is called a *pseudo-embedding* if $\dim[g(B^2) \cap S(X)] \leq 0$ and the set $\{x \in X \mid x \in g(B^2) \text{ and } g^{-1}(x) \text{ is not a singleton}\}$ is contained in $S(X)$ (so, roughly speaking, $g$ is an embedding "away from" $S(X)$).

**Theorem 4.1** (virtual loop theorem)**.** *Assume that $X$ is a generalized 3-manifold with boundary, $R$ is a connected surface in $\partial X$, $G$ is a normal subgroup of $\pi_1(R)$, and $f\colon (B^2, \partial B^2) \to (X, R)$ is a map such that $[f|\partial B^2] \not\subset G$. If $S(X)$ has general-position dimension one in $X$; then there exists a pseudo-embedding $g\colon (B^2, \partial B^2) \to (X, R)$ such that $[g|\partial B^2] \not\subset G$.*

**Remarks.** For simplicity, we have stated this theorem in less than the full generality found in [27]. The virtual loop theorem found in [25], similarly scaled down, would have the same statement but for the single additional hypothesis that $\dim S(X) \leq 0$.

A very brief and oversimplified sketch of the proof of Theorem 3.9 (from the VLT), intended to communicate the spirit of the proof, follows. Assume that $f\colon B^2 \to X$ is a "generic," "nontrivial" singular disk in $X$ (i.e., $\dim[f(B^2) \cap S(X)] = 0$ and $f$ cannot be $\varepsilon$-homotoped off of $S(X)$). By persistent use of the VLT, one obtains a family $\{g_i\colon B^2 \to X\}_{i=1}^{n}$ of pseudo-embeddings with pairwise-disjoint images such that $[f|\partial B^2]$ belongs to the normal closure of the union of the conjugacy classes in $\pi_1(U)$ determined by $\{g_i|\partial B^2\}_{i=1}^{n}$, where $U$ is some "small" neighborhood of $f(B^2)$ minus $S(X)$. Now, for each $i$, let $N_i$ be a regular neighborhood of $g_i(B^2) \setminus S(X)$ in $M(X)$. Note that, for each $i$, $\mathring{N}_i$ is homeomorphic to the product of the open unit interval $(0,1)$ with $\mathring{B}^2 \setminus Z_i$, where $Z_i$ is some compact, 0-dimensional subspace of $\mathring{B}^2$. Attach $B^2 \times (0,1)$ to $X \setminus [g_i(B^2) \cap S(X)]$ by using this homeomorphism. Repeat this procedure for each $i$ to obtain a noncompact space, such that $f|\partial B^2$ is null-homotopic in $M(V) = V \setminus S(V)$. The end point compactification of $V$ (denoted $X_2$) is again a generalized 3-manifold whose singular set has general-position dimension one. Furthermore, if we denote $X = X_1$, there is an easily defined "projection" map $p_1\colon X_2 \to X_1$, which is cell-like. Iterating this construction (using appropriately chosen generic, nontrivial singular disks), we obtain the following inverse sequence of cell-like maps on generalized 3-manifolds:

$$X_1 \xleftarrow{p_1} X_2 \xleftarrow{p_2} X_3 \xleftarrow{p_3} \dots .$$

The projection from $X_\infty = \lim X_i$ to $X_1$ is a cell-like map defined on a generalized 3-manifold and, furthermore, any singular disk in $X_\infty$ can be approximated by one with image in $M(X_\infty)$ (hence $S(X_\infty)$ is 1-LCC). By Theorem 3.3, $X_\infty$ is a 3-near manifold, and the proof is completed.

The long (close to 50 pages) proof of the VLT is impossible to even outline adequately in this survey, but we briefly indicate some of the principal ideas (however, note that the distinction between the proofs of the VLT stated above and the less general version found in [25] is too technical to touch upon here). First, recall the proof of the classical loop theorem [19] (whose hypotheses are the same as the VLT stated above except for the fact that the ambient space $X$ is a 3-manifold and whose conclusion is the same except for the fact that $g$ is an embedding). The following commutative diagram (called a "tower") is constructed:

$$\begin{array}{ccccccccc}
X \supset N_1 & \longleftarrow & \tilde{N}_1 \supset N_2 & \longleftarrow & \tilde{N}_2 \supset N_3 & \longleftarrow & \cdots & \longleftarrow & \tilde{N}_{n-1} \supset N_n \\
{\scriptstyle f_1}\uparrow & & {\scriptstyle f_2}\uparrow & & {\scriptstyle f_3}\uparrow & & & & {\scriptstyle f_n}\uparrow \\
B^2 & = & B^2 & = & B^2 & = & \cdots & = & B^2
\end{array} .$$

Here $f_1 = f$ and $N_1$ is a regular neighborhood of $f(B^2)$. For all $i$, $N_i$ is a regular neighborhood of $f_i(B^2)$ in $\tilde{N}_{i-1}$, where $\tilde{N}_i$ is a connected double covering of $N_{i-1}$ and $f_i$ is a lift of $f_{i-1}$ to $\tilde{N}_{i-1}$. It can be shown that the tower must "terminate" (i.e., $N_n$ has no connected double covering) from which one concludes (using duality) that the genus of $\partial N_n$ is zero. Then one easily extracts from $f_n(\partial B^2)$ an embedded "subloop" which (when projected down to $X$) "avoids" the normal subgroup $G$. Of course, this subloop bounds a disk in $\partial N_n$. This disk is "projected" step by step down the tower ("down" is to the left in our diagram). However, if self-intersections have been introduced at a given level, Dehn cuts are immediately performed to restore injectivity (since the projection maps are two to one such Dehn cuts are easily effected). Finally, at the bottom of the tower (in $X$), one has the desired embedding.

In a more general setting of the VLT, Dehn cuts become problematic (even for maps whose self-intersections in $M(X)$ are simple double lines) and an alternate approach was devised. First, we illustrate the use of it in the context of the classical loop theorem but only to conclude the existence of a map $g$ such that $g|\partial B^2$ is injective. As above, construct a tower but with these changes: the domain of the maps $f_i$ is, instead of $B^2$, the space $Z$ obtained by identifying two points $x$ and $y$ in $\partial B^2$ such that $f(x) = f(y)$ (and $f$ restricted to one of the arcs $\partial B^2 \setminus \{x, y\}$ is injective). The map $f_1$ is chosen to be the obvious factor of $f$ having domain $Z$. Each covering $\tilde{N}_i$ is the "largest" covering to which $f_1$ lifts (i.e., $\pi_1(\tilde{N}_i) = (f_i)_\sharp(\pi_1(Z))$). Again, the tower terminates but one now concludes, since $(f_n)_\sharp$ is surjective (and using duality), that the genus of $\partial N_n$ is no greater than one. Denoting by $e\colon \partial B^2 \to Z$ the identification map restricted to $\partial B^2$ and taking into account the "homological" conditions on $f_n \circ e\colon \partial B^2 \to \partial N_n$ (where we assume that $\partial N_n$ is a torus), one can conclude the existence of a disk in $\partial N_n$. This newly discovered disk is projected down to $X$ (with no Dehn cuts made). This map either satisfies the hypothesis of the loop theorem and has fewer boundary self-intersections or can be used together with the original map to establish such (via "cut and paste"). Repetition completes the proof.

The same approach is used to "eliminate" the singularities of $f|\partial B^2$ in the proof of the VLT. Instead of regular neighborhoods (which, of course, will not exist in general), we use neighborhoods which are regular with respect to some "large" compact polyhedral part of $f(B^2) \setminus S(X)$. In this way, one obtains a map of a compact planar surface into $M(X)$ extending arbitrarily far out toward $f(B^2) \cap S(X)$. One pieces these maps together (via a combinatorial argument originally used in [6], which is by now a fundamental device in the study of noncompact 3-manifolds) to obtain the desired map (i.e., one with fewer boundary self-intersections). It remains to "convert" the so-obtained "Dehn" singular disk (i.e., one with no boundary self-intersections) to a pseudo-embedding. This "desingularization" is carried out by a transfinite induction which makes use of the above ideas to produce an exotic variant of the standard compression procedure used in the study of noncompact 3-manifolds (and typically applied to exhaustions of such 3-manifolds). Here this "virtual compression procedure" is applied to an exhaustion of a regular neighborhood $N$ of $f(B^2) \cap M(X)$ in $M(X)$ to obtain, ultimately, a new ambient 3-manifold $N'$, in which the boundary of the Dehn map (a simple loop) is null-homotopic and hence, by the classical Dehn's lemma, bounds an embedded disk. Restricting this embedded disk to $N \cap N'$, one obtains in $N$ an embedded compact planar surface reaching "arbitrarily far toward infinity" in $N$ (where how "far" is determined by the neighborhood of infinity in $N$ to which the virtual compression procedure was confined). The same combinatorial argument used before is now used to piece together such "planar embeddings" to obtain the desired pseudo-embeddings.

Even the most general of the above resolution theorems requires that the singular set of the generalized 3-manifold be very "small," and its proof relies on techniques coming from the study of noncompact 3-manifolds. This approach may have now run its course; further progress might require some fundamentally new ideas.

However, further refinement of the techniques used in the proof of Theorem 3.9 might result in a stronger resolution theorem taking as its hypothesis (instead of "$S(X)$ has general-position dimension one in $X$") the following property (subsequently referred to as $P$): any map of a 2-disk into $X$ can be approximated by a map such that the preimage of $S(X)$ under the approximating map is 0-dimensional. We leave it to the reader to verify that $P$ is no more restrictive than the hypothesis of Theorem 3.9.

We are indebted to R. J. Daverman for the following example, which then demonstrates that $P$ is, in fact, less restrictive (so the example is a generalized 3-manifold $X$ with $P$ such that $S(X)$ fails to have general-position dimension less than or equal to one in $X$). The space $X$ is the decomposition space of a cell-like decomposition of $S^3$. The nondegenerate elements of the decomposition are all subsets of the subspace $Z$ of $S^3$ constructed as follows. Denoting the Cantor set and Whitehead compactum by $C$ and $W$, respectively, let $C \times W \subset S^3$ be the embedding obtained by "ramifying" the defining sequence for the standard embedding of $W$ in $S^3$. Choose a point $x \in W$ and ("tamely") embed the mapping cylinder of a 2-to-1 surjective map $f\colon C \times \{x\} \to L$, where $L$ is a tame arc in $S^3$ disjoint from $C \times W$. Then $Z$ is the union of $C \times W$ with this mapping cylinder. Denoting by $F\colon Z \to L$ the obvious retraction, each nondegenerate element of the decomposition is $F^{-1}(p)$, for some $p \in L$ (note that for each $p \in L$, $F^{-1}(p)$ is either a single component of $C \times W$ wedged with a tame arc or a pair of components connected by a tame arc). Now let $\alpha$ be a simple loop in $S^3$ which avoids $Z$ and links $T$ (the first solid torus in a defining sequence for $C \times W$). Let $\beta$ be the image of $\alpha$ under the decomposition map. Clearly, $S(X)$ is the (homeomorphic) image of $L$ under the decomposition map. If a null-homotopy of $\beta$ intersected $S(X)$ in a 0-dimensional subspace, then $\varepsilon$-lifting would yield the existence of a null-homotopy of $\alpha$ missing at least some nondegenerate elements of the decomposition. However, $\alpha$ (geometrically) links all of them. On the other hand, since any singular disk in $S^3$ can be approximated by one which intersects $Z$ in a 0-dimensional subspace, $X$ has property $P$.

## 5. Conclusion

The following resolution theorem, recently proven by Daverman and Thickstun [17], takes as its hypothesis not a restriction on the "size" of the singular set but rather a rich supply of polyhedral subspaces. This theorem was inspired by the characterizations of 3-manifolds among resolvable generalized 3-manifolds found in [14] and, together with one of those theorems, yields a characterization, modulo the Poincaré conjecture, of 3-manifolds among generalized 3-manifolds.

**Theorem 5.1** ([17]). *A generalized* 3*-manifold* $X$ *has a near resolution if there exists a sequence* $\{f_i\colon R_i \to X\}_{i=1}^{\infty}$ *of maps of closed surfaces satisfying the following two conditions*:

(1) *for all* $k \in \mathbb{N}$, $\bigcup_{i=1}^{k} f_i(R_i)$ *is a polyhedron*;

(2) *for distinct points $p$ and $q$ in $X$, there exists $i \in \mathbb{N}$ such that $p$ and $q$ are homologically separated by $f_i$.*

*Sketch of the proof.* One constructs an inverse sequence

$$X = X_0 \xleftarrow{\varphi_1} X_1 \xleftarrow{\varphi_2} X_2 \xleftarrow{\varphi_3} \dots$$

of generalized 3-manifolds and conservative, cell-like maps such that, for each $k$, all $f_i$ for which $i \le k$ $\varepsilon$-lift into $M(X_k)$ (under $\varphi_1 \circ \varphi_2 \circ \cdots \circ \varphi_k$). This property implies that for every $i$, $f_i$ $\varepsilon$-lifts into $X_\infty$ (the inverse limit), which is easily seen to imply that $\dim S(X_\infty) = 0$. Furthermore, the inverse limit of generalized 3-manifolds under cell-like maps is a generalized 3-manifold (and the projection map is cell-like). Hence $X_\infty$ is a generalized 3-manifold with 0-dimensional singular set and so by [25] has a near resolution. The composition of that near resolution with the projection from $X_\infty$ to $X$ is then the desired near resolution of $X$.

**Definition 5.2.** A generalized 3-manifold has the *relative simplicial approximation property* (RSAP) if, given any compact subpolyhedron $K$ of $B^2$ and map $f\colon B^2 \to X$ such that $f|K$ is simplicial, $f$ can be approximated by a simplicial map $g\colon B^2 \to X$ such that $f|K = g|K$. (*Note*: By Nicholson's theorem [21] and classical relative simplicial approximation for maps with polyhedral domain and target, all 3-manifolds have the RSAP.)

**Corollary 5.3.** *Any generalized* 3*-manifold having the RSAP is near-resolvable.*

*Sketch of the proof.* It is fairly straightforward to demonstrate that any generalized 3-manifold with the RSAP satisfies the hypothesis of the above theorem.

**Theorem 5.4.** *The Poincaré conjecture implies that a generalized* 3*-manifold is a* 3*-manifold if and only if it satisfies the RSAP.*

*Proof.* Combine Corollary 5.3 with [15, Theorem 3.1], noting that the RSAP is stronger than the SSAP.

**Remarks.** Note that although every generalized 3-manifold is conjecturally resolvable, the condition (RSAP) used above to ensure resolvability (modulo the Poincaré conjecture) is a priori stronger than the condition (SSAP) required for approximability of that resolution. This is an inherently unsatisfactory situation which inevitably suggests the following two questions. Does the SSAP imply the RSAP? Failing an affirmative answer to the first question, can it be shown that generalized 3-manifolds with the SSAP are near resolvable?

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